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Periodic Solution of the Viscous Modified Camassa-Holm Equation

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Abstract: This paper is concerned with time periodic solution of the viscous modified Camassa-Holm equation with a periodic boundary condition. The existence and uniqueness of a time periodic solution is presented.

Keywords: viscous modified Camassa-Holm equation; periodic solution; Galerkin method; Leray-Schauder fixed point theorem

1 Introduction

In 2001, Liu and Qian [1] discussed the peakons and their bifurcations when the integral constants taken as zero of the following generalized Camssa-Holm equation (GCH equation in short):

$$u_t + 2ku_x - u_{xxt} + au^m u_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

where $a > 0$, $k \in \mathrm{R}$, $m \in \mathrm{N}$ and m is called the strength of nonlinearity. They introduced Eq. (1) from the mathematical point of view. In the case of $m = 1, 2, 3$ and $k \neq 0$, they gave the explicit expression for the peakons. Tian and Song [2] derived some new peaked solitary wave solutions of Eq. (1) for $m = 1, 2, 3$. Shen and Xu [3] analyzed the dynamical behavior of travelling wave solutions of Eq. (1) by using the bifurcation theory and the method of phase portraits analysis. They showed that Eq. (1) has compactons and cuspwaves for arbitrary integer m.

For $m = 1$ and $a = 3$, Eq. (1) becomes the well-known Camassa-Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, $u(t, x)$ standing for the fluid velocity at time $t \geq 0$ in the spatial x direction and k being a nonnegative parameter related to the critical shallow water speed [4-5]. The Camassa-Holm equation has a bi-Hamiltonian structure [6] and is completely integrable [4,7]. Its solitary waves are smooth if $k > 0$ and peaked in the limiting case $k = 0$ [8]. The peaked solitons are orbital stable [9]. The explicit interaction of the peaked solitons is given in [10]. The Cauchy problems of the Camassa-Holm equation have been extensively studied: the equation is locally well-posed [11-14] for the initial data $u_0 \in H^s(I)$ with $s > 3/2$, where $I = \mathrm{R}$ or $I = \mathrm{R}/\mathrm{Z}$. More interestingly, it has global strong solutions [11,13,15] and also blow-up solutions in finite time [11,13,15,16]. On the other hand, it has global weak solutions in $H^1(I)$ [13,17,18]. It is worth pointing out that the Camassa-Holm equation models breaking waves [8,16].

With $m = 2$, $a = 3$ and $k = 0$ in Eq. (1) we find the following equation

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx}, \tag{2}$$

which was presented by Wazwaz [19]. Since then, Eq. (2) has been also investigated by many authors. Tian and Song [2] gave some physical explanation and obtained peakons composed of hyperbolic function $\tanh z$ for Eq. (2). Wazwaz [20] showed that Eq. (2) has the following solitary wave solution:

$$u(x, t) = -2 \sec h^2 \frac{(x - 2t)}{2}. \tag{3}$$

In [21], using the bifurcation method of planar systems and numerical simulation of differential equations, Liu and Ouyang showed the following fact: for the wave speed $c = 2$, the solitary wave and the peakon coexist in Eq. (2). The solitary wave is given by (3) and the peakon is expressed by $u(x, t) = \frac{2}{(\cosh(\frac{x}{2} - t) + \sqrt{2} \sinh|\frac{x}{2} - t|)^2}$.

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Through some special phase orbits, Wang and Tang [22] showed that Eq. (2) has the following solitary wave and peakon solutions: $u(x,t) = \frac{1}{3}[1 - 4\sec h^2 \frac{1}{\sqrt{6}}(x - \frac{t}{3})]$, and $u(x,t) = \frac{8}{(\sqrt{2}+|x-3t|)^2} - 1$. There are other research papers about this equation, to name just a few.

In this paper, we would like to consider the following viscous version of Eq. (2):

$$u_t - u_{txx} - \gamma(u_{xx} - u_{xxxx}) + 3u^2 u_x - 2u_x u_{xx} - uu_{xxx} = f(t,x), \quad t > 0, \quad x \in \mathrm{R}, \tag{4}$$

$$u(t, x + L) = u(t,x), \quad t > 0, \quad x \in \mathrm{R}, \tag{5}$$

$$u(t + \omega, x) = u(t,x), \quad t > 0, \quad x \in \mathrm{R}, \tag{6}$$

where $\gamma > 0$ is a viscosity constant and the forcing term f is ω-periodic in time t and L-periodic in spatial x. Without loss of generality, we assume further $\int_\Omega f(t,x)dx = 0$, where $\Omega = [0, L]$. Eq. (4) is the generalized form of the viscous Camassa-Holm equation which can be viewed as a one dimensional version of the three dimensional Navier-Stokes-alpha model for turbulence [23]. We name Eq. (4) the viscous modified Camassa-Holm equation. We have studied the optimal control problem for Eq. (4) [24].

When system is periodically dependent on time t, we want to know whether there exists time-periodic solution with the same period for the system. In many nonlinear evolution equations, the study of time-periodic solution has attracted considerable interest (for example [25-27]). To finish this object, we shall prove that Eqs. (4-6) have a time-periodic solution based on the Galerkin method and Leray–Schauder fixed point theorem [25].

Our paper is organized as follows. In Sec. 2, we give some definition of space and inequalities used throughout paper. In Sec. 3, the existence of the approximate solution is discussed. The convergence of a sequence of the approximate solution is proved using uniform a priori estimates. Sec. 4 is devoted to the proof of the existence and uniqueness of time-periodic solution for Eqs. (4-6).

2 Preliminaries

Let X be a Banach space, for $1 \le p \le \infty$, the space $L^p(\omega; X)$ is the set ω-periodic X-valued measurable functions on R such that $\|u\|_{L^p(\omega;X)} = \begin{cases} (\int_0^\omega \|u\|_X^p \, dt)^{1/p} < \infty, 1 \le p < \infty \\ \sup\limits_{0 \le t \le \omega} \|u\|_X < \infty, p = \infty \end{cases}$.

The space $W^{k,p}(\omega; X)$ denote the set of functions which belong to $L^p(\omega; X)$ together with their derivatives up to order k, and we write $W^{k,2}(\omega; X) = H^k(\omega; X)$ in particular when X is a Hilbert space.

The following inequalities (see [28]) will be used in the proofs later:

$$\|u\|_\infty \le k_1 \|u\|_{H^1}. \tag{7}$$

$$\left\|D^j u\right\|_p \le k_2 \|u\|_{H^m}^\theta \|u\|^{1-\theta}, \tag{8}$$

where $D^j u = \frac{\partial^j u}{\partial x^j}$, $\frac{1}{p} = j + \theta(\frac{1}{2} - m) + (1-\theta)\frac{1}{2}$ as $0 \le j < m$, j/m $\le \theta \le 1$.

$$\|u\| \le k_3 \|u_x\|, \quad \int_\Omega u(x)dx = 0. \tag{9}$$

3 A uniform priori estimates

We denote the unbounded linear operator $Au = -u_{xx}$ on $X = L^2 \cap \{u \, |u(x + L) = u(x), \int_\Omega u dx = 0\}$, then the set of all linearly independent eigenvectors $\{w_j\}_{j=0}^\infty$ of A, i.e., $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty$, is an orthonormal basis of $L^2(\Omega)$. For any n and a group of function $\{a_{jn}(t)\}_{j=1}^n$, where $a_{jn}(t)(j = 1, 2, \cdots, n) \in C^1(\omega; \mathrm{R})$, the function $u_n = \sum\limits_{j=1}^n a_{jn}(t) w_j \in C^1(\omega; H_n)$ is called an approximate solution to Eqs. (4)-(6) if it satisfies the equation as follows:

$$(u_{nt} - u_{nxxt} - \gamma(u_{nxx} - u_{nxxxx}), w_j) = (Nu_n + f, w_j), \quad j = 1, \cdots, n, \tag{10}$$

where $Nu_n = -3u_n^2 u_{nx} + 2u_{nx} u_{nxx} + u_n u_{nxxx}$ and $S_n = span\{w_1, w_2, \cdots, w_n\}$. By the classical theory of ordinary differential equations, for any fixed $v_n(t) = \sum\limits_{j=1}^n b_{jn}(t) w_j \in C^1(\omega; S_n)$, the equation $(u_{nt} - u_{nxxt} - \gamma(u_{nxx} -$

$u_{nxxxx}), w_j) = (Nv_n + f, w_j)$, $j = 1, \cdots, n$, has a unique ω-periodic solution u_n and the mapping $F : v_n \to u_n$ is continuous and compact in $C^1(\omega; S_n)$. Hence from Leray-Schauder fixed point theorem, we want to prove the existence of an approximate solution only to show that $\sup\limits_{0\le t\le\omega} \|u_n\|^2 \le c$ for all possible solution of Eq. (10) replaced by $\lambda N u_n$ $(0 \le \lambda \le 1)$ instead of nonlinear term Nu_n, where c is a constant which only depends on L, ω, ε, γ, k, k_1, k_2, k_3, f.

Lemma 3.1 *Assume that $f \in C^1(\omega; H^{-1}(\Omega))$, then $\sup\limits_{0\le t\le\omega} (\|u_n\|^2 + \|u_{nx}\|^2) \le c_1$, where c_1 is a constant which depends on L, ω, ε, γ, λ_1, f, $M = \sup\limits_{0\le t\le\omega} \{\|f(t,x)\|^2_{H^{-1}(\Omega)}\}$ and $d_1 = \min\{2\gamma\lambda_1, 2\gamma - \varepsilon\} > 0$.*

Proof. Multiplying Eq. (10) by $a_{jn}(t)$ and summing up over j from 1 to n, we obtain $(u_{nt} - u_{nxxt} - \gamma(u_{nxx} - u_{nxxxx}), u_n) = (Nu_n + f, u_n)$.

Then, we can get

$$\frac{1}{2}\frac{d}{dt}(\|u_n\|^2 + \|u_{nx}\|^2) + \gamma(\|u_{nx}\|^2 + \|u_{nxx}\|^2) = (Nu_n + f, u_n). \tag{11}$$

Notice that $-3\int_\Omega u_n^3 u_{nx}dx = 0$, $2\int_\Omega u_n u_{nx} u_{nxx}dx + \int_\Omega u_n^2 u_{nxxx}dx = 0$ and $\|u_{nxx}\| = \int_\Omega \left|(\sum\limits_{j=1}^n a_{jn}(t)w_j)_{xx}\right|^2 dx = \int_\Omega \left|\sum\limits_{j=1}^n \lambda_j a_{jn}(t)w_j\right|^2 dx \ge \lambda_1 \|u_n\|^2$. From Young's inequality, we have $\int_\Omega f u_n dx \le \frac{\varepsilon}{2}\|u_{nx}\|^2 + \frac{M}{2\varepsilon}$, where $\varepsilon > 0$ is a constant and $M = \sup\limits_{0\le t\le\omega} \{\|f(t,x)\|^2_{H^{-1}(\Omega)}\}$.

According to the above analysis, we can deduce from Eq. (11) that

$$\frac{d}{dt}(\|u_n\|^2 + \|u_{nx}\|^2) + d_1(\|u_n\|^2 + \|u_{nx}\|^2) \le \frac{M}{\varepsilon}, \tag{12}$$

where $d_1 = \min\{2\gamma\lambda_1, 2\gamma - \varepsilon\} > 0$.

Considering the time periodicity of u_n and integrating (12) over $[0, \omega]$, we get $d_1\int_0^\omega (\|u_n\|^2 + \|u_{nx}\|^2)dt \le \frac{\omega M}{\varepsilon}$. Hence there exists $t^* \in [0, \omega)$ such that $\|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2 \le \frac{M}{d_1\varepsilon}$. From (12), we have $\frac{d}{dt}(\|u_n\|^2 + \|u_{nx}\|^2) \le \frac{M}{\varepsilon}$. Integrating the above inequality with respect to t from t^* to $t \in [t^*, t^* + \omega]$, we obtain that $\|u_n(t)\|^2 + \|u_{nx}(t)\|^2 \le \|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2 + \frac{\omega M}{\varepsilon} \le (\frac{1}{d_1} + \omega)\frac{M}{\varepsilon}$.

Hence, we infer

$$\sup_{0\le t\le\omega} (\|u_n\|^2 + \|u_{nx}\|^2) \le (\frac{1}{d_1} + \omega)\frac{M}{\varepsilon} \triangleq c_1, \tag{13}$$

which concludes our proof. ■

From Lemma 3.1 and Leray-Schauder fixed point theorem, Eq. (10) has solution $\{u_n\}_{n=1}^\infty$, which is also a sequence of approximate solutions of Eqs. (4-6). In order to obtain the convergence of sequence $\{u_n\}_{n=1}^\infty$, we need to give a priori estimates for the high order derivers of $\{u_n\}_{n=1}^\infty$.

Lemma 3.2 *If $f \in C^1(\omega; H^{-1}(\Omega))$, then $\sup\limits_{0\le t\le\omega} (\|u_{nx}\|^2 + \|u_{nxx}\|^2) \le c_2$, where c_2 is a constant which only depends on L, ω, ε, γ, f, k_1, k_2 and c_1.*

Proof. Multiplying Eq. (10) by $-\lambda_j a_{jn}(t)$ and summing up over j from 1 to n, we have $(u_{nt} - u_{nxxt} - \gamma(u_{nxx} - u_{nxxxx}), u_{nxx}) = (Nu_n + f, u_{nxx})$. The above equation gives

$$-\frac{1}{2}\frac{d}{dt}(\|u_{nx}\|^2 + \|u_{nxx}\|^2) - \gamma(\|u_{nxx}\|^2 + \|u_{nxxx}\|^2) = (Nu_n + f, u_{nxx}) \tag{14}$$

From Young's inequality, we get

$$\left|\int_\Omega f u_{nxx}dx\right| \le \frac{\varepsilon}{2}\|u_{nxxx}\|^2 + \frac{M}{2\varepsilon}, \tag{15}$$

where $\varepsilon > 0$ is a constant and $M = \sup\limits_{0\le t\le\omega} \{\|f(t,x)\|^2_{H^{-1}(\Omega)}\}$.

From (7), (13), Young's inequality, we can deduce that

$$\begin{aligned}\left|\int_\Omega u_n^2 u_{nx}u_{nxx}dx\right| &\le \|u_n\|_\infty \|u_n\|_\infty \int_\Omega |u_{nx}u_{nxx}|\,dx \le k_1^2\|u_n\|_{H^1}^2\int_\Omega |u_{nx}u_{nxx}|\,dx\\ &\le k_1^2c_1\left(\frac{\varepsilon}{2k_1^2c_1}\|u_{nxx}\|^2+\frac{k_1^2c_1}{2\varepsilon}\|u_{nx}\|^2\right)\le \frac{\varepsilon}{2}\|u_{nxx}\|^2+\frac{k_1^4c_1^3}{2\varepsilon}\end{aligned} \tag{16}$$

From (8), (13), Cauchy-Schwarz inequality, Young's inequality and Lemma 1, we get

$$\begin{aligned}\left|\int_\Omega u_n u_{nxx}u_{nxxx}dx\right| &= \left|-\frac{1}{2}\int_\Omega u_{nx}u_{nxx}^2dx\right| \le \frac{1}{2}\|u_{nx}\|\,\|u_{nxx}\|_4^2 \le \frac{1}{2}c_1^{1/2}k_2^2\|u_n\|^{1/2}\|u_n\|_{H^3}^{3/2}\\ &\le \frac{3}{4}\varepsilon\|u_n\|_{H^3}^2+\frac{c_1^3k_2^8}{64\varepsilon^3}\le \frac{3}{4}\varepsilon c_1+\frac{3}{4}\varepsilon\|u_{nxx}\|^2+\frac{3}{4}\varepsilon\|u_{nxxx}\|^2+\frac{c_1^3k_2^8}{64\varepsilon^3}\end{aligned} \tag{17}$$

Taking (15-17) into account and choosing ε small enough such that $\frac{15}{4}\varepsilon < \frac{\gamma}{2}$, then (14) yields

$$\frac{d}{dt}(\|u_{nx}\|^2+\|u_{nxx}\|^2)+\gamma(\|u_{nxx}\|^2+\|u_{nxxx}\|^2)\le \frac{M}{\varepsilon}+\frac{3k_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3c_1^3k_2^8}{32\varepsilon^3}. \tag{18}$$

Integrating (18) about t from 0 to ω and considering the time periodicity of u_n, we get there exists $t^*\in[0,\omega)$ such that $\|u_{nxx}(t^*)\|^2+\|u_{nxxx}(t^*)\|^2\le\frac{1}{\gamma}(\frac{M}{\varepsilon}+\frac{3k_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3c_1^3k_2^8}{32\varepsilon^3})$.

From(18), we have $\|u_{nx}(t)\|^2+\|u_{nxx}(t)\|^2\le\|u_{nx}(t^*)\|^2+\|u_{nxx}(t^*)\|^2+(\frac{M}{\varepsilon}+\frac{3k_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3c_1^3k_2^8}{32\varepsilon^3})\omega$.
Then we can obtain

$$\sup_{0\le t\le\omega}(\|u_{nx}\|^2+\|u_{nxx}\|^2)\le(\frac{1}{\gamma}+\omega)(\frac{M}{\varepsilon}+\frac{3k_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3c_1^3k_2^8}{32\varepsilon^3})\triangleq c_2. \tag{19}$$

■

In the following, we continue to establish a priori estimates for high order derivers of the approximate solution $\{u_n\}_{n=1}^\infty$ by an inductive argument.

Lemma 3.3 *For any $k\ge 0$, if $f\in C^1(\omega;H^{k-1}(\Omega))$, then $\sup_{0\le t\le\omega}(\|D^{k+1}u_n\|^2+\|D^{k+2}u_n\|^2)\le c$, where c is a constant which depends on L, ω, ε, γ, k, k_1, k_2, k_3, f.*

Proof. The conclusion of Lemma 3.3 holds for $k=0$ from Lemma 3.2. Assume that for $k\le m-1$ $(m\ge 2)$ the conclusion of Lemma 3 holds, we want to prove that the same statement holds for $k=m$ also.

Multiplying Eq. (10) by $(-1)^{m+1}\lambda_j^{m+1}a_{jn}(t)$ and summing up over j from 1 to n, we have

$$(-1)^{m+1}\frac{1}{2}\frac{d}{dt}(\|D^{m+1}u_n\|^2+\|D^{m+2}u_n\|^2)+(-1)^{m+1}\gamma(\|D^{m+2}u_n\|^2+\|D^{m+3}u_n\|^2)=(Nu_n+f,D^{2(m+1)}u_n). \tag{20}$$

From Young's inequality, we get

$$\left|\int_\Omega fD^{2(m+1)}u_ndx\right|=\left|\int_\Omega D^{m-1}fD^{m+3}u_ndx\right|\le\frac{\varepsilon}{2}\|D^{m+3}u_n\|^2+\frac{1}{2\varepsilon}\|D^{m-1}f\|^2. \tag{21}$$

From the conclusion of Lemma 3 for $k\le m-1$, (7), (9) and Young's inequality, we can deduce that

$$\begin{aligned}&\left|\int_\Omega u_n^2u_{nx}D^{2(m+1)}u_ndx\right|\le\|u_n\|_\infty\left|\int_\Omega(\sum_{i=0}^{m+1}C_{m+1}^iD^iu_nD^{m+1-i}u_{nx})D^{m+1}u_ndx\right|\\ &\le\|u_n\|_\infty\int_\Omega|u_nD^{m+2}u_nD^{m+1}u_n|\,dx+\|u_n\|_\infty\int_\Omega\left|(\sum_{i=1}^{m+1}C_{m+1}^iD^iu_nD^{m+1-i}u_{nx})D^{m+1}u_n\right|dx\\ &\le\varepsilon\|D^{m+2}u_n\|^2+\frac{[k_1^2k_3^2c]^2}{4\varepsilon}\|D^{m+1}u_n\|^2+k_1\|u_n\|_{H^1}\int_\Omega\left|(\sum_{i=1}^{m+1}C_{m+1}^iD^iu_nD^{m+1-i}u_{nx})D^{m+1}u_n\right|dx\\ &\le\varepsilon\|D^{m+2}u_n\|^2+c(\varepsilon,k_1,k_3,m).\end{aligned} \tag{22}$$

$$\left|\int_\Omega u_{nx}u_{nxx}D^{2(m+1)}u_n dx\right| \le \|u_{nx}\|_\infty \int_\Omega \left|D^{m+1}u_n D^{m+3}u_n\right| dx$$
$$+\sum_{i=1}^{m-1} C_{m-1}^i \left\|D^{i+1}u_n\right\|_\infty \left\|D^{m+1-i}u_n\right\|_\infty \int_\Omega \left|D^{m+3}u_n\right| dx \le 2\varepsilon \left\|D^{m+3}u_n\right\|^2 + c(m,k_1,\varepsilon,L). \tag{23}$$

Similarly, we can also deduce that

$$\left|\int_\Omega u_n u_{nxxx}D^{2(m+1)}u_n dx\right| \le \int_\Omega \left|u_n D^{m+3}u_n D^{m+2}u_n\right| dx + \int_\Omega \left|C_m^1 Du_n (D^{m+2}u_n)^2\right| dx$$
$$+\int_\Omega \left|C_m^2 D^2u_n D^{m+1}u_n D^{m+2}u_n\right| dx + \int_\Omega \left|(\sum_{i=3}^{m} C_m^i D^i u_n D^{m+3-i}u_n)D^{m+2}u_n\right| dx$$
$$\le c(k_1,k_3,m)(\int_\Omega \left|D^{m+3}u_n D^{m+2}u_n\right| dx + \int_\Omega \left|D^{m+2}u_n\right|^2 dx + \int_\Omega \left|D^{m+1}u_n D^{m+2}u_n\right| dx + \int_\Omega \left|D^{m+2}u_n\right| dx). \tag{24}$$

From the conclusion of Lemma 3.3 for $k \le m-1$, Young's inequality and (8), we have

$$c(k_1,k_3,m)\int_\Omega \left|D^{m+3}u_n D^{m+2}u_n\right| dx \le \varepsilon \left\|D^{m+3}u_n\right\|^2 + c(k_1,k_2,k_3,m,\varepsilon)\|u_n\|_{H^{m+3}}^{2(m+2)/m+3} \|u_n\|^{2(1-m+2/m+3)}$$
$$\le \varepsilon \left\|D^{m+3}u_n\right\|^2 + \varepsilon\|u_n\|_{H^{m+3}}^2 + c(k_1,k_2,k_3,m,\varepsilon)\|u_n\|^2 \le 2\varepsilon \left\|D^{m+3}u_n\right\|^2 + \varepsilon \left\|D^{m+2}u_n\right\|^2 + c(k_1,k_2,k_3,m,\varepsilon),$$

$$c(k_1,k_3,m)\int_\Omega \left|D^{m+2}u_n\right|^2 dx \le c(k_1,k_2,k_3,m)\|u_n\|_{H^{m+3}}^{2(m+2)/m+3} \|u_n\|^{2(1-m+2/m+3)}$$
$$\le \varepsilon \left\|D^{m+3}u_n\right\|^2 + \varepsilon \left\|D^{m+2}u_n\right\|^2 + c(k_1,k_2,k_3,m,\varepsilon),$$

$$c(k_1,k_3,m)\int_\Omega \left|D^{m+1}u_n D^{m+2}u_n\right| dx \le \varepsilon \left\|D^{m+2}u_n\right\|^2 + c(k_1,k_3,m,\varepsilon)\left\|D^{m+1}u_n\right\|^2 \le \varepsilon \left\|D^{m+2}u_n\right\|^2 + c(k_1,k_3,m,\varepsilon),$$

and $c(k_1,k_3,m)\int_\Omega \left|D^{m+2}u_n\right| dx \le \varepsilon \left\|D^{m+2}u_n\right\|^2 + c(k_1,k_3,m,\varepsilon,L)$.

Combining (24) and the above inequality, we can get

$$\left|\int_\Omega u_n u_{nxxx}D^{2(m+1)}u_n dx\right| \le 3\varepsilon \left\|D^{m+3}u_n\right\|^2 + 4\varepsilon \left\|D^{m+2}u_n\right\|^2 + c(k_1,k_2,k_3,m,\varepsilon,L). \tag{25}$$

Taking (20-23) and (25) into account, we can deduce that $\frac{d}{dt}(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2) + 2\gamma(\left\|D^{m+2}u_n\right\|^2 + \left\|D^{m+3}u_n\right\|^2) \le 15\varepsilon \left\|D^{m+3}u_n\right\|^2 + 14\varepsilon \left\|D^{m+2}u_n\right\|^2 + c(k_1,k_2,k_3,m,\varepsilon,f,L)$

Choosing ε small enough such that $15\varepsilon < \gamma$, we have

$$\frac{d}{dt}(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2) + \gamma(\left\|D^{m+2}u_n\right\|^2 + \left\|D^{m+3}u_n\right\|^2) \le c(k_1,k_2,k_3,m,\varepsilon,f,L). \tag{26}$$

Integrating (26) about t from 0 to ω, there exists $t^* \in [0,\omega)$ such that

$$\left\|D^{m+2}u_n(t^*)\right\|^2 + \left\|D^{m+3}u_n(t^*)\right\|^2 \le \frac{c(k_1,k_2,k_3,m,\varepsilon,f,L)}{\gamma}. \tag{27}$$

From (26), we have $\frac{d}{dt}(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2) \le c(k_1,k_2,k_3,m,\varepsilon,f,L)$.

Integrating the above inequality from t^* to $t \in [t^*, t^*+\omega]$ and with (18), we can easily obtain

$$\sup_{0\le t\le\omega}(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2) \le (\frac{k_3^2}{\gamma} + \omega)c(k_1,k_2,k_3,m,\varepsilon,f,L) \triangleq c.$$

The proof is completed. ■

Lemma 3.4 *For any $k \geq 0$, if $f \in C^1(\omega; H^{k+1}(\Omega))$, then $\sup_{0\leq t\leq\omega}(\left\|D^k u_{nt}\right\|^2 + \left\|D^{k+1}u_{nt}\right\|^2) \leq c$, where c is a constant which only depends on L, ω, ε, γ, λ_n, k, k_1, k_2, k_3 and f.*

Proof. We first prove the conclusion of Lemma 4 holds for $k = 0$. Multiplying Eq. (10) by $a'_{jn}(t)$ and summing up over j from 1 to n, we have

$$\|u_{nt}\|^2 + \|u_{nxt}\|^2 = (Nu_n + f + \gamma(u_{nxx} - u_{nxxxx}), u_{nt}). \tag{28}$$

By Lemma 3, if $f \in C^1(\omega; H^1(\Omega))$, then we have $\|u_n\|^2_{H^4} \leq c$. Hence

$$|(Nu_n + f + \gamma(u_{nxx} - u_{nxxxx}), u_{nt})| \leq \|Nu_n + f + \gamma(u_{nxx} - u_{nxxxx})\| \, \|u_{nt}\| \leq c\,\|u_{nt}\|\,. \tag{29}$$

Therefore, by (28) and (29), it is easy to know that $\sup_{0\leq t\leq\omega}(\|u_{nt}\|^2 + \|u_{n\text{x}t}\|^2) \leq c$. So, the conclusion of Lemma 3.4 exists for $k = 0$. Now we assume that the conclusion of Lemma 3.4 holds for $k \leq m \quad (m \geq 1)$, we want to prove that the conclusion of Lemma 3.4 also holds for $k = m + 1$.

Multiplying Eq. (10) by $(-1)^{m+1}\lambda_j^{m+1}a'_{jn}(t)$ and summing up over j from 1 to n, we have

$$(-1)^{m+1}(\left\|D^{m+1}u_{nt}\right\|^2 + \left\|D^{m+2}u_{nt}\right\|^2) = (Nu_n + f + \gamma(u_{nxx} - u_{nxxxx}), D^{2(m+1)}u_{nt}). \tag{30}$$

By Lemma 3.3, if $f \in C^1(\omega; H^{m+2}(\Omega))$, then $\left\|D^k u_n\right\|^2 \leq c$ for $k \leq m + 5$. Hence

$$\begin{aligned}\left|(Nu_n + f + \gamma(u_{nxx} - u_{nxxxx}), D^{2(m+1)}u_{nt})\right| &\leq \left\|D^{m+1}[Nu_n + f + \gamma(u_{nxx} - u_{nxxxx})]\right\| \left\|D^{m+1}u_{nt}\right\| \\ &\leq c\left\|D^{m+1}u_{nt}\right\|\,.\end{aligned} \tag{31}$$

Taking (30) and (31) into account, it follows $\sup_{0\leq t\leq\omega}(\left\|D^{m+1}u_{nt}\right\|^2 + \left\|D^{m+2}u_{nt}\right\|^2) \leq c$.

This completes the proof of Lemma 3.4 by an inductive argument. ■

4 Existence and uniqueness of time-periodic solution

We have proved that Eqs. (4-6) have a sequence of approximate solutions $\{u_n\}_{n=1}^{\infty}$. In this section, we want to prove that the sequence converges and the limit is a solution of Eqs. (4-6).

By Lemma 3.1-3.4 and standard compactness arguments, we conclude that there is a subsequence which we denote also by $\{u_n\}$ such that for any $K \geq 0$, if $f \in C^1(\omega; H^{k+1}(\Omega))$, we have

$$u_n(t) \to u(t),\ \ weakly*in\ \ L^\infty(\omega; H^{k+4}(\Omega)); u_n(t) \to u(t),\ \ strongly\ \ in\ \ L^\infty(\omega; H^{k+3}(\Omega)),$$

$$u_{nt}(t) \to u_t(t),\ \ weakly*in\ \ L^\infty(\omega; H^{k+1}(\Omega)); u_{nt}(t) \to u_t(t),\ \ strongly\ \ in\ \ L^\infty(\omega; H^k(\Omega)).$$

From the above conclusions, we know that the nonlinear terms are well defined

$\left\|u_n^2 u_{nx} - u^2 u_x\right\| \leq \left\|u_n^2(u_{nx} - u_x)\right\| + \left\|u_x(u_n^2 - u^2)\right\| \leq \|u_n\|^2_\infty \|u_{nx} - u_x\| + \|u_x\|_\infty \|u_n + u\|_\infty \|u_n - u\| \to 0$, as $n \to \infty$, uniformly in t.

$\|u_{nx}u_{nxx} - u_x u_{xx}\| \leq \|u_{nx}(u_{nxx} - u_{xx})\| + \|u_{xx}(u_{nx} - u_x)\| \leq \|u_{nx}\|_\infty \|u_{nxx} - u_{xx}\| + \|u_{xx}\|_\infty \|u_{nx} - u_x\| \to 0$, as $n \to \infty$, uniformly in t.

$\|u_n u_{nxxx} - u u_{xxx}\| \leq \|u_n\|_\infty \|u_{nxxx} - u_{xxx}\| + \|u_n - u\|_\infty \|u_{xxx}\| \leq \|u_n\|_\infty \|u_{nxxx} - u_{xxx}\| + k_1 \|u_n - u\|_{H^1} \|u_{xxx}\|$ $\to 0$,as $n \to \infty$, uniformly in t.

Consequently, it follows $(u_t - u_{xxt} - \gamma(u_{xx} - u_{xxxx}), \eta) = (Nu + f, \eta), \quad \eta \in L^2_{per}$.

Thanks to the estimates obtained in the previous section, we have $u_t - u_{xxt} - \gamma(u_{xx} - u_{xxxx}) = Nu + f,\ a.e.\ on\ \mathrm{R}^1 \times \Omega$.

So we obtain that the existence of time periodic solution for the viscous modified Camassa-Holm equation (4-6), which is the following theorem.

Theorem 4.1 *Given $f \in C^1(\omega; H^{k+1}(\Omega))$, $k \geq 0$, there exist a time periodic solution $u(t,x)$ to Eqs. (4-6), such that $u(t,x) \in L^\infty(\omega; H^{k+4}(\Omega)) \cap W^{1,\infty}(\omega; H^k(\Omega))$.*

Under the assumption of Theorem 4.1 we are unable to prove the uniqueness of the solution for Eqs. (4-6). But if we assume that M is sufficiently small, then the result can be obtained.

Theorem 4.2 *Suppose that the assumption in Theorem 1 holds. If M is sufficiently small, then the time periodic solution of Eqs. (4-6) in Theorem 1 is unique.*

Proof. Let u and $\bar{u}$ be any two time periodic solutions of Eqs. (4-6).
With $v = u - \bar{u}$, we can get from Eq. (1) that

$$v_t - v_{xxt} - \gamma(v_{xx} - v_{xxxx}) = Nu - N\bar{u}. \tag{32}$$

Taking the inner product of (32) with v, we have

$$\frac{1}{2}\frac{d}{dt}(\|v\|^2 + \|v_x\|^2) + \gamma(\|v_x\|^2 + \|v_{xx}\|^2) = (Nu - N\bar{u}, v). \tag{33}$$

By the inequality (9), (33) yields

$$\frac{1}{2}\frac{d}{dt}(\|v\|^2 + \|v_x\|^2) + \frac{\gamma}{2k_3^2}\|v\|^2 + \frac{\gamma}{2}\|v_x\|^2 + \gamma\|v_{xx}\|^2 \le (Nu - N\bar{u}, v). \tag{34}$$

Since

$$\begin{aligned} &\left|(-3u^2u_x + 3\bar{u}^2\bar{u}_x, v)\right| \le \left|(-3u^2v_x, v)\right| + \left|(-3(u+\bar{u})v\bar{u}_x, v)\right| \\ &\le \frac{3}{2}\|u\|_\infty^2(\|v\|^2 + \|v_x\|^2) + 3\|\bar{u}_x\|_\infty\|u+\bar{u}\|_\infty\|v\|^2 \le (\frac{3k_1^2c_1}{2} + 6k_1^2c_1^{1/2}c_2^{1/2})\|v\|^2 + \frac{3k_1^2c_1}{2}\|v_x\|^2. \end{aligned} \tag{35}$$

$$\begin{aligned} &|(2u_xu_{xx} - 2\bar{u}_x\bar{u}_{xx}, v)| \le \|u_x\|_\infty(\|v\|^2 + \|v_{xx}\|^2) + 2\|\bar{u}_x\|_\infty\|v_x\|^2 + \|\bar{u}_x\|_\infty(\|v\|^2 + \|v_{xx}\|^2) \\ &\le 2k_1c_2^{1/2}\|v\|^2 + 2k_1c_2^{1/2}\|v_x\|^2 + 2k_1c_2^{1/2}\|v_{xx}\|^2. \end{aligned} \tag{36}$$

$$\begin{aligned} &|(uu_{xxx} - \bar{u}\bar{u}_{xxx}, v)| \le \int_\Omega |u_xvv_{xx}|\,dx + \int_\Omega |uv_xv_{xx}|\,dx + 2\int_\Omega \left|\bar{u}_xv_x^2\right|dx + 2\int_\Omega |\bar{u}_xvv_{xx}|\,dx \\ &\le \frac{1}{2}\|u_x\|_\infty(\|v\|^2 + \|v_{xx}\|^2) + \frac{1}{2}\|u\|_\infty(\|v_x\|^2 + \|v_{xx}\|^2) + 2\|\bar{u}_x\|_\infty\|v_x\|^2 + \|\bar{u}_x\|_\infty(\|v\|^2 + \|v_{xx}\|^2) \\ &\le \frac{3}{2}k_1c_2^{1/2}\|v\|^2 + (\frac{1}{2}k_1c_1^{1/2} + 2k_1c_2^{1/2})\|v_x\|^2 + (\frac{1}{2}k_1c_1^{1/2} + \frac{3}{2}k_1c_2^{1/2})\|v_{xx}\|^2. \end{aligned} \tag{37}$$

Hence, if M is sufficient small such that

$$\frac{3k_1^2c_1}{2} + 6k_1^2c_1^{1/2}c_2^{1/2} + \frac{7}{2}k_1c_2^{1/2} \le \frac{\gamma}{4k_3^2}, \quad \frac{3k_1^2c_1}{2} + 4k_1c_2^{1/2} + \frac{1}{2}k_1c_1^{1/2} \le \frac{\gamma}{4}, \quad \frac{1}{2}k_1c_1^{1/2} + \frac{7}{2}k_1c_2^{1/2} \le \frac{\gamma}{2}. \tag{38}$$

Then, it follows from (34-38) that $\frac{d}{dt}(\|v\|^2 + \|v_x\|^2) + \rho(\|v\|^2 + \|v_x\|^2) \le 0$, where $\rho \ge 0$ is suitable constant.
Applying Gronwall's inequality, we derive that$(\|v(t)\|^2 + \|v_x(t)\|^2) \le (\|v(0)\|^2 + \|v_x(0)\|^2)e^{-\rho t}$,for any $t \ge 0$.
Since v is ω-periodic in t, then for any positive integer m, we have $\|v(t)\|^2 + \|v_x(t)\|^2 = \|v(t+m\omega)\|^2 + \|v_x(t+m\omega)\|^2$.
Then we can infer that $(\|v(t)\|^2 + \|v_x(t)\|^2) \le (\|v(0)\|^2 + \|v_x(0)\|^2)e^{-\rho(t+m\omega)}$.
It follows from $v(0) = v_x(0) = 0$ that $u(t,x) = \bar{u}(t,x)$, which completes the proof of Theorem 4.2. ■

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