ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.18(2014) No.1,pp.71-77

Spectral Collocation Method for the Numerical Solution of the Gardner and Huxley Equations

S. Ashrafi[1], M. Alineia[2], H. Kheiri[2], G. Hojjati[2] *,
[1] Islamic Azad University, Maragheh Branch, Maragheh, Iran
[2] Faculty of Mathematical sciences, University of Tabriz , Tabriz, Iran
(Received 05 August 2013, accepted 15 July 2014)

Abstract: In this paper, we apply a spectral collocation method on Chebyshev polynomials of spatial derivatives for numerical solutions for Gardner and Huxley equations. The problems are reduced to a system of ordinary differential equations that are solved by `RK45` solver. The numerical experiments show efficiency of the proposed methods.

Keywords: Gardner and Huxley equations, Spectral collocation method, Chebyshev polynomials.

1 Introduction

Nonlinear equations have attracted much attention in connection with the important problems that arise in scientific applications. Nonlinear wave phenomena frequently appear in many domains of physics such as fluid dynamics, plasma, physics, mathematical biology, etc [6, 7].

Many powerful methods, such as inverse scattering method, Hirota bilinear forms, pseudo spectral method, the tanh-sech method, the sine-cosine method, and the Riccati equation expansion method have been used to investigate the solutions of these types of equations ([14]-[2]).

In this study, we consider Gardner and Huxley equations. These equations are formulated as follows:

- Gardner equation:
$$u_t + (\alpha u + \mu u^2)u_x + \beta u_{xxx} = 0, \tag{1}$$
with the initial condition
$$u(x,0) = f(x), \quad x \in D \tag{2}$$
and the boundary conditions
$$\begin{aligned} &u(x,t) = g(t), \\ &u_x(x,t) = h(t), \qquad (x,t) \in \delta D \times [0,T], \end{aligned} \tag{3}$$
where $D = \{x : a < x < b\}$ and δD is its boundary.

- Huxley equation:
$$u_t = u_{xx} + u(k-u)(u-1), \quad k \neq 0 \tag{4}$$
with the initial condition
$$u(x,0) = \bar{f}(x), \quad x \in D \tag{5}$$
and the boundary conditions
$$u(x,t) = \bar{g}(t), \qquad (x,t) \in \delta D \times [0,T], \tag{6}$$
where $D = \{x : a < x < b\}$ and δD is its boundary.

*Corresponding author. *E-mail address*: ghojjati@tabrizu.ac.ir; ghojjati@yahoo.com

IJNS.2014.08.15/822

The Gardner equation (1), as the Kortewegde Vries equation, is an integrable model. The solutions of (1) depend cardinally on the sign of the coefficient of the cubic non-linear term, μ. In particular, if $\mu < 0$ there is one family of solitary wave (solitons) only, if $\mu > 0$ there are two families of the solitons, and also breathers (oscillating wave packet) exist [11].

Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory [4, 15]. This equation occurred in the theory of interfacial waves in the stratified shear flow at some certain conditions on the medium stratification. It also describes a variety of wave phenomena in plasma and solid state [2, 5]. The Huxley equation describes nerve pulse propagation in nerve fibres and wall motion in liquid crystals [1].

We aim in this work to employ the Chebyshev spectral collocation (ChSC) method for solving Gardner and Huxley equations. The ChSC method is accomplished through, starting with Chebyshev approximation for the approximate solution and generating approximations for the higher-order derivatives through successive differentiation of the approximate solution. This method has introduced by Khater et al. [7] for solving Burgers' type equations.

Chebyshev polynomials [12] are well-known family of orthogonal polynomials on the interval $[-1, 1]$ of the real line. These polynomials present, among others, very good properties in the approximation of functions. Therefore, spectral methods based on Chebyshev polynomials as basis functions for solving numerically differential equations have been used by many authors [3, 8].

The paper is organized as follows: In Section 2, the Chebyshev spectral collocation method is applied to obtain numerical solutions of Gardner and Huxley equations. In Section 3, we give numerical experiments which confirm the efficiency of the proposed method. The paper is closed in Section 4 by concluding.

2 Spectral collocation method

While, historically, finite-difference methods are the standard numerical techniques for solving partial differential equations, newer alternative methods can be more effective in certain contexts. In particular, we consider here methods founded on orthogonal expansions, the so-called spectral collocation method, with special reference to methods based on expansions in Chebyshev polynomials.

In a typical finite-difference method, the unknown function $u(x)$ is represented by a table of numbers $\{y_0, y_1, ..., y_n\}$ approximating its values at a set of discrete points $\{x_0, x_1, ..., x_n\}$, so that $y_j \approx u(x_j)$. (The points are almost always equal spaced through the range of integration, so that $x_{j+1} - x_j = h$ for some small fixed h). But, in a spectral method, in contrast, the function $u(x)$ is represented by an infinite expansion $u(x) = \sum_k c_k\phi_k(x)$, where $\{\phi_k\}$ is a chosen sequence of prescribed basis functions. One then proceeds somehow to estimate as many as possible of the coefficients $\{c_k\}$, thus approximating $u(x)$ by a finite sum such as

$$u_n(x) = \sum_{k=0}^{n} c_k\phi_k(x).$$

One clear advantage that the spectral methods have over finite-difference methods is that, once approximate spectral coefficients have been found, the approximate solution can immediately be evaluated at any point in the range of integration, whereas the evaluation of a finite-difference solution at an intermediate point requires a further step of interpolation.

Chebyshev spectral collocation methods have become increasingly popular for solving differential equations and also they are very useful in providing highly accurate solutions to differential equations. In this method, the solution u is approximated as

$$u(x,t) = {\sum_{j=0}^{N}}'' a_j T_j^\star(x), \tag{7}$$

where $T_j^\star(x_n) = T_j((2x_n - (b + a))/(b - a))$ is the jth Chebyshev polynomial of the first kind. A summation symbol with double primes denotes a sum with the first and last term halved.
We apply the discrete orthogonality relation [10]

$$ {\sum_{n=0}^{N}}'' T_i^\star(x_n)T_j^\star(x_n) = \alpha_i\delta_{ij}, \tag{8}$$

where the constants α_i are

$$\alpha_i = \begin{cases} N/2, & i \neq 0, N \\ N, & i = 0, N \end{cases}$$

and using (7), the coefficients a_j are defined as

$$a_j = \frac{2}{N}\sum_{n=0}^{N}{}'' T_j^{\star}(x_n)u(x_n,t). \tag{9}$$

The first three derivatives of approximate solution u at the Chebyshev-Gauss-Lobatto points in the interval $[a,b]$

$$x_n = \frac{1}{2}\left((a+b)-(b-a)\cos\left(\frac{\pi n}{N}\right)\right), \quad n = 0,1,\cdots,N,$$

are given by

$$u_x(x_i,t) = \sum_{j=0}^{N}{}'' a_j {T'}_j^{\star}(x_i) = \sum_{n=0}^{N}{}'' \left(\frac{2}{N}\sum_{j=0}^{N}{}'' {T'}_j^{\star}(x_i)T_j^{\star}(x_n)\right)u(x_n,t) = \sum_{n=0}^{N}[A_x]_{in}u(x_n,t), \tag{10}$$

where

$$[A_x]_{in} = \frac{2c_n}{N}\sum_{j=0}^{N}{}'' {T'}_j^{\star}(x_i)T_j^{\star}(x_n), \qquad i,n = 0,1,\cdots,N,$$

$$c_0 = c_N = \frac{1}{2}, \quad c_n = 1, \quad n = 1,2,\cdots,N.$$

Similarly,

$$u_{xx}(x_i,t) = \sum_{n=0}^{N}[A_x]_{in}u_x(x_n,t) = \sum_{j=0}^{N}\left(\sum_{n=0}^{N}[A_x]_{in}[A_x]_{nj}\right)u(x_j,t) = \sum_{j=0}^{N}[B_x]_{ij}u(x_j,t), \tag{11}$$

where $B_x = A_x^2$ and the elements of the matrix B_x are given by

$$[B_x]_{ij} = \sum_{n=0}^{N}[A_x]_{in}[A_x]_{nj}, \qquad i,j = 0,1,\cdots,N, \tag{12}$$

and

$$u_{xxx}(x_i,t) = \sum_{n=0}^{N}[B_x]_{in}u_x(x_n,t) = \sum_{j=0}^{N}\left(\sum_{n=0}^{N}[A_x]_{in}[B_x]_{nj}\right)u(x_j,t) = \sum_{j=0}^{N}[C_x]_{ij}u(x_j,t), \tag{13}$$

where

$$[C_x]_{ij} = \sum_{n=0}^{N}[A_x]_{in}[B_x]_{nj}, \qquad i,j = 0,1,\cdots,N. \tag{14}$$

Using the boundary conditions (3) and (10)-(13), we have

$$\begin{aligned} u_x(x_i,t) &= d_i(t) + \sum_{n=1}^{N-1}[A_x]_{in}u_n(t), \\ u_{xx}(x_i,t) &= \bar{d}_i(t) + \sum_{n=1}^{N-1}[B_x]_{in}u_n(t), \\ u_{xxx}(x_i,t) &= \bar{\bar{d}}_i(t) + \sum_{n=1}^{N-1}[C_x]_{in}u_n(t), \end{aligned} \tag{15}$$

where

$$\begin{aligned} d_i(t) &= [A_x]_{i0}u_0(t) + [A_x]_{iN}u_N(t), \\ \bar{d}_i(t) &= [A_x]_{i0}u_0^*(t) + [A_x]_{iN}u_N^*(t) + [B_x]_{i0}u_0(t) + [B_x]_{iN}u_N(t), \\ \bar{\bar{d}}_i(t) &= [B_x]_{i0}u_0^*(t) + [B_x]_{iN}u_N^*(t) + [C_x]_{i0}u_0(t) + [C_x]_{iN}u_N(t). \end{aligned}$$

With substituting (15) into Gardner equation (1), we obtain

$$\begin{aligned}&\dot{u}_i(t) + \alpha u_i(t)\sum_{n=1}^{N-1}[A_x]_{in}u_n(t) + \mu u_i^2(t)\sum_{n=1}^{N-1}[A_x]_{in}u_n(t) + \beta\sum_{n=1}^{N-1}[C_x]_{in}u_n(t)\\ &+\alpha u_i(t)d_i(t) + \mu u_i^2(t)d_i(t) + \beta\bar{\bar{d}}_i(t) = 0,\\ &u_i(0) = f(x_i).\end{aligned} \tag{16}$$

Then the system (16) can be written the follwing form

$$\dot{u}(t) = F(t, u(t)), u(0) = u_0, \tag{17}$$

where

$$\begin{aligned}&u(t) = [u_1(t), u_2(t), \cdots, u_{N-1}(t)]^T, \quad \dot{u}(t) = [\dot{u}_1(t), \dot{u}_2(t), \cdots, \dot{u}_{N-1}(t)]^T,\\ &u_0 = [u_1(0), u_2(0), \cdots, u_{N-1}(0)]^T, F(t, u(t)) = [F_1(t, u(t)), F_2(t, u(t)), \cdots, F_{N-1}(t, u(t))]^T,\end{aligned}$$

and

$$\begin{aligned}F_i(t, u(t)) =& -\alpha u_i(t)\sum_{n=1}^{N-1}[A_x]_{in}u_n(t) - \mu u_i^2(t)\sum_{n=1}^{N-1}[A_x]_{in}u_n(t) - \beta\sum_{n=1}^{N-1}[C_x]_{in}u_n(t)\\ &-\alpha u_i(t)d_i(t) - \mu u_i^2(t)d_i(t) - \beta\bar{\bar{d}}_i(t), \qquad i = 1, 2, \cdots, N-1.\end{aligned}$$

Eq.(17) forms a system of ordinary differential equations (ODEs) in time. Therefore, to advance the solution in time, we use ODE solver such as the `RK45` because it is an explicit method which gives a good accuracy and extends trivially to nonlinear.

Similarly with substituting (15) into Huexly equation (4), we obtain

$$\begin{aligned}\dot{u}(t) =& \sum_{n=1}^{N-1}[B_x]_{in}u_n(t) + \bar{d}_i(t) + u_i(t)(k - u_i(t))(u_i(t) - 1),\\ u_i(0) =& \bar{f}(x_i),\end{aligned} \tag{18}$$

Then the system (18) can be written the follwing form

$$\begin{aligned}&\dot{u}(t) = F(t, u(t)),\\ &u(0) = u_0,\end{aligned} \tag{19}$$

where

$$\begin{aligned}&u(t) = [u_1(t), u_2(t), \cdots, u_{N-1}(t)]^T, \quad \dot{u}(t) = [\dot{u}_1(t), \dot{u}_2(t), \cdots, \dot{u}_{N-1}(t)]^T,\\ &u_0 = [u_1(0), u_2(0), \cdots, u_{N-1}(0)]^T, F(t, u(t)) = [F_1(t, u(t)), F_2(t, u(t)), \cdots, F_{N-1}(t, u(t))]^T,\end{aligned}$$

and

$$F_i(t, u(t)) = \sum_{n=1}^{N-1}[B_x]_{in}u_n(t) + \bar{d}_i(t) + u_i(t)(k - u_i(t))(u_i(t) - 1), \qquad i = 1, 2, \cdots, N-1.$$

Again the system (18) is solved by using the `RK45`.

3 Numerical results

In this section we apply the Chebyshev spectral collocation method to Gardner and Huxley equations. We report the absolute error of the proposed method. The initial and boundary conditions are taken from the exact solutions.

Table 1: Absolute errors for various values of x, t and β for $N = 10$.

x	α	β	$t = 0.001$	$t = 0.006$	$t = 0.03$
1.04529	1	2	$5.88E-5$	$5.73E-5$	$4.25E-5$
	1	3	$8.00E-6$	$6.57E-6$	$7.48E-6$
3.09017	1	2	$1.58E-7$	$9.41E-7$	$4.31E-6$
	1	3	$9.88E-8$	$5.81E-7$	$2.53E-6$
5.00000	1	2	$3.88E-5$	$2.99E-5$	$2.22E-5$
	1	3	$4.54E-6$	$3.54E-6$	$5.08E-5$

Table 2: Absolute errors for various values of x, t and β for $N = 15$.

x	α	β	$t = 0.001$	$t = 0.006$	$t = 0.03$
1.04529	1	2	$2.23E-9$	$4.41E-7$	$2.45E-6$
	1	3	$2.10E-9$	$4.29E-7$	$2.38E-6$
3.09017	1	2	$2.40E-8$	$3.97E-7$	$3.30E-6$
	1	3	$2.35E-8$	$3.80E-7$	$1.18E-6$
5.00000	1	2	$7.18E-8$	$3.95E-7$	$5.34E-6$
	1	3	$6.97E-8$	$3.71E-7$	$5.09E-6$

3.1 Numerical solution of the Gardner equation

Consider the Gardner equation

$$u_t + 2\alpha u u_x - 3\beta u^2 u_x + u_{xxx} = 0, \qquad \alpha, \beta > 0, \tag{20}$$

with the kink solutions [13],

$$u(x,t) = \frac{\alpha}{3\beta}\left(1 + tanh\left(\frac{\alpha}{3\sqrt{2\beta}}(x - \frac{2\alpha^2}{9\beta}t)\right)\right), \tag{21}$$

in the region $D = \{x : a < x < b\}$, α and β are arbitrary constants.

In Tables 1 and 2, we give the absolute error between the exact kink solution and the result obtained by the present method for $a = -10$,$b = 10$, $N = 10$ and $N = 15$, respectively.

We plot the numerical and exact solutions of the Gardner equation for $N = 10$, $t = 0.1$, $\beta = 2$ and $\alpha = 1$ in Fig.1. In Fig.2 , we display the absolute error of the method for the Gardner equation at time $t = 0.1$, for $a = -8$, $b = 8$ and $N = 10$.

3.2 Numerical solution of the Huxley equation

Consider the Huxley equation

$$u_t = u_{xx} + u(1-u)(u-1),$$

with the exact solutions[1],

$$u(x,t) = \frac{1}{2} + \frac{1}{2}tanh\left(\frac{1}{2\sqrt{2}}(x - \frac{t}{\sqrt{2}})\right). \tag{22}$$

In Table 3 and Table 4, we give the absolute error between the exact solution and the result obtained by the present method for various values of t, x, a, b for $N = 10$ and $N = 20$, respectively .
In Fig. 3, we display the numerical and exact solutions of Huxley equation for $N = 20$, $t = 0.5$, $a = -15$, $b = 10$. In Fig. 4, , we display the absolute error of Example 2 at time $t = 1$, for $a = -14$, $b = 10$ and $N = 20$.

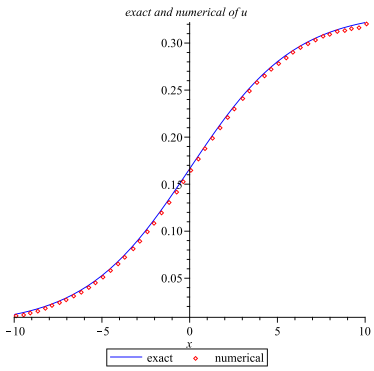

Figure 1: The numerical and the exact solutions of Gardner equation .

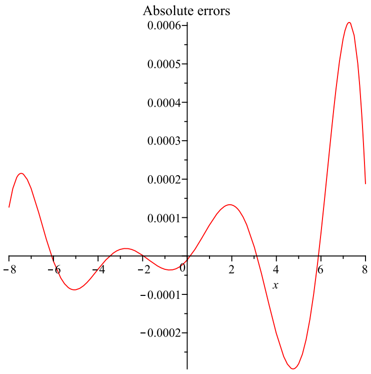

Figure 2: The absolute error of the method for Gardner equation .

Table 3: Absolute errors for various values of x, t, a and b for $N = 10$.

a	b	t	x	Error
-2	2	0.002	0.050	$2.22E-3$
-2	2	0.100	0.700	$1.78E-3$
-5	5	0.001	0.500	$1.69E-2$
-5	5	0.001	2.500	$9.90E-3$
-10	10	0.002	0.010	$4.05E-4$
-10	10	0.100	1.000	$3.17E-2$

Table 4: Absolute errors for various values of x, t, a and b for $N = 20$.

a	b	t	x	Error
-2	2	0.002	0.050	$3.99E-4$
-2	2	0.100	0.700	$1.65E-3$
-5	5	0.001	0.500	$8.97E-3$
-5	5	0.001	2.500	$8.93E-3$
-10	10	0.002	0.010	$2.44E-4$
-10	10	0.100	1.000	$9.76E-3$

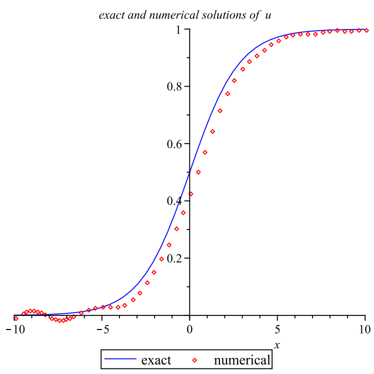

Figure 3: The numerical and the exact solutions of Huxley equation .

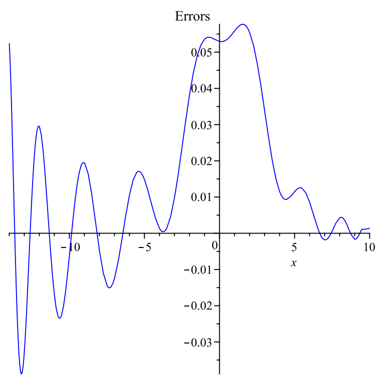

Figure 4: The absolute error of the method for the Huxley equation.

4 Conclusions

In this paper the Chebyshev spectral collocation methods were extended to obtain numerical solutions for Gardner and Hexley equations in a bounded domain. Using these methods, the problem is reduced to a system of ODEs that are solved by RK45 method. The obtained approximate numerical solutions maintains a good accuracy compared with the exact solution for the best choice of different values of parameters α and β.

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