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Numerical Solution of Hyperbolic Telegraph Equation Using Method of Weighted Residuals

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Abstract: In this article, the method of weighted residuals is used to solve some hyperbolic telegraph equations. A complete description of this method is highlighted and the convergence of this method is shown via sample problems. The solution procedure is simple and the obtained results were accurate.

Keywords:method of weighted residuals; Hyperbolic telegraph equation; residual function, collocation method.

1 Introduction

In this work the numerical approximation of hyperbolic telegraph equation governed by the following second order partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} + \beta u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \tag{1}$$

where α and β are known constant coefficients is considered. This telegraph equation is usually used in signal analysis for transmission and propagation of electrical signals and other related field. In recent years, attention has been given to the development, analysis and numerical solution of second order hyperbolic telegraph equations, (see [4] and references therein). In 2007, Dehghan and Shokri [4] solved the equation using radial basis function. In this paper we will use the method of weighted residuals to solve the problem. The highlight of the presentation is as follows: A brief description of the method is presented in section 2. Section 3 deals with the numerical applicability of the method while section 4 gives the conclusion.

2 Methods of solution

The idea of method of weighted residuals is to seek an approximate solution, in form of a polynomial, to the differential equation of the form

$$L[u(x)] = h \text{ in the domain } \Omega, \tag{2}$$

$$B_\mu[u] = \gamma_\mu, \text{ on } \partial\Omega, \tag{3}$$

where $L[u]$ denotes a general differential operator (linear or nonlinear) involving spatial derivatives of dependent variable u, h is a known function of position, $B_\mu[u]$ represents the appropriate number of boundary conditions and Ω is the domain with boundary $\partial\Omega$. The function u (i.e solution) is not only required to satisfy the operator equation (2), it is also required to satisfy the boundary conditions (3). A trial function of the form

$$u = u_0 + \sum_{i=1}^{n} c_i u_i, \tag{4}$$

where c_i are constants to be determined which satisfies the given boundary conditions (3), is assumed to be the solution of (1). The trial function is chosen in such a way that it satisfies all the boundary conditions. Substitution of equation (2)

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into equation (1) resorts to residual function R. The idea is to make this residual function as small as possible. If $R = 0$, then the trial function is the exact solution. In method of weighted residuals the parameters $c_{i's}$ are determined by setting the integral (over the domain) of weighted residual of the approximation to zero.

$$\int_{\Omega} w_i(x)R(x,t,c_i)dxdt = 0, i = 1,2,\cdots,n \tag{5}$$

where $w_{i's}$ are weight functions. The weight functions can be chosen in many ways and each choice corresponds to a different criterion of method of weighted residuals.
(i) The Galerkin Method (GM)
In this method, the weight functions are given by

$$w_i = \frac{\partial u}{\partial c_i}, i = 1,2,\cdots,n. \tag{6}$$

(ii) The Collocation Method (CM)
The collocation method seeks approximate solution to (2) in the form (4) by requiring the residual in the equation to be identically zero at n selected points $x_i = (x^i, t^i), i = 1,2,\cdots,n$ in the domain Ω:

$$R(x^i, t^i, c_i) = 0, i = 1,2,\cdots,n. \tag{7}$$

(iii) The Moment Method (MM)
The weighting functions w_i are chosen as $1, x, x^2, \cdots$ for the differential equations.
(iv) The Partition Method (PM)
In PM we divide the domain Ω into n smaller subdomains, Ω_i and choose

$$w_i = \begin{cases} 1, x \in \Omega_i \\ 0, x \notin \Omega_i \end{cases}. \tag{8}$$

The differential equation integrated over the subdomains are then equated to zero.
(v) The Least-Squares Method (LSM)
In this method the weight functions are given by

$$w_i = \frac{\partial R}{\partial c_i}, i = 1,2,\cdots,n. \tag{9}$$

In this work, to minimize the residual function along the whole domain the collocation method and partition method were used. For the collocation method the residual is then collocated at equally space point and equated to zero while for the partition method, the domain is subdivided into subdomain and the residual are integrated over these subdomain and equated to zero. The resulting system of equations are then solved to determine the parameters c_k in both cases. The polynomial u which is the approximate solution is then determined. For more expositions on the method, see [1-3,5-8]. This method takes into consideration the whole domain unlike the other methods such as the shooting method which can only be used in some part of the domain.

3 Application

3.1 Example 1

We consider the case in which $\alpha = 4$, $\beta = 2$ and $f(x,t) = 0$, equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial u}{\partial t} + 2u = \frac{\partial^2 u}{\partial x^2}, \tag{10}$$

subject to

$$\begin{cases} u(x,0) = \sin(x), 0 \le x \le \pi, \\ u_t(x,0) = -\sin(x), 0 \le x \le \pi, \\ u(0,t) = 0, u(\pi,t) = 0, t > 0. \end{cases} \tag{11}$$

Table 1: Errors due to approximation using method of weighted residuals (mwr) at $t = 0.01$ for example 1

x	$\|u_e - u_c\|$	$\|u_e - u_p\|$
0.0	0.0	0.0
$\frac{11}{14}$	0.9398×10^{-6}	0.2206×10^{-6}
$\frac{11}{7}$	1.5949×10^{-6}	0.025537×10^{-6}
$\frac{33}{14}$	0.9456×10^{-6}	0.2272×10^{-6}
$\frac{22}{7}$	0.014682×10^{-6}	0.013137×10^{-6}
rms	0.929656×10^{-6}	1.150811928×10^{-6}

The exact solution is given by $u_e = e^{-t}\sin(x)$. An approximate solution to the differential equation (5), which has been made to satisfy equation (6), is assumed in the form

$$u(x,t) = (1-t)\sin(x) + c_1x^2(x-\pi)t^2 + c_2x(x-\pi)^2t^2, \tag{12}$$

where $c_k, k = 1,2$ are constants to be determined.
We then substitute equation (12) into equation (10) to have the residual function R given by

$$\begin{aligned} R = \; & 2c_1x^2(x-\pi) + 2c_2(x-\pi)^2x - 4\sin(x) + 8c_1x^2(x-\pi)t + \\ & 8c_2(x-\pi)^2xt + 3(1-t)\sin(x) + 2c_1x^2(x-\pi)t^2 + 2c_2(x-\pi)^2xt^2 - \\ & 2c_1(x-\pi)t^2 - 4c_1xt^2 - 2c_2xt^2 - 4c_2(x-\pi)t^2. \end{aligned} \tag{13}$$

We now minimize equation (13) by collocation at equally spaced points $(x = \frac{\pi}{3}, t = 0.01)$, $(x = \frac{2\pi}{3}, t = 0.01)$ and equate to zero. Solving these system of equations we have $c_1 = -0.0623097746$, $c_2 = 0.06223097746$. Hence we have the approximate solution using collocation method $(u_c(x,t))$ given by

$$u_c(x,t) = (1-t)\sin(x) - 0.0623097746x^2(x-\pi)t^2 + 0.0623097746x(x-\pi)^2t^2. \tag{14}$$

Also, if we minize eqution (14) using partition method by integrating over equally spaced points $(x = 0..\frac{\pi}{2}, t = 0.01)$ and $(x = \frac{\pi}{2}..\pi, t = 0.01)$ and equate to zero. Solving these system of equations we have $c_1 = -0.06099419508$, $c_2 = 0.06099419510$. Hence we have the approximate solution using partition method $(u_c(x,t))$ given by

$$u_p(x,t) = (1-t)\sin(x) - 0.06099419508x^2(x-\pi)t^2 + 0.06099419510x(x-\pi)^2t^2. \tag{15}$$

The root mean square (rms) of the error is given by rms $= \sqrt{\frac{\sum|u_e - mwr|^2}{n}}$, where n is the number of partion of the interval.

3.2 Example 2

We next consider the case in which $\alpha = 6$, $\beta = 2$ and $f(x,t) = 2e^{-t}\sin(x)$, equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} + 6\frac{\partial u}{\partial t} + 2u = \frac{\partial^2 u}{\partial x^2} + 2e^{-t}\sin(x), \tag{16}$$

subject to

$$\begin{cases} u(x,0) = \sin(x), 0 \le x \le \pi, \\ u_t(x,0) = -\sin(x), 0 \le x \le \pi, \\ u(0,t) = 0, u(\pi,t) = 0, t > 0. \end{cases} \tag{17}$$

The exact solution is given by $u_e = e^{-t}\sin(x)$. An approximate solution to this is also assumed in the form

$$u(x,t) = (1-t)\sin(x) + c_1x^2(x-\pi)t^2 + c_2x(x-\pi)^2t^2, \tag{18}$$

where $c_k, k = 1,2$ are constants to be determined.
We then substitute equation (18) into equation (16) to have the residual function R given by

$$\begin{aligned} R = \; & 2c_1x^2(x-\pi) + 2c_2(x-\pi)^2x - 6\sin(x) + 12c_1x^2(x-\pi)t + \\ & 12c_2(x-\pi)^2xt + 3(1-t)\sin(x) + 2c_1x^2(x-\pi)t^2 + 2c_2(x-\pi)^2xt^2 - \\ & 2c_1(x-\pi)t^2 - 4c_1xt^2 - 2c_2xt^2 - 4c_2(x-\pi)t^2 - 2e^{-t}\sin(x). \end{aligned} \tag{19}$$

Table 2: Errors due to approximation using method of weighted residuals (mwr) at $t = 0.01$ for example 2

x	$\lvert u_e - u_c\rvert$	$\lvert u_e - u_p\rvert$
0.0	0.0	0.0
$\frac{11}{14}$	1.374588×10^{-4}	1.340265×10^{-4}
$\frac{11}{7}$	1.803815×10^{-4}	1.758062×10^{-4}
$\frac{33}{14}$	1.373180×10^{-4}	1.338894×10^{-4}
$\frac{22}{7}$	0.00307781×10^{-4}	0.00300412×10^{-4}
rms	1.185653702×10^{-4}	1.155832032×10^{-4}

We now collocate at equally spaced points $(x = \frac{\pi}{3}, t = 0.01)$, $(x = \frac{2\pi}{3}, t = 0.01)$ and equate to zero. Solving these system of equations, we have $c_1 = -0.2969917491$, $c_2 = 0.2969917491$. Hence we have the approximate solution using collocation method $(u_c(x,t))$ given by

$$u_c(x,t) = (1-t)\sin(x) - 0.2969917491x^2(x-\pi)t^2 + 0.2969917491x(x-\pi)^2t^2. \tag{20}$$

Also, if we minize eqution (19) using partition method by integrating over equally spaced points $(x = 0..\frac{\pi}{2}, t = 0.01)$ and $(x = \frac{\pi}{2}..\pi, t = 0.01)$ and equate to zero. Solving these system of equations we have $c_1 = -0.2910893968$, $c_2 = 0.2910893967$. Hence we have the approximate solution using partition method $(u_p(x,t))$ given by

$$u_p(x,t) = (1-t)\sin(x) - 0.2910893968x^2(x-\pi)t^2 + 0.2910893967x(x-\pi)^2t^2. \tag{21}$$

3.3 Example 3

We now consider the case in which $\alpha = 1$, $\beta = 1$ and $f(x,t) = (2-2t+t^2)(x-x^2)e^{-t} + 2t^2e^{-t}$, we have

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2} + (2-2t+t^2)(x-x^2)e^{-t} + 2t^2e^{-t}, \tag{22}$$

subject to

$$\begin{cases} u(x,0) = 0, 0 \le x \le 1, \\ u_t(x,0) = 0, 0 \le x \le 1, \\ u(0,t) = 0, u(1,t) = 0, t > 0. \end{cases} \tag{23}$$

The exact solution is given by $u_e = (x-x^2)t^2e^{-t}$. An approximate solution to this is also assumed in the form

$$u(x,t) = c_1x^2(x-1)t^2 + c_2x(x-1)^2t^2, \tag{24}$$

where $c_k, k = 1,2$ are constants to be determined.
We then substitute equation (24) into equation (22) to have the residual function R given by

$$\begin{aligned} R = \;& 2c_1x^2(x-1) + 2c_2(x-1)^2x + 2c_1x^2(x-1)t + 2c_2(x-1)^2xt + \\ & c_1x^2(x-1)t^2 + c_2(x-1)^2xt^2 - 2c_1(x-1)t^2 - \\ & 4c_1xt^2 - 2c_2xt^2 - 4c_2(x-1)t^2 - (2-2t+t^2)(x-x^2)e^{-t} - 2t^2e^{-t}. \end{aligned} \tag{25}$$

We now collocate at equally spaced points $(x = \frac{1}{3}, t = 0.01)$, $(x = \frac{2}{3}, t = 0.01)$ and equate to zero. Solving these system of equations, we have $c_1 = -0.9704545869$, $c_2 = 0.9704545869$. Hence we have the approximate solution using collocation method $(u_c(x,t))$ given by

$$u_c(x,t) = -0.9695728009x^2(x-1)t^2 + 0.9704545869x(x-1)^2t^2. \tag{26}$$

Also, if we minize eqution (24) using partition method by integrating over equally spaced points $(x = 0..\frac{1}{2}, t = 0.01)$ and $(x = \frac{1}{2}..1, t = 0.01)$ and equate to zero. Solving these system of equations we have $c_1 = -0.9704574948$, $c_2 = 0.9704574964$. Hence we have the approximate solution using partition method $(u_c(x,t))$ given by

$$u_p(x,t) = -0.9704574948x^2(x-1)t^2 + 0.9704574964x(x-1)^2t^2. \tag{27}$$

Table 3: Errors due to approximation using method of weighted residuals (mwr) at $t = 0.01$ for example 3.

x	$\lvert u_e - u_c \rvert$	$\lvert u_e - u_p \rvert$
0.0	0.0	0.0
$\frac{1}{4}$	0.36741087×10^{-6}	0.36735633×10^{-6}
$\frac{1}{2}$	0.48988116×10^{-6}	0.48980846×10^{-6}
$\frac{3}{4}$	0.36741087×10^{-6}	0.36735635×10^{-6}
1	0.0	0.0
rms	$0.3193634436\times10^{-6}$	$0.3193160467\times10^{-6}$

Table 4: Errors due to approximation using method of weighted residuals (mwr) at $t = 0.01$ for example 4.

x	$\lvert u_e - u_c \rvert$	$\lvert u_e - u_p \rvert$
0.0	0.0	0.0
$\frac{1}{4}$	0.0	0.0
$\frac{1}{2}$	0.0	0.0
$\frac{3}{4}$	0.0	0.0
1	0.0	0.0
rms	0.0	0.0

3.4 Example 4

We also consider the case in which $\alpha = 1$, $\beta = 1$ and $f(x,t) = x^2 + t - 1$, we have

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2} + x^2 + t - 1, \tag{28}$$

subject to

$$\begin{cases} u(x,0) = x^2, 0 \le x \le 1, \\ u_t(x,0) = 1, 0 \le x \le 1, \\ u(0,t) = t, u(1,t) = 1 + t, t > 0. \end{cases} \tag{29}$$

The exact solution is given by $u_e = x^2 + t$. An approximate solution to this is also assumed in the form

$$u(x,t) = x^2 + t + c_1 x^2 (x-1) t^2 + c_2 x (x-1)^2 t^2, \tag{30}$$

where $c_k, k = 1, 2$ are constants to be determined.
We then substitute equation (30) into equation (28) to have the residual function R given by

$$\begin{aligned} R = \quad & 2c_1 x^2 (x-1) + 2c_2 (x-1)x + 2c_1 x^2 (x-1)t + \\ & 2c_2 (x-1)xt + c_1 x^2 (x-1)t^2 + c_2 (x-1)xt^2 - \\ & 2c_1 (x-1)t^2 - 4c_1 xt^2 - 2c_2 t^2. \end{aligned} \tag{31}$$

We now collocate at equally spaced points $(x = \frac{1}{3}, t = 0.01)$, $(x = \frac{2}{3}, t = 0.01)$ and equate to zero. Solving these system of equations, we have $c_1 = 0.0$, $c_2 = 0.0$. Hence we have the approximate solution using collocation method $(u_c(x,t))$ given by

$$u_c(x,t) = x^2 + t, \tag{32}$$

which coincide with the exact solution.
Also, if we minize eqution (30) using partition method by integrating over equally spaced points $(x = 0..\frac{1}{2}, t = 0.01)$ and $(x = \frac{1}{2}..1, t = 0.01)$ and equate to zero. Solving these system of equations we have $c_1 = 0.0$, $c_2 = 0.0$. Hence we have the approximate solution using partition method $(u_c(x,t))$ given by

$$u_p(x,t) = x^2 + t, \tag{33}$$

which also coincide with the exact solution.

4 Conclusion

In this article, the method of weighted residuals is used to solve some second order hyperbolic telegraph equations, which is proven to be very simple and effective. The solution obtained are valid in the whole solution domain and very accurate. It is worthy to note that aside from collocation and partition methods used in this article, other methods listed in Section 2 produced similar results.

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