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Coupling of Adomian's Decomposition and Taylor's Series for Advection Problems

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Abstract: In this paper, we apply decomposition method coupled with Taylors series to solve linear and nonlinear Advection problems. It is observed that the proposed technique is highly suitable for such problems and overcomes some of the basic deficiencies of traditional decomposition method. Several examples are given to re-confirm the efficiency of the suggested algorithm.

Keywords: advection problems; nonlinear problems; adomians polynomials; Taylors series

1 Introduction

The rapid development of nonlinear sciences witnesses a wide range of analytical and numerical techniques by various scientists. Most of the developed schemes have their limitations like limited convergence, divergent results, linearization, discretization, unrealistic assumptions and non-compatibility with the versatility of physical problems [1-11]. In the similar context, G. Adomian developed Adomian's decomposition method which has been modified for time and again by different Scientists, see [1-11] and the references therein. The basic motivation of present study is the extension of traditional Modified Decomposition Method coupled with the Taylor's series [10, 11] to tackle linear and nonlinear Advection problems which arise very frequently in applied and engineering sciences. It has been observed that the coupling of decomposition method with Taylor's series enhances its efficiency and reduces the computational work to a tangible level. Moreover, this version is more user-friendly and it overcomes the complexities of selection of initial value. Several examples are given which reveal the efficiency and reliability of the proposed algorithm.

2 Modified adomian's decomposition method (MADM)

Consider the differential equation [7-11]

$$Lu + Ru + Nu = g, \tag{1}$$

where$L$ is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order lesser order than L, $N\,u$ represents the nonlinear terms and g is the source term. Applying the inverse operator L^{-1} to both sides of (1) and using the given conditions, we obtain

$$u = f - L^{-1}(R\,u) - L^{-1}(N u), \tag{2}$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method [10, 11] defines the solution $u\,(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3}$$

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where the components $u_n(x)$ are usually determined recurrently by using the relation

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= L^{-1}(R u_k) - L^{-1}(N u_k), \quad k \geq 0. \end{aligned}$$

The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n,$$

where $A_n$are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in [10, 11] which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) N\left(\sum_{i=0}^{n}\left(\lambda^i u_i\right)\right)_{\lambda=0}, \quad n = 0, 1, 2, \cdots.$$

or equivalently,

$$A_0 = F(u_0),$$

$$A_1 = u_1 F'(u_0),$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0),$$

$$A_3 = u_3 F'(u_0) + u_2 u_1 F''(u_0) - \frac{1}{3!} u_1^3 F'''(u_0),$$

$$\begin{aligned} &A_4 = u_4 F'(u_0) + \left(\tfrac{1}{2!} u_2^2 + u_3 u_1\right) F''(u_0) - \tfrac{1}{2!} u_1^2 u_2 F'''(u_0) + \tfrac{1}{4!} u_1^4 F^{(iv)}(u_0), \\ &A_5 = u_5 F'(u_0) + \left(u_2 u_3 + u_4 u_1\right) F''(u_0) + \left(\tfrac{1}{2!} u_1 u_2^2 + \tfrac{1}{2!} u_3 u_1^2\right) F'''(u_0) \\ &- \tfrac{1}{3!} u_1^3 u_2 F^{(iv)}(u_0) + \tfrac{1}{5!} u_1^5 F^{(v)}(u_0), \\ &\vdots \end{aligned}$$

3 Numerical application

In this section, we apply Adomian's Decomposition Method (ADM) using Taylor's series to solve nonlinear Advection problems. Numerical results are very encouraging.

Example 3.1

Consider the following partial differential equation,

$$u_t + \frac{1}{2} u_x^2 = e^x + t^2 e^{2x},$$

with initial conditions,

$$u(x, 0) = 0.$$

Applying Adomian's decomposition method, we get

$$u(x,t) = t e^x - \frac{1}{3} t^3 e^{2x} - \frac{1}{2} L_t^{-1} L_x\left(u^2\right).$$

Applying Taylor series, we get

$$u(x,t) = t + \frac{1}{3} t^3 + \left(t + \frac{2}{3} t^3\right) x + \left(\frac{t}{2} + \frac{2}{3} t^3\right) x^2 + \left(\frac{1}{6} t + \frac{4}{9} t^3\right) x^3 + \cdots - \frac{1}{2} L_t^{-1} L_x\left(u^2\right).$$

According to the proposed technique, we have the following recurrence relation,

$$u_0(x,t) = t + \frac{1}{3} t^3,$$

$$u_{k+1}(x,t) = f_{k+1}(x) - \frac{1}{2}\int_0^t \frac{\partial}{\partial x}\left(u_k^2\right) dt, \quad k = 0.$$

Consequently, following approximants are obtained,

$$u_0\left(x,t\right)=t+\frac{1}{3}t^3,$$

$$u_1\left(x,t\right)=\left(t+\frac{2}{3}t^3\right)x-\frac{1}{2}\int_0^t\frac{\partial}{\partial x}\left(u_0^2\right)\right)dt=tx+\frac{2}{3}xt-\frac{1}{2}.$$

$$u_2\left(x,t\right)=\left(\frac{t}{2}+\frac{2}{3}t^3\right)x^2-\frac{1}{2}\int_0^t\frac{\partial}{\partial x}\left(2u_0u_1\right)\right)dt,$$

$$=\frac{1}{2}tx^2+\frac{2}{3}x^2t^3-\frac{2}{63}t^7-\frac{1}{5}t^7-\frac{1}{3}t^3.$$

$$u_3\left(x,t\right)=\left(\frac{1}{6}t+\frac{4}{9}t^3\right)x^3-\frac{1}{2}\int_0^t\frac{\partial}{\partial x}\left(u_1^2+2u_0u_2\right)\right)dt,$$

$$=\frac{1}{6}tx^3+\frac{4}{9}x^3t^3-\frac{8}{63}xt^7-\frac{3}{5}xt^5-\frac{2}{3}xt^3.$$

The series solution is given by

$$u\left(x,t\right)=t+\frac{1}{3}t^3+tx+\frac{2}{3}xt-\frac{1}{2}+\frac{1}{2}tx^2+\frac{2}{3}x^2t^3-\frac{2}{63}t^7-\frac{1}{5}t^7-\frac{1}{3}t^3+\frac{1}{6}tx^3+\frac{4}{9}x^3t^3-\frac{8}{63}xt^7-\frac{3}{5}xt^5-\frac{2}{3}xt^3+\ldots,$$

and closed form solution is given by,

$$u\left(x,t\right)=te^x.$$

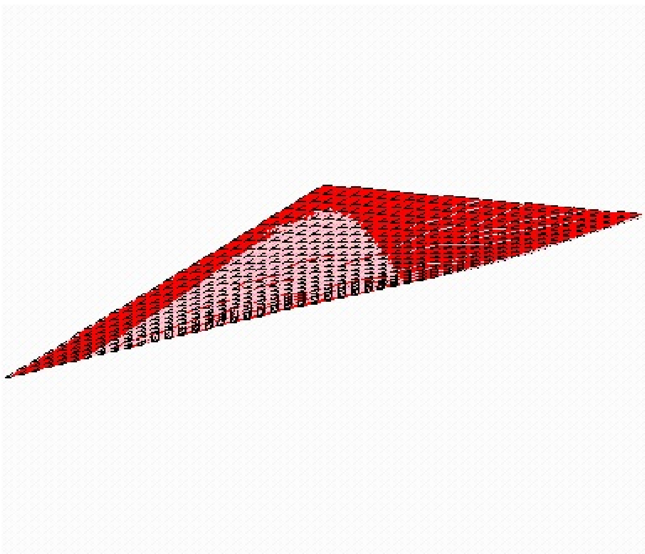

Figure 1: Comparison of exact and approximate solutions of Example 3.1.

Example 1 *Consider the following partial differential equation,*

$$u_t-\frac{1}{2}u_x^2=-\sin\left(x+t\right)\ -\frac{1}{2}sin2(x+t),$$

With initial conditions,

$$u\left(x,0\right)=cosx.$$

Table 1: Error estimates for exact and series solution example 3.1

x=t	Exact solution	Series solution	*Error(absolute)
0.00,0,00	0.00000000	0.00000000	0.00000000
0.01,0.01	0.01010050	0.01010050	0.00000000
0.02,0.02	0.02040426	0.20404028	0.00000002
0.03,0.03	0.03091363	0.03091364	0.00000001
0.04,0.04	0.04163243	0.04163247	0.00000004
0.05,0.05	0.05256355	0.05256368	0.00000013
0.06,0.06	0.06371019	0.06371051	0.00000032
0.07,0.07	0.07507557	0.07507627	0.00000070
0.08,0.08	0.08666296	0.08666431	0.00000135
0.09,0.09	0.09847568	0.09847810	0.00000242

Applying Adomian's decomposition method, we get

$$u\left(x,t\right)=\cos\left(x+t\right)\ +\frac{1}{4}cos2\left(x+t\right)-\frac{1}{4}cos2x-\frac{1}{2}\int_0^t L_x(u^2)dt,$$

Apply the Taylor series, we get

$$u\left(x,t\right)=cosx+\left(-sinx-\frac{1}{2}sin2x\right)t+\left(\frac{-1}{2}cosx-\frac{1}{2}cosx\right)t^2+\cdots-\frac{1}{2}\int_0^t L_x(u^2)dt,$$

According to the proposed technique, we have the following recurrence relation,

$$u_0\left(x,t\right)=cosx,$$

$$u_{k+1}\left(x,t\right)=f_{k+1}\left(x\right)-\frac{1}{2}\int_0^t L_x\left(A_k\right)dt,\quad k=0.$$

Consequently, following approximation obtained,

$$u_0\left(x,t\right)=cosx,$$

$$u_1\left(x,t\right)=\left(-sinx-\frac{1}{2}sin2x\right)t-\frac{1}{2}\int_0^t\frac{\partial}{\partial x}(u_0^2)dt,$$

$$=-sinx\cdot t-\frac{1}{2}sin2x\cdot t+cosxsinx\cdot t.$$

$$u_2\left(x,t\right)=\left(\frac{-1}{2}cosx-\frac{1}{2}cosx\right)t^2-\frac{1}{2}\int_0^t\frac{\partial}{\partial x}(2u_0u_1)dt,$$

$$\begin{aligned}=\ &\frac{-1}{2}cosx\cdot t^2-\frac{1}{2}cos2x{\cdot}t^2-\frac{1}{4}[-2sinx\left\{-sinx-\frac{1}{2}sin2x+cosxsinx\right\}\\&+2cosx\{-cosx-cos2x-sin^2x+cos^2x]t^2\end{aligned}$$

Therefore we have series solution,

The solution is given by,

$$\begin{aligned}u\left(x,t\right)\ =\ &cosx-sinx\cdot t-\frac{1}{2}sin2x\cdot t+cosxsinx\cdot t+\frac{-1}{2}cosx\cdot t^2-\frac{1}{2}cos2x\cdot t^2\\&-\frac{1}{4}[-2sinx\left\{-sinx-\frac{1}{2}sin2x+cosxsinx\right\}\\&+2cosx\left\{-cosx-cos2x-\sin^2x+cos^2x\right]t^2+\ldots.\end{aligned}$$

Since the exact solution is

$$y\left(x,t\right)=\cos(x+t)$$

Example 2 *Consider the following partial differential equation,*

$$u_t + uu_x - u = e^t,$$

With initial conditions,

$$u(x,0) = x + 1,.$$

Applying Adomian's decomposition method, we get

$$u(x,t) = x + e^x - L_t^{-1}(uu_x - u)).$$

Apply the Taylor series, we get

$$u(x,t) = x + 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots - L_t^{-1}(uu_x - u)),$$

According to the proposed technique, we have the following recurrence relation

$$u_0(x,t) = x + 1,$$

$$u_{k+1}(x,t) = f_{k+1}(x) - \int_0^t (u_k \frac{\partial}{\partial x}(u_k) - u_k)\, dt, \quad k = 0.$$

Consequently, following approximation obtained,

$$u_0(x,t) = x + 1,$$

$$u_1(x,t) = t - \int_0^t [(x+1) - (x+1)]dt = t.$$

$$u_2(x,t) = \frac{1}{2}t^2 - \int_0^t [u_0 \frac{\partial}{\partial x}u_1 + u_1 \frac{\partial}{\partial x}u_0]dt = \frac{1}{2}t^2.$$

Therefore we have series solution,

$$u(x,t) = x + 1 + t + \frac{t^2}{2!} + \ldots,$$

and the closed form is given by,

$$u(x,t) = x + e^t.$$

Table 2: Error estimates for exact and series solution example 2.

x=t	Exact solution	Series solution	*Error(absolute)
0.00,0,00	1.00000000	1.00000000	0.00000000
0.01,0.01	0.99980000	0.99980000	0.00000000
0.02,0.02	0.99920019	0.99920007	0.00000012
0.03,0.03	0.99820053	0.99820037	0.00000016
0.04,0.04	0.99680170	0.99680117	0.00000053
0.05,0.05	0.99500416	0.99500284	0.00000132
0.06,0.06	0.99280863	0.99280593	0.00000270
0.07,0.07	0.99021599	0.99021100	0.00000499
0.08,0.08	0.98722728	0.98721876	0.00000852
0.09,0.09	0.98384369	0.98383005	0.00001364

4 Conclusions

Decomposition method coupled with Taylors series is applied to solve linear and nonlinear Advection problems. Numerical results clearly reflect the accuracy and efficiency of the proposed algorithm. Moreover, suggested technique makes the selection of initial value extremely simple and hence enhances its efficiency.

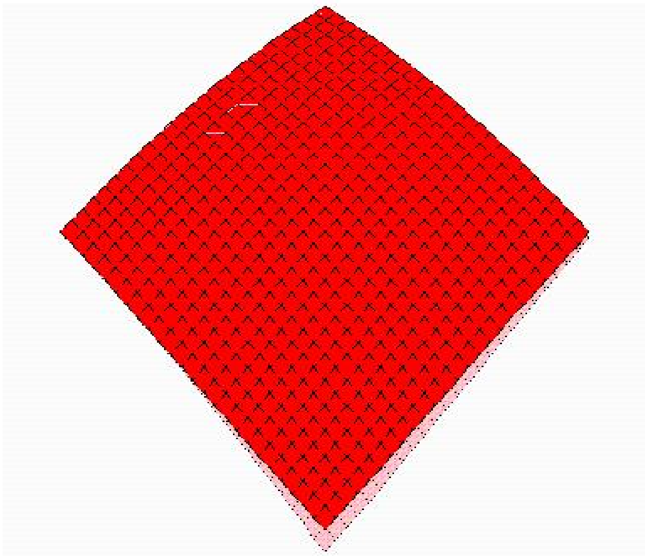

Figure 2: Comparison of exact and approximate solutions of example 2.

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