A Piecewise Approximation for Linear Two Dimensional Volterra Integral Equation by Chebyshev Polynomials

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Abstract: In this paper, we investigate piecewise approximate solution for linear two dimensional Volterra integral equation, based on the interval approximation of the true solution by truncated Chebyshev series. By discretization respect to spatial and time variables, the solution is approximated by using collocation method. Analysis of discretization error is discussed and efficiency of the method is shown by applying it on some problems.

Keywords: Two dimensional Volterra integral equation; Chebyshev approximation; Collocation methods.

1 Introduction

Consider the following linear two dimensional Volterra integral equation

\[ u(t, x) = f(t, x) + \int_0^t \int_0^x G(t, z, x, y) u(z, y) dydz, \]  \hspace{1cm} (1)

where \( x \in J := [0, X] \) and \( t \in I := [0, T] \). We assume that the given real-valued function \( f := f(t, x) \) and \( G := G(t, x, z, y) \) are at least continuous on \( D := I \times J \) and \( S = \{(t, z, x, y); 0 \leq z \leq t \leq T, 0 \leq y \leq x \leq X\} \), respectively. Equation of type (1) often arises from the mathematical modeling of the mechanical problems. Analysis of existence and uniqueness of the solution may be found in [5]. There are no more tasks for numerical solution of this type of problems. Recently some authors applied differential transform methods (DTM) for solving (1.1) [6]. Here, we propose an approximation by using Chebyshev polynomials.

This article is organized as follows: In section 2, the method is described and in section 3, the order of convergence of the method is determined. In section 4, numerical examples are given to confirm the theoretical results.

2 Description of the method

Let \( \Pi_M : 0 = x_0 < x_1 < \cdots < x_M = X \) and \( \Pi_N : 0 = t_0 < t_1 < \cdots < t_N = T \) be partitions of \([0, X]\) and \([0, T]\) with constant step size \( h = x_{i+1} - x_i \), \( i = 0, 1, \cdots, M - 1 \) and \( \tau = t_{j+1} - t_j \), \( j = 0, 1, \cdots, N - 1 \). We approximate the solution \( u(t, x) \) in the subinterval \((t_i, t_{i+1}] \times (x_i, x_{i+1}]\), say \( U_{i, i}(t, x) \), that is in \( S_p^0(\Pi_M) \times S_p^0(\Pi_N) \) where \( S_p^0(\Pi_M) := \{ v : v|_{[x_{m}, x_{m+1}]} \in \pi_p, \ 0 \leq m \leq M - 1 \} \) is a real polynomial spline space of degree \( p \) with continuity property in nodes. Here the approximated solution is represented by means of Chebyshev polynomials. The approximate solution for (1) in every subinterval \((t_i, t_{i+1}] \times (x_i, x_{i+1}]\) is determined by using the collocation method. The approximations in the subintervals \([(0, T] \times [0, x_i]) \cup [(0, t_i] \times (x_i, x_{i+1}]) \) have been obtained from the previous stages, thus we have

\[ U_{i, i}(t, x) = F_{i, i}(t, x) + \int_{t_i}^{t} \int_{x_i}^{x} G(t, z, x, y) U_{i, i}(z, y) dydz, \]  \hspace{1cm} (2)

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where \((t, x) \in (t_i, t_{i+1}] \times (x_i, x_{i+1}]\) and
\[
F_{i,j}(t, x) = f(t, x) + \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} \int_{x_r}^{x_{r+1}} \int_{x_s}^{x_{s+1}} G(t, z, x) U_{r, s}(z, y) dy dz \\
+ \sum_{r=0}^{i-1} \int_{t_1}^{t} \int_{t_r}^{t} G(t, z, x) U_{r,i}(z, y) dy dz \\
+ \sum_{s=0}^{j-1} \int_{t_1}^{t} \int_{t_s}^{t} G(t, t, x) U_{i,s}(z, y) dy dz. \tag{3}
\]

2.1 Discretization in spatial variable

We rewrite \(U_{i,j}(t, x)\) in terms of new variable \(\alpha \in (-1, 1]\) instead of \(x \in (x_i, x_{i+1}]\) defined by
\[
x = x_i + \frac{h}{2}(1 + \alpha).
\]
Thus we can write
\[
U_{i,j}(t, x) = U_{i,j}(t, x_i + \frac{h}{2}(1 + \alpha)) = \bar{U}_{i,j}(t, \alpha).
\]
The solution of integral equation (1.1) with respect to the second variable can be approximated by the following finite sum (see [4])
\[
\bar{U}_{i,j}(t, \alpha) \simeq \sum_{k=0}^{p} \sum_{j=0}^{p} \sum_{j=0}^{p} a_k(t) T_k(\alpha), \tag{4}
\]
where
\[
a_k(t) = 2 \sum_{w=0}^{p} U_{i,j}(t, x_i + \xi_w h) T_k(\alpha_w), \quad k = 0, 1, \ldots, p,
\]
\[
\alpha_j = \cos(p - j) \frac{\pi}{p}, \quad \xi_j = \frac{1 + \alpha_j}{2}, \quad j = 0, 1, \ldots, p,
\]
and \(T_k(\alpha)\) is the standard Chebyshev polynomial of degree \(k\). Double prime on \(\sum\) indicates that both of the first and the last terms in the summation are halved. So, by changing of variable \(y = x_i + \frac{h}{2}(1 + \alpha)\) in (2), substituting from (4) and (5), and collocating the both sides of obtained equation in the points \(x_i + \xi_j h, \quad j = 1, \ldots, p\), it leads to a system of Volterra integral equations that contains \(p\) equations and \(p\) unknown functions \(U_{i,j}(t, x_i + \xi_w h), \quad w = 1, 2, \ldots, p\) in the form of
\[
U_{i,j}(t, x_i + \xi_j h) = F_{i,j}(t, x_i + \xi_j h) + h \sum_{k=0}^{p} \sum_{w=0}^{p} T_k(\alpha_w) \int_{t_i}^{t} \int_{z_i}^{z_{j+1}} G(t, z, x_i + \xi_j h, z) \\
+ h \sum_{w=0}^{p} T_k(\alpha_w) U_{i,j}(z, x_i + \xi_w h) dz. \tag{6}
\]
For simplifying, set
\[
U_{i,j}^{[w]} = U_{i,j}(z, x_i + \xi_w h), \quad w = 0, 1, \ldots, p,
\]
\[
F_{i,j}^{[j]} = F_{i,j}(t, x_i + \xi_j h), \quad j = 1, 2, \ldots, p.
\]
As the approximated value of \(u(t, x_i)\) is obtained from the previous stage, for continuity of approximation solution, we set \(U_{i-1,j}(t, x_i)\) instead of \(U_{i,j}(t, x_0)\). Thus
\[
U_{i,j}^{[j]}(t) = F_{i,j}^{[j]}(t) + h \sum_{w=1}^{p} K_{j,w}(t, z) U_{i,j}^{[w]}(z) dz, \quad j = 1, 2, \ldots, p. \tag{7}
\]
where

\[
F^{[j]}_{i,l} = F^{[j]}_{i,l,0} + \frac{h}{2p} \sum_{k=0}^{p} T_k(\alpha_0) \int_{t_i}^{t'} \int_{z_{i-1}}^{\alpha_j} G(t, z, x_i + \xi_j, x_i) \, dz \, dr + \frac{h}{2} (1 + \alpha) T_k(\alpha) U^{[p]}_{i-1,l}(z) \, dz \cdot j = 1, \ldots, p,
\]

and

\[
K_{j,w}(t, z) = \sum_{k=0}^{p} T_k(\alpha_w) \int_{z_{i-1}}^{\alpha_j} G(t, z, x_i + \xi_j, x_i) \, dz \, dr + \frac{h}{2} (1 + \alpha) T_k(\alpha) \, da \cdot j = 1, \ldots, p, \quad w = 1, \ldots, p.
\]

Prime on \( \sum \) indicates that the last term in summation is halved. Note that when \( i = 0 \), we set \( f(z, 0) \) instead of \( U^{[p]}_{i-1,l}(z) \).

### 2.2 Discretization in time variable

In subsection 2.1, a system of linear Volterra integral equations with \( p \) unknown functions \( U^{[w]}_{i,l} \), \( w = 1, \ldots, p \) was obtained. Now we approximate the solution of this system for \( t \in [t_i, t_{i+1}] \) by Chebyshev polynomials. By using the same manner, we rewrite \( U^{[j]}_{i,l}(t) \) in terms of new variable \( \beta \) that is defined by

\[
t = t_i + \frac{\tau}{2} (1 + \beta).
\]

For solving (7), we seek an approximate solution in the form of following finite sum

\[
U^{[j]}_{i,l}(t) = U^{[j]}_{i,l}(\beta) = \sum_{r=0}^{q} b_r T_r(\beta),
\]

where

\[
b_r = \frac{2}{q} \sum_{s=0}^{q} U^{[j]}_{i,l}(t_i + \tau \xi_s) T_r(\beta), \quad r = 0, 1, \ldots, q,
\]

\[
\beta_j = \cos(q - j) \frac{\pi}{q}, \quad \xi_j = \frac{1 + \beta_j}{2}, \quad j = 0, 1, \ldots, q.
\]

By changing of variable \( z = t_i + \frac{\tau}{2} (1 + \beta) \), substituting (10) and (11) in (7) and collocating in the points \( t_i + \xi_d \tau, \quad d = 1, 2, \ldots, q \), the following algebraic system is yield

\[
U^{[j,d]}_{i,l} = F^{[j,d]}_{i,l} + \frac{h \tau}{2pq} \sum_{a=1}^{p} \sum_{w=1}^{q} T_r(\beta_w) U^{[w,a]}_{i,l,0} \int_{-1}^{\beta_a} K_{j,w}(t_i + \xi_d \tau, t_i + \frac{\tau}{2} (1 + \beta)) T_r(\beta) \, d\beta.
\]

Here, \( G_{j,w,d,s} \) for \( j, w = 1, 2, \ldots, p \) and \( d, s = 1, 2, \ldots, q \) is in the form

\[
G_{j,w,d,s} = \sum_{r=0}^{q} T_r(\beta_s) \int_{-1}^{\beta_d} K_{j,w}(t_i + \xi_d \tau, t_i + \frac{\tau}{2} (1 + \beta)) T_r(\beta) \, d\beta.
\]

Similar to discretization in spatial, for continuity of the solution, the value of \( U^{[j,0]}_{i,l} \) is substituted by approximated value from the previous stage i.e \( U^{[j]}_{i-1,l} \) and for \( l = 0 \), the value of \( U^{[w,0]}_{i,l} \) is substituted by \( f(0, x_i + \xi_w h) \). Thus we have the approximated solution \( U_{i,l}(t, x) \) in the form of

\[
U_{i,l}(t, x) = \frac{4}{pq} \sum_{k=0}^{p} \sum_{w=0}^{q} \sum_{r=0}^{q} \sum_{s=0}^{q} T_k(\alpha_w) T_r(\beta_s) T_p(2(t - t_i) - 1) T_k(\frac{2h}{\tau} (x - x_i) - 1) U^{[w,s]}_{i,l,0}.
\]
# 3 Convergence analysis

The collocation methods for one dimensional Volterra integral equations have order $p$ for any choice of collocation parameters, $0 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq 1$, and can be achieved local superconvergence in the mesh points by opportunely choosing the collocation parameters, as stated by the following theorem.

**Theorem 1** [2] Suppose that the given functions in the following VIE

$$u(t) = g(t) + \int_0^t k(t, \tau, u(\tau))d\tau, \quad t \in I := [0, T],$$

satisfy $k \in C^{(p)}(D \times \mathbb{R})$, $g \in C^{(p)}(I)$, where $D := \{(t, \tau) : 0 \leq \tau \leq t \leq T\}$. Then, for any choice of the collocation parameters $0 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq 1$, the error $\epsilon(t) = u(t) - u_p(t)$, which $u_p(t)$ is the approximated solution by the collocation method, satisfies

$$||\epsilon||_\infty = O(h^p).$$

Suppose moreover that $k \in C^{(2p-\nu)}(D \times \mathbb{R})$, $g \in C^{(2p-\nu)}(I)$, for some $\nu \in \{0, 1, 2\}$, then

If the collocation parameters are the Radau II points for $[0, 1]$ we have, for $\nu = 1$

$$\max_{n=0,\ldots,N} |\epsilon(t_n)| = O(h^{2p-1}).$$

If the collocation parameters are p Lobatto points for $[0, 1]$ or $p-1$ Gauss-Legendre points for $[0, 1]$ with $\xi_p = 1$, we have, for $\nu = 2$,

$$\max_{n=0,\ldots,N} |\epsilon(t_n)| = O(h^{2p-2}).$$

**Theorem 2** Assume that $f(t, x)$ and $G(t, z, y)$ are sufficiently differentiable functions of their variables and $\bar{U}(t, x)$ is the approximated solution of (1) in $S_N^p(H_N) \times S_N^q(H_M)$ that is of degree $p$ and $q$ with respect to spatial and temporal variables, respectively. Then the total error $E(t, x)$ satisfies

$$\|E\|_\infty = O(\tau^{q+1}) + O(h^{p+1}), \quad (t, x) \in D,$$

where $E_{i, j}(t, x) := E(t, x)|_{(t, t, \tau, x, x_i, x_{i+1})} = \bar{U}_{i, j}(t, x) - u(t, x)$.

**Proof.** By approximating the solution of (2) with respect to the second variable by collocation methods, we have

$$\sup_{\alpha \in [-1, 1]} |u(t, x_i + \frac{h}{2}(1 + \alpha)) - U_{i, j}(t, x_i + \frac{h}{2}(1 + \alpha))| = O(h^{p+1}).$$

Also, by assuming $\bar{U}(t_l + \frac{\tau}{2}(1 + \beta), x_i + \xi_w h)$ as approximate solution of VIEs (6), we have

$$\sup_{\beta \in [-1, 1]} |\bar{U}_{i, j}(t_l + \frac{\tau}{2}(1 + \beta), x_i + \xi_w h) - U_{i, j}(t_l + \frac{\tau}{2}(1 + \beta), x_i + \xi_w h)| = O(\tau^{q+1}).$$

Hence

$$\|u - \bar{U}\| \leq \sup_{\alpha \in [-1, 1]} |u(t, x_i + \frac{h}{2}(1 + \alpha)) - U_{i, j}(t, x_i + \frac{h}{2}(1 + \alpha))| +$$

$$\sup_{\beta \in [-1, 1]} |\bar{U}_{i, j}(t_l + \frac{\tau}{2}(1 + \beta), x_i + \xi_w h) - U_{i, j}(t_l + \frac{\tau}{2}(1 + \beta), x_i + \xi_w h)|$$

$$= O(h^{p+1}) + O(\tau^{q+1}).$$
4 Numerical examples

In this section, we present the numerical results obtained using our new scheme on some problems. The aim of this investigation is to demonstrate numerically the generally superior performance of the new method.

Example 1 Consider the following linear integral equation

\[ u(t, x) = x \cos(t) - t + \frac{1}{2} xt^2 - x \sin(x) \sin(t) + x^2 \cos(x) \sin(t) \]

\[ -\frac{1}{2} xt^2 \cos(x) - \frac{1}{4} x^2 t \sin^2(t) + xt \sin(t) - xt^2 \cos(t) \]

\[ + \int_0^t \int_0^x (x \sin(y) + t \sin(z)) u(z, y) dy dz, \]

where \( t \in [0, 1], x \in [0, 1] \) with exact solution \( u(t, x) = x \cos(t) - t \). We use the method by \( p = q = 3 \) and \( m = M = 4, 8, 16, 32 \). Absolute error of the method in some points is reported in Table 1. Figure 1 shows the numerical solution of problem that is continues. Figure 2 represents the error function where it is seen that the error growths when \( t \) and \( x \) increase.

<table>
<thead>
<tr>
<th>((t, x))</th>
<th>(M = N = 4)</th>
<th>(M = N = 8)</th>
<th>(M = N = 16)</th>
<th>(M = N = 32)</th>
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<td>(5.03 \times 10^{-6})</td>
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<td>(7.27 \times 10^{-5})</td>
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</table>

Figure 1: The plot of the numerical solution of Example 1

Figure 2: The plot of the error function of Example 1

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Example 2 Consider the problem

\[ u(t, x) = \int_0^t \int_0^x xt(xy + tz)u(z, y)dydz + \sin(xt) + xt - x^3t - t^5x^3/6 - t^3x + x^2\sin(xt) - \frac{t^3x^5}{6} + t^2\sin(xt), \]

where \( t, x \in [0, 1] \). The exact solution of this problem is \( u(t, x) = \sin(xt) + xt \). In Table 2, the absolute error of the method in some points is reported. In Figures 3 and 4, the plot of numerical solution and the function of error are shown, respectively.

<table>
<thead>
<tr>
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</table>

Example 3 Consider the following linear integral equation

\[ u(t, x) = xe^t + te^x - x^2t + xt^2e^{-x} - xt^2 + x^2t^2e^{-(x+t)} + x^2te^{-t} + \int_0^t \int_0^x xte^{-(z+y)}u(z, y)dydz, \]

where \( t \in [0, 1], x \in [0, 1] \). The exact solution of this problem is \( u(t, x) = xe^t + te^x \). We use the method by \( p = q = 3 \) and \( m = M = 4, 8, 16, 32 \). The absolute error of the method in some points is reported in Table 3. Figures 5 and 6 show the numerical solution of the problem and its function error, respectively.
Table 3: Absolute error of the method for Example 3.

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<tr>
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5 Conclusion

The presented method can be applied on the two dimensional Volterra integral equations to obtain a solution in the form of piecewise polynomials that is continues in discretization nodes with good accuracy properties. Although, theoretically for getting higher accuracy we can set the method with larger values of $M$ and $N$ and also larger of the degree of approximation, $p$ and $q$, but it leads to solving $MN$ linear systems of size $pq \times pq$, that have its difficulties.

References


