Existence Results for Nonhomogeneous System of Elliptic Equations with Lack of Compactness

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Abstract: We establish the existence of a nontrivial solution for inhomogeneous quasilinear elliptic systems, governed by two Pseudo-Laplacian operators

\[
\begin{aligned}
- \Delta_p u + m(x) |u|^{p-2} u & = \lambda (\alpha + 1) h(x) |u|^{\alpha-1} |v|^{\beta+1} + f & \quad & \text{in } \Omega, \\
- \Delta_q v + l(x) |v|^{q-2} v & = \lambda (\beta + 1) h(x) |u|^{\alpha+1} |v|^{\beta-1} + g & \quad & \text{in } \Omega,
\end{aligned}
\]

\[(u, v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega), \]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) with \( N \geq 2 \), \( 1 < p, q < N \), \( \alpha > -1 \), \( \beta > -1 \), \( \lambda \) is a positive parameter, the functions \( m(x) \) and \( h(x) \) are smooth functions with change sign on \( \Omega \), \( (f, g) \in L^p(\Omega) \times L^q(\Omega) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \). We propose to show that under condition \( \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1 \), where \( p^* = \frac{Np}{N-p} \) and \( q^* = \frac{Nq}{N-q} \), our result is depending on the local minimization method. Our result is depending on the local minimization method.

Keywords: Quasilinear Elliptic systems; Nehari manifold; Local minimization; Ekeland Variational principle

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1 Introduction

In this paper we are interested in the problem

\[
\begin{aligned}
- \Delta_p u + m(x) |u|^{p-2} u & = \lambda (\alpha + 1) h(x) |u|^{\alpha-1} |v|^{\beta+1} + f & \quad & \text{in } \Omega, \\
- \Delta_q v + l(x) |v|^{q-2} v & = \lambda (\beta + 1) h(x) |u|^{\alpha+1} |v|^{\beta-1} + g & \quad & \text{in } \Omega,
\end{aligned}
\]

\[(u, v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega), \]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) with \( N \geq 2 \), \( 1 < p, q < N \), \( \alpha > -1 \), \( \beta > -1 \), \( \lambda \) is a positive parameter, the functions \( m(x) \) and \( h(x) \) are smooth functions with change sign on \( \Omega \), \( (f, g) \in L^p(\Omega) \times L^q(\Omega) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \). We propose to show that under condition \( \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1 \), where \( p^* = \frac{Np}{N-p} \) and \( q^* = \frac{Nq}{N-q} \), our result is depending on the local minimization method. Our result is depending on the local minimization method.

For \( p \geq 1 \), \( \Delta_p u \) is the p-Laplacian defined by \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) and \( W^{1,p}_0(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) equipped with the norm \( \| \nabla u \|_p \), where \( \| \cdot \|_p \) represents the norm of Lebesgue space \( L^p(\Omega) \). Let \( W^{-1,p'}(\Omega) \) be the dual space to \( W^{1,p}_0(\Omega) \) and we will write \( \| f \|_{-1,p'} \) for the norm in \( W^{-1,p'}(\Omega) \). We denote by \( \langle \cdot, \cdot \rangle \) the natural duality paring between \( W^{1,p}_0(\Omega) \) and \( W^{-1,p'}(\Omega) \). For all \( p > 1 \), \( S_p = \inf \{ \| \nabla u \|_p^p; \| u \|_{p'} = 1, u \in W^{1,p}_0(\Omega) \} \) is the best Sobolev constant of immersion \( W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega) \).

In nonlinear elliptic problems involving critical nonlinearities, one of the major difficulties is to recover the compactness of Palais-Smale sequences of the associated Euler-Lagrange functional. The concentration-compactness principle due to Lions [10] is widely used to overcome these difficulties. In [11], the author considered the system

\[
\begin{aligned}
- \Delta_p u & = |u|^{\alpha-1} u |v|^{\beta+1} + f & \quad & \text{in } \Omega, \\
- \Delta_q v & = |u|^{\alpha+1} u |v|^{\beta-1} + g & \quad & \text{in } \Omega, \\
u & = v = 0 & \quad & \text{on } \partial \Omega,
\end{aligned}
\]

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where $\Omega$ is a regular bounded set of $\mathbb{R}^N$, $\alpha > -1$, $\beta > -1$, $(f, g) \in W^{-1,p}_0(\Omega) \times W^{-1,q}_0(\Omega)$. He show that under condition $\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} = 1$, (1.1) admits a solution.

Other methods, based on the convergence almost everywhere of the gradients of Palais-Smale sequences, can be also used to recover the compactness.

J. Chabrowski [7] studied the following system
\[
\begin{align*}
-\Delta_p u &= \lambda u|u|^\alpha v^\beta + f \quad \text{in } \Omega, \\
-\Delta_q v &= \lambda v|v|^\beta u^\alpha + g \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. In the case $p = q$, he developed a method that can be used to find norm-estimates of $f$ and $g$ guaranteeing the solvability of system.

K. Adiouchi et al. [1] considered the system
\[
\begin{align*}
-\Delta_p u &= \lambda f(x, u) + u|u|^\alpha v^\beta + f \quad \text{in } \Omega, \\
-\Delta_q v &= \lambda g(x, u) + u|u|^\beta v^\alpha + g \quad \text{in } \Omega,
\end{align*}
\]
in bounded domain with Dirichlet or mixed boundary conditions. The functions $f$ and $g$ are two Caratheodory functions with subcritical conditions on the level corresponding to the energy of Palais-Smale sequences which guarantees their relative compactness.

In [5] S. Benmouloud et al. studied system (1.2) in open subset of $\mathbb{R}^N$ with lack of compactness. They used the method based on preliminary results on the convergence almost everywhere of the gradients of Palais-Smale sequences.

Motivated by paper [5], the object of this article is to study the existence of weak solution of system (1.1). Here, we borrow some ideas from that work.

Let us define $X = W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$ equipped with the norm $\|(u, v)\|_X = \max(\|\nabla u\|_p, \|\nabla v\|_q)$ which gives to $X$ Banach space properties, reflexivity and separability ([11]).

**Definition 1** We say that $(u, v) \in X$ is a weak solution of system (1.1) if and only if
\[
\begin{align*}
\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla w_1 dx + \int_{\Omega} m(x)|u|^{p-2}w_1 dx &= \lambda(\alpha + 1) \int_{\Omega} h(x)|u|^\alpha v^\beta \Delta w_1 dx + \int_{\Omega} f w_1 dx, \\
\int_{\Omega} |\nabla v|^{q-2}\nabla v \cdot \nabla w_2 dx + \int_{\Omega} l(x)|v|^{q-2}w_2 dx &= \lambda(\beta + 1) \int_{\Omega} h(x)|u|^\beta v^\alpha \Delta w_2 dx + \int_{\Omega} g w_2 dx,
\end{align*}
\]
for all $(w_1, w_2) \in X$.

The associated Euler-Lagrange functional to system (1.1) $J : X \rightarrow \mathbb{R}$ is defined by
\[
J(u, v) = \frac{1}{p} P(u) + \frac{1}{q} Q(v) - \lambda R(u, v) - \langle f, u \rangle - \langle g, v \rangle,
\]
where
\[
P(u) = \|\nabla u\|_p^p + \int_{\Omega} m(x)|u|^p dx, \quad Q(v) = \|\nabla v\|_q^q + \int_{\Omega} l(x)|v|^q dx,
\]
and
\[
R(u, v) = \int_{\Omega} h(x)|u|^\alpha v^\beta + \|\nabla u\|_p^p dx.
\]

It is well known if $J$ is bounded below and $J$ has a minimizer on $X$, then this minimizer is a critical point of $J$. However, the Euler function $J(u, v)$, associated with the problem (1.1), is not bounded below on the whole space $X$, but is bounded on an appropriate subset, and has a minimizer on this set (if it exists) gives rise to solution to (1.1). Clearly, the critical points of $J$ are the weak solutions of problem (1.1).

Consider the Nehari manifold associated to problem (1.1) given by
\[
\Lambda = \{(u, v) \in X \setminus \{(0, 0)\}; \langle J'(u, v), (u, v) \rangle = 0\}
\]

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We set
\[ m_1 = \inf_{(u,v) \in A} J(u,v), \]
and for all \( r > 0 \) and \( t > 0 \)
\[
\begin{align*}
 a(t) &= \frac{1}{t} - \frac{1}{\alpha + \beta + 2}, \\
b(t) &= \frac{t - 1}{(\alpha + \beta + 2)(\alpha + \beta + 1)}, \\
c(t) &= \frac{\alpha + \beta + 2 - t}{\alpha + \beta + 1}, \\
d(r, t) &= \frac{1}{p^r + q^r t^r},
\end{align*}
\]
and
\[
\varepsilon = d(\theta, \gamma) |c(p)| - \frac{\theta^p}{q} \frac{b(p)\min(S_p^+, S_q^+)}{c_0 \lambda} p^p q^q,
\]
where \( c_0 = \max_{x \in \Omega} h(x) \) and \( \theta, \gamma \) are fixed numbers such that
\[
0 < \theta < \frac{pc(p)}{1}, \quad \text{and} \quad 0 < \gamma < \frac{qc(q)}{1}.
\]

### 2 Main result

Our main result is the following:

**Theorem 1** Suppose that \((f, g) \in W^{-1,p'}_0(\Omega) \times W^{-1,q'}_0(\Omega)\), non of the functions \( f \) and \( g \) is identically to zero on \( \Omega \) and

(a) \( \frac{\alpha + 1}{p^*} + \frac{\beta + 1}{q^*} = 1 \), \quad (b) \( \max(p, q) < \alpha + \beta + 2 \), \quad (c) \( \frac{\theta^p}{q} \frac{b(p)\min(S_p^+, S_q^+)}{c_0 \lambda} p^p q^q < \min(\varepsilon_1, \varepsilon_2, 1) \).

Then for any \( \lambda > 0 \) there exists a pair \((u^*, v^*) \in A\) such that \((u^*, v^*)\) is a solution of system (1.1) satisfies the property \( J(u^*, v^*) < 0 \).

**Definition 2** We say that the functional \( J \) satisfies the Palais-Smale condition at level \( c \in R \) (in short form \( (PS)_c \)) if every sequence \((\{u_m, v_m\}) \subset X\) such that \( J(u_m, v_m) \to c \) and \( J'(u_m, v_m) \to 0 \) in \( X^* \) as \( m \to \infty \) is relatively compact in \( X \).

**Lemma 2** Suppose \( \alpha + \beta + 2 > \max(p, q) \). Then, there exists a sequence \((u_m, v_m) \in A\) such that \( \lim_{m \to \infty} J(u_m, v_m) = \inf_{(u,v) \in A} J(u,v) \) and

\[ \|J'(u_m, v_m)\|_{X^*} \leq \frac{1}{m}. \]

**Proof.** We show that \( J \) is bounded below on \( A \). Let \((u, v)\) be an arbitrary element in \( A \). We have

\[ J(u, v) = a(p)P(u) + a(q)Q(v) - a(1)\langle f, u \rangle - a(1)\langle g, v \rangle. \]

Using successively the Holder’s inequality and the Young’s inequality on the terms \( \langle f, u \rangle \) and \( \langle g, v \rangle \), we can write

\[
J_{\lambda} \geq a(p)\|\nabla u\|_p^p - \theta^p\|\nabla u\|_p^p + [a(q)\|\nabla u\|_q^q - \theta^q\|\nabla u\|_q^q - \theta^{-p'}[a(1)]\|\nabla u\|_p^p]\theta^{-r} - \gamma^{-q'}[a(1)]\|g\|_{-1,q'} \gamma. \]

Since the real numbers \( \theta \) and \( \gamma \) being arbitrary, a suitable choice of \( \theta \) and \( \gamma \) assure that \( J \) is bounded below on \( A \). The Ekeland Variation principle ensures the existence of such sequence.

We shall show that each minimizing sequence contains a Palais-Smale sequence when \( f, g \) satisfied in condition (c).

For all \((u, v) \in X\) we consider

\[ I(u, v) = \langle J'(u, v), (u, v) \rangle = P(u) + Q(v) - \lambda(\alpha + \beta + 2)R(u, v) - \langle f, u \rangle - \langle g, v \rangle. \]

We want to establish that \( J'(u_m, v_m) \to 0 \) in \( X^* \) as \( m \to \infty \). It suffice to show that
Lemma 3 (see [11], Lemma 4.6, Proposition 5.1) Under condition (c), we have
(i) \( \langle P'(u, v), (u, v) \rangle \neq 0 \) for all \((u, v) \in \Lambda\).
(ii) There exists \( \delta \) such that \(|J'(u_m, v_m), (u_m, v_m)| > \delta > 0, \forall n \geq n_0 \) for some \( n_0 \in \mathbb{N} \).

Lemma 4 (see [5]) The critical value of \( J \) on \( \Lambda, m_1 = \inf_{(u, v) \in \Lambda} J(u, v) \), has the following property
\[ m_1 < \min \left[ -\frac{\alpha + 1}{p'}, \frac{\beta + 1}{q'} \|g\|_{-1, q'}^q \right]. \]

Lemma 5 Let \( c \in \mathbb{R} \). Then each \((PS)_c\)–sequence for \( J \) is bounded.

Proof. Let \( \{(u_m, v_m)\} \) be such sequence, that is
\[ J(u_m, v_m) = c + o_m(1), \quad \text{and} \quad J'(u_m, v_m) = o_m \left( \|u_m, v_m\|_X \right). \]
We can write
\[ J(u_m, v_m) = \frac{1}{\alpha + \beta + 2} \langle J'(u_m, v_m), (u_m, v_m) \rangle \]
\[ = \frac{a(p)\|\nabla u_m\|^p + a(q)\|\nabla v_m\|^q}{\alpha + \beta + 2} \leq \frac{a(p)\|\nabla u_m\|^p + a(q)\|\nabla v_m\|^q}{\alpha + \beta + 2} \]
Using successively the Holder’s inequality and the Young’s inequality on the terms \( \langle f, u_m \rangle \) and \( \langle g, v_m \rangle \), we have
\[ \|u_m\|_p \leq c + o_m \left( \|u_m, v_m\| \right). \]
At this stage we can assume, up to a subsequence, that
\[ u_m \to u \quad \text{in} \quad W^{1, p}_0(\Omega), \]
\[ v_m \to v \quad \text{in} \quad W^{1, q}_0(\Omega), \]
\[ u_m \to u \quad \text{a.e. in} \quad \Omega, \]
\[ v_m \to v \quad \text{a.e. in} \quad \Omega. \]

Let
\[ J_0(u, v) = \frac{1}{p} P(u) + \frac{1}{q} Q(v) - \lambda R(u, v), \]
and
\[ \Lambda_0 = \{(u, v) \in X \setminus \{(0, 0)\}; D_1 J_0(u, v) = D_2 J_0(u, v) = 0\}, \]
where \( D_1 J_0 \) (resp. \( D_2 J_0 \)) denotes the Gateaux derivative of \( J_0 \) with respect to its first (resp. second) variable. Set
\[ m_0 = \inf_{(u, v) \in \Lambda_0} J_0(u, v). \]

Theorem 6 The functional \( J \) satisfies \((PS)_c\) with
\[ c \in (-\infty, m_0 + m_1). \]

Proof. By standard argument, we can show that the pair \((u, v)\) in (2.1) and (2.2) is a critical point of \( J \). Now, we set
\[ X_m = u_m - u. \]
\[ Y_m = v_m - v. \]

From Brezis-Lieb’s lemma [6], we have

\[
\begin{align*}
P(X_m) &= P(u_m) - P(u) + o_m(1), \\
Q(Y_m) &= Q(v_m) - Q(v) + o_m(1), \\
R(X_m, Y_m) &= R(u_m, v_m) - R(u, v) + o_m(1).
\end{align*}
\]

It follows that

\[
\begin{align*}
P(X_m) - R(X_m, Y_m) &= o_m(1), \\
Q(Y_m) - R(X_m, Y_m) &= o_m(1), \\
J_0(X_m, Y_m) &= c - J(u, v) + o_m(1).
\end{align*}
\]

Let \( P(X_m), Q(Y_m) \) and \( R(X_m, Y_m) \) have the same limit \( l \). We will show that \( l = 0 \). Assume for the sake of contradiction, \( l \neq 0 \). Let \((s_0(u_m, v_m), t_0(u_m, v_m)) \in \mathbb{R}^2\) satisfy the following system

\[
\begin{align*}
\frac{\partial}{\partial s} J_0(s_0 X_m, t_0 Y_m) &= 0, \\
\frac{\partial}{\partial t} J_0(s_0 X_m, t_0 Y_m) &= 0.
\end{align*}
\]

Let \( r = \frac{q(\alpha+1)}{q-(\beta+1)} \), we get \( p < r \). An easy computation shows that

\[
s_0(u_m, v_m) = \left[ \frac{P(X_m)Q(Y_m)^{\frac{q-(\beta+1)}{(\alpha+1)(\beta+1)}}}{(\alpha+1)(\beta+1)} \right]^{\frac{1}{q-(\beta+1)}}.
\]

and

\[
t_0(u_m, v_m) = s_0^\frac{q}{r} (u_m, v_m) \left[ \frac{\lambda(\beta+1)R(X_m, Y_m)}{Q(Y_m)^{\frac{q}{r}}(\alpha+1)} \right]^{\frac{1}{q-(\beta+1)}}.
\]

It is clear that for suitable choice of \( \lambda \) when \( \alpha, \beta \) are sufficiently small, we have

\[
\lim_{m \to \infty} s_0(u_m, v_m) = 1 = \lim_{m \to \infty} t_0(u_m, v_m)
\]

and the pair \((s_0 X_m, t_0 Y_m) \in A_0\) which together with (2.4), implies that

\[
c - J(u, v) = \lim_{m \to \infty} J_0(X_m, Y_m) = \lim_{m \to \infty} J_0(s_0 X_m, t_0 Y_m) \geq m_0,
\]

and consequently

\[
c \geq m_0 + m_1.
\]

This leads to contradiction with (2.3). \( \blacksquare \)

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**References**


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