Majorization for Certain Classes of Analytic Functions Defined by Generalized Fractional Calculus Operators

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Abstract: In this paper, we investigate a majorization problem involving starlike multivalent function of complex order belonging to a certain subclasses of multivalent function defined by generalized fractional calculus operator. Moreover, we point out some new or known consequences of our main result.

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1 Introduction

Let \( f \) and \( g \) be analytic in the open unit disk

\[
\Delta = \{ z : z \in \mathbb{C} , |z| < 1 \}.
\]

We say that \( f \) is majorized by \( g \) in \( \Delta \) (see [1]) and write

\[
f(z) \ll g(z) \quad (z \in \Delta),
\]

if there exists a function \( \varphi \), analytic in \( \Delta \) such that

\[
|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \Delta).
\]

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions.

For two functions \( f \) and \( g \), analytic in \( \Delta \), we say that the function \( f \) is subordinate to \( g \) in \( \Delta \), and we write

\[
f(z) \prec g(z),
\]

if there exists a Schwarz function \( \omega \), which is analytic in \( \Delta \) with

\[
\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \Delta)
\]

such that

\[
f(z) = g(\omega(z)) \quad (z \in \Delta).
\]

Furthermore, if the function \( g \) is univalent in \( \Delta \), then we have the following equivalence,

\[
f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).
\]
Let $A_p$ denote the class of functions of the form
\[ f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \]
which are analytic in the open unit disk $\Delta$. For simplicity, we write $A_1 := A$.

In the following, we recall the Definition of generalized fractional derivative.

**Definition 1.1 (\cite{12}, see also \cite{11,13}).** Let $0 \leq \lambda < 1$ and $\mu, \eta \in \mathbb{R}$. Then
\[ J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz} \left( \frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-t)^{-\lambda} \frac{1}{\Gamma(\mu+\eta+1)} \frac{1}{\Gamma(1-\lambda)} f(t) dt \right), \]
where the function is analytic in a simply connected region of the $z$-plane containing the origin, with the order $f(z) = O(|z|^\varepsilon)(z \to 0)$, and $\varepsilon > \max\{0, \mu - \eta\} - 1$.

It is understood that $(z-t)^{-\lambda}$ denotes the principal value for $0 \leq \arg(z-t) < 2\pi$. The function occurring in the right-hand side of (1.5) is the familiar Gaussian hypergeometric function.

**Definition 1.2 (\cite{12}).** Under the hypothesis of Definition 1.1, a fractional calculus operator $J_{0,z}^{\lambda+\mu+m,\mu+m,\eta+m}$ is defined by,
\[ J_{0,z}^{\lambda+\mu+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z) \quad (z \in \Delta; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \]

We observe that
\[ D_z^\lambda f(z) = J_{0,z}^{\lambda,\mu,\eta} f(z)(0 \leq \lambda < 1), \]
and
\[ D_z^{\lambda+m} f(z) = J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z)(0 \leq \lambda < 1; m \in \mathbb{N}_0), \]
where $D_z^{\lambda+m} f(z)$ is the well known fractional derivative operator (see \cite{14} and many others). Furthermore, in terms of Gamma functions Definition 1.1, readily yields

**Lemma 1.1.** (Srivastava et al. \cite{15}). If $0 \leq \lambda < 1; \mu, \eta \in \mathbb{R}$ and $k > \max\{0, \mu - \eta\} - 1$, then
\[ J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k+\mu+1)}{\Gamma(k-\eta+1)\Gamma(k+\mu+\eta+1)} z^{k-\mu}. \]

**Definition 1.3** A function $f \in A_p$ is said to be in the class of $M_{p,m}^{\lambda,\mu,\eta}[A,B;\gamma]$, of $p$-valent function of complex order $\gamma \neq 0$ in $\Delta$ if and only if
\[ 1 + \frac{1}{\gamma} \left( \frac{z J_{0,z}^{\lambda+\mu+m,\mu+m,\eta+m} f(z)}{J_{0,z}^{\lambda,\mu,\eta} f(z)} - (\mu - m) \right) < \frac{1 + A z}{1 + B z} \]
\[(z \in \Delta, p \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda < 1, \mu < 1 \text{ and } \eta > \max(\lambda, \mu - p - 1)\]

Clearly we have the following relationship.

1. $M_{p,m}^{\lambda,\mu,\eta}[A,B;\gamma] \equiv M_{p,m}^{\lambda,\mu,\eta}[A,B;\gamma]$ where $M_{p,m}^{\lambda,\mu,\eta}[A,B;\gamma]$ represents the class of functions $f \in A_p$ satisfying the condition
\[ 1 + \frac{1}{\gamma} \left( \frac{z D_z^{\lambda+1} f(z)}{D_z^1 f(z)} - (\mu - m) \right) < \frac{1 + A z}{1 + B z} \]
2. $M_{1,0}^{0,0,\eta}[A,B;\gamma] \equiv S^*\{A,B;\gamma\}$
3. $M_{1,0}^{1,1,\eta}[1,-1;\gamma] \equiv K(\gamma)$
4. $M_{1,0}^{0,0,\eta}[1,-1;\gamma] \equiv S(\gamma)$
5. $M_{1,0}^{0,0,\eta}[1,-1;1-\alpha] \equiv S^*(\alpha)$

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The class $S^*[A, B; \gamma]$ is studied by Polatoglu [10], which is a well-known class of starlike function. The classes $S(\gamma)$ and $K(\gamma)$ are classes of starlike and convex of complex order $\gamma \neq 0$ in $\Delta$ which were considered by Naser and Aouf [8], and Wiatrowski [16]. The class of starlike functions of order $S^*(\alpha)$ in $\Delta$.

A majorization problem for the class $S(\gamma)$ has recently been investigated by Altinas et al. [1]. Also, majorization problems for the class $S^* = S^*(0)$ have been investigated by MacGregor [7]. Further, majorization problems for different classes have been studied by Goyal and Goswami [5], Goyal at el. [6], Goswami and Aouf [2], Goswami and Wang [3] and Goswami et al. [4]. In the present paper, we investigate a majorization problem for the class $M_{p, m}^\lambda, \mu, \eta [A, B, \gamma]$.

2 Majorization problem for the class $M_{p, m}^\lambda, \mu, \eta [A, B, \gamma]$.

We begin by proving

**Theorem 2.1.** Let the function $f \in A_p$ and suppose that $g \in M_{p, m}^\lambda, \mu, \eta [A, B, \gamma]$. If $J_{p,m}^\lambda, \mu, \eta, m + m g(z)$ is majorized by $J_{0,m}^\lambda, \mu, \eta, m + m g(z)$ in $\Delta$, then

$$\left| \frac{J_{0,m}^\lambda, \mu, \eta, m + m g(z)}{J_{0,m}^\lambda, \mu, \eta, m + m g(z)} \right| \leq \frac{1 + Aw(z)}{1 + Bw(z)},$$

for $|z| = r_1$, where $r_1$ is the smallest positive root of the following equation

$$|\gamma A - B(\gamma - p + \mu + m)| r^3 - (|p - \mu - m| + 2B) r^2$$

$$- (|\gamma A - B(\gamma - p + \mu + m)| + 2)r + |p - \mu - m| = 0$$

where $(z \in \Delta, p \in \mathbb{N}, m, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda < 1, \mu < 1, |p - \mu - m| \geq |\gamma A - B(\gamma - p + \mu + m)|$ and $\eta > \text{max}(\lambda, \mu) - p - 1$.

**Proof.** Since $g \in M_{p, m}^\lambda, \mu, \eta [A, B, \gamma]$, we have

$$1 + \frac{1}{\gamma} \left( \frac{zJ_{0,m}^\lambda, \mu, \eta, m + m g(z)}{J_{0,m}^\lambda, \mu, \eta, m + m g(z)} - (p - \mu - m) \right) = \frac{1 + Aw(z)}{1 + Bw(z)}.$$ (2.3)

We note that $w(z) = c_1 z + c_2 z^2 + \ldots \in \mathcal{P}$, where $\mathcal{P}$ denotes the well-known class of bounded analytic function in $\Delta$ (see [9]) and satisfies the conditions

$$w(0) = 0 \text{ and } |w(z)| \leq |z| \ (z \in \Delta).$$

From (2.3), we get

$$zJ_{0,m}^\lambda, \mu, \eta, m + m g(z) = \frac{w(z)\{\gamma A - B(\gamma - p + \mu + m)\} + (p - \mu - m)}{1 + Bw(z)},$$

which yields

$$\left| \frac{J_{0,m}^\lambda, \mu, \eta, m + m g(z)}{J_{0,m}^\lambda, \mu, \eta, m + m g(z)} \right| \leq \frac{|(1+Bz)|}{|(p-\mu -m)| - |w(z)\{\gamma A - B(\gamma - p + \mu + m)\}|} \left| \frac{J_{0,m}^\lambda, \mu, \eta, m + m g(z)}{J_{0,m}^\lambda, \mu, \eta, m + m g(z)} \right|. $$ (2.5)

Next, since $J_{0,m}^\lambda, \mu, \eta, m + m f(z)$ is majorized by $J_{0,m}^\lambda, \mu, \eta, m + m g(z)$ i.e.

$$J_{0,m}^\lambda, \mu, \eta, m + m f(z) = \phi(z)J_{0,m}^\lambda, \mu, \eta, m + m g(z)$$

Differentiating the above equation with respect to '$z'$, we get

$$J_{0,m}^\lambda, \mu, \eta, m + m f(z) = \phi'(z)J_{0,m}^\lambda, \mu, \eta, m + m g(z) + \phi(z)J_{0,m}^\lambda, \mu, \eta, m + m g(z).$$ (2.6)

Since $\phi \in \mathcal{P}$ satisfies the inequality (see eg. Nehari [9])

$$|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}, \ (z \in \Delta),$$
by using it in (2.6), we easily get

\[ \left| f_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} f(z) \right| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \left| J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z) \right| + |\phi(z)| \left| J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} g(z) \right|. \]

Using (2.5) in above equation, we obtain

\[ \left| f_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} f(z) \right| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \left| J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z) \right| + |\phi(z)| \left| J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} g(z) \right|. \] (2.7)

Let \(|\phi(z)| = \rho (0 \leq \rho \leq 1)\) and \(|z| = r\), then

\[ \left| J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} f(z) \right| \leq \frac{\psi(r,\rho)}{(1-r^2)(|(p-\mu-m)| - r |(\gamma A - B(\gamma - p + \mu + m))|)} \left| J_{0,z}^{\lambda+m,\mu+m,\eta+m} g(z) \right|. \] (2.8)

where

\[ \psi(r,\rho) = -\rho^2 r(1 + Br) + \rho [(1 - r^2) |(p - \mu - m)| - r |(\gamma A - B(\gamma - p + \mu + m))|] + r(1 + Br). \]

Now we have to prove

\[ \left| J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} f(z) \right| \leq \left| J_{0,z}^{\lambda+m+1,\mu+m+1,\eta+m+1} g(z) \right|. \]

To prove it, it is sufficient to show that

\[ \psi(\rho) \leq 1, \]

which is equivalent

\[ (1 - \rho)(-1 + \rho)r(1 + Br) + (1 - r^2) |(p - \mu - m)| - r |(\gamma A - B(\gamma - p + \mu + m))| \geq 0, \]

this implies

\[ u(r,\rho) = [(1 - r^2) |(p - \mu - m)| - r |(\gamma A - B(\gamma - p + \mu + m))|] - (1 + \rho)r(1 + Br) \geq 0. \]

while the function \(u(r,\rho)\) takes its minimum values at \(\rho = 1\), i.e.

\[ \min\{u(r,\rho) : \rho \in [0,1]\} = u(r, 1) = v(r), \]

where

\[ v(r) = |\gamma A - B(\gamma - p + \mu + m)| r^3 - (|p - \mu - m| + 2B) r^2 \]

\[ -(|\gamma A - B(\gamma - p + \mu + m)| + 2) r + |p - \mu - m| \]

It follows that \(v(r) \geq 0\) for all \(r \in [0,r_1]\), where \(r_1\) is the smallest positive root of the equation given by (2.2). 

Upon setting \(\lambda = \mu\), we get

\textbf{Corollary 2.1.} Let the function \(f \in A_p\) and suppose that \(g \in M^{\lambda}_A[A, B; \gamma]\). If \(D^{\lambda+m}_{0,z} g(z)\) is majorized by \(D^{\lambda+m}_{0,z} g(z)\) in the unit disk \(\Delta\), then

\[ \left| D^{\lambda+m+1}_{0,z} f(z) \right| \leq \left| D^{\lambda+m+1}_{0,z} g(z) \right| \quad \text{for} |z| \leq r_2 \]

where \(r_2\) is the smallest positive root of the following equation,

\[ |\gamma A - B(\gamma - p + \lambda + m)| r^3 - (|p - \lambda - m| + 2B) r^2 \]

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\[ - (|\gamma A - (\gamma - p + \lambda + m)| + 2) r + |p - \mu - m| = 0 \]

\( (z \in \Delta, p \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda < 1, \text{ and } \eta > \max \lambda - p - 1) \).

Putting \( \lambda = \mu = m = 0 \) and \( p = 1 \), we have

**Corollary 2.2.** \( f \in A_p \) and suppose that \( g \in S^*[A, B; \gamma] \). If \( f(z) \) is majorized by \( g(z) \), then

\[ |f'(z)| \leq |g'(z)| \quad \text{for} \quad |z| \leq r_3, \]

where \( r_3 \) is the smallest positive root of the following equation,

\[ |\gamma A - B(\gamma - 1)| r^3 - (1 + 2B) r^2 - (|\gamma A - B(\gamma - 1)| + 2) r + 1 = 0. \]

**Remarks :**

(i) Putting \( \lambda = \mu = m = p = 1 \), and \( A = 1, B = -1 \), we have a known result obtain by Altinas et al. [1],

(ii) Putting \( \lambda = m = 0, \mu = p = 1 \) and \( A = 1, B = -1 \), we have the known result obtained by Mac-Gregor [7].

**References**


