Computation of Eigenvalues of Singular Sturm-Liouville Problems using Modified Adomian Decomposition Method

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Abstract: In this paper, we present a novel method for computation of eigenvalues and eigenfunctions for a class of singular Sturm-Liouville boundary value problems using modified Adomian decomposition method. The proposed method can be applied to any type of regular as well as singular Sturm-Liouville problems. This current method is capable of finding any \( n \)-th eigenvalues and eigenfunctions of the problem. The efficiency of the method is tested by considering four singular and one regular examples and the results are compared with previous known results. The proposed scheme gives eigenvalue and eigenfunction simultaneously. Numerical results show that the method is simple, however powerful and effective.

Keywords: Singular Sturm-Liouville Problems (SSLPs); Modified Adomian Decomposition Method (MADM); Adomian Polynomials; Eigenvalues; Normalized Eigenfunctions.

1 Introduction

Consider the following class of singular Sturm-Liouville two-point boundary value problems

\[-(p(x)y'(x))' + q(x)y(x) = w(x)y(x); \quad 0 < x \leq 1;\]

with boundary conditions

\[y(0) = 0; \quad \text{or} \quad y'(0) = 0 \quad \text{and} \quad ay(1) + by'(1) = 0;\]

where \( a > 0 \) and \( b \geq 0 \) are any finite constants. The parameter \( \lambda \) is not specified in the problem (1), and to find the value of \( \lambda \) for which there exists a nontrivial solution of (1) satisfying the boundary conditions is called the Sturm-Liouville problem. Such values of \( \lambda \) are called the eigenvalues and nontrivial solutions are called the eigenfunctions of this problem. If \( p(0) = 0 \) or at least one of these \( q \) or weight function \( w \) is discontinuous at \( x = 0 \) then the problem (1) is called singular problem. Consider the problem (1) with the following conditions on \( p, q \) and \( w \)

\[(A_1) \quad p \in C[0; 1], \quad p \in C^1(0; 1) \quad \text{with} \quad p > 0 \quad \text{in} \quad (0; 1] \quad \text{and} \quad \int \frac{1}{p(x)} \, dx < \infty;\]

\[(A_2) \quad q > 0, \quad w > 0 \quad \text{in} \quad (0; 1] \quad \text{and} \quad q, \quad w \in L^1(0; 1) \quad \text{and} \quad \int \frac{1}{p(x)} \left( \int R(x; y; \lambda) \, dx \right) \, dx < \infty;\]

It can be noted that all the eigenvalues of the Sturm-Liouville problem (1) are real and simple. Further, the eigenvalues form an infinite sequence and can be ordered according to increasing magnitudes so that \( \lambda_1 < \lambda_2 < \ldots \); \( \lambda \in \mathbb{Z}^+ \). Moreover, \( \lambda_i \to \infty \) as \( i \to \infty \). The singular Sturm-Liouville problem (1) plays a very important role in both theory and applications. They are used to describe a large number of physical, biological and chemical phenomena. In particular, if \( p = x, \quad q = n^2/x, \quad \text{and} \quad w = x \) the problem (1), i.e,

\[-(xy')' + \frac{n^2}{x} y = xy; \quad 0 < x \leq 1;\]

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is known Bessel’s equation (see Boyce et al. [1]).

The singular Sturm-Liouville problem (1) can be solved by using either analytical or numerical methods. For some problems, we may find all the eigenvalues and the corresponding eigenfunctions using analytical methods. However, by applying numerical methods first few eigenvalues and eigenfunctions can be calculated, but it may be difficult to determine n-th (n ≥ 2) eigenvalue and eigenfunctions of the problem (1) [2].

Some attempt has been made for solving nonsingular Sturm-Liouville problems by several researchers [2–7]. Chen and Ho [3] used differential transformation method to compute approximate eigenvalues and corresponding eigenfunctions of the Strum-Liouville problems, and consider following two problems

\[-y'' = y(x); \]
\[y(0) = y'(0); \quad y(1) + y'(1) = 0; \]
\[-y'' = 2xy(x); \]
\[y(0) = 0; \quad y(1) = 0; \] (3)

Later, Hassan [4] also applied the differential transformation technique to the problems (3), (4) and one more extra example to calculate the eigenvalues and corresponding normalized eigenfunctions. The differential transformation technique is very simple and efficient for such types of problems. However, the equation of the form

\[-(x^2y')' = (2y(x) + \sin(ln x)); \] (5)

may be difficult to handle by the differential transformation method. One can note that the differential transformation for the left-hand side of (5) can be easily obtained but it is not so for the right-hand side. Since the differential transformation of \(\sin(ln x)\) is not known, it is only available for the elementary functions, e.g. \(x^n; e^x; \ldots\) which may be one of the disadvantages of the differential transformation method. In [5], the Adomian method was applied to compute eigenvalues of regular Sturm-Liouville problems.

The code SLEIGN was specially designed to calculate the eigenvalues of regular or singular Sturm-Liouville problems with regular, separated, self-adjoint boundary conditions was introduced by Bailey et al. [8, 9]. But in singular case the code automatically select boundary conditions user does not have the option of specifying boundary conditions. Later, Baily et al. [10] proposed an improved version of the algorithm SLEIGN 2 which was tested for regular and singular Sturm-Liouville boundary value problems. Here user has complete freedom in specifying boundary conditions.

Throughout this paper we assume that the conditions (A1) and (A2) hold. When any additional conditions are needed they will be specified. Here, we are interested in numerical computation of eigenvalues and corresponding approximate eigenfunctions of regular Sturm-Liouville problems.

In this paper, we extend the work of [3–5] for a large class of the problems of the type (1)-(2). Here, we are interested in a class of singular Sturm-Liouville two-point boundary value problem. The modified recursive scheme is established by modifying zeroth component and inverse integral operator in Adomian decomposition method. Then this modified recursive scheme is applied for computing the approximate eigenvalues and corresponding normalized eigenfunctions of the problem (1)-(2). This current algorithm scheme allows us to solve large class of regular as well as singular Sturm-Liouville problems. Numerical results are presented to demonstrate the effectiveness of the method.

The rest of the paper is organized as follows. In subsection 2.1, a brief review of standard Adomian decomposition method is given. The implementation of modified Adomian decomposition method to problem (1)-(2) is discussed in subsection 2.2. The numerical results and discussionis will be given in section 3.

2 Analysis of Methods

2.1 Adomian decomposition method (ADM)

In this section, we briefly describe standard Adomian decomposition method.

Recently, many researchers [11–23] have shown interest to the study of ADM for different scientific models. The ADM gives the solution in the form of a series and on decomposing the nonlinear operator into a series in which the terms are calculated recursively using the Adomian polynomials [11].

Let us consider nonlinear second order ordinary differential equation of the following form

\[Ly + Ry + Ny = g(x); \] (6)

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where $L \equiv \frac{d^2}{dx^2}$ is the second-order linear derivative operator, $R$ is the linear remainder operator, $N$ represents the nonlinear term, and $g(x)$ is a source term. The above equation can be rewritten as

$$Ly = g(x) - R y - Ny$$

(7)

The inverse operator of $L$ is defined as

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dxdx.$$  

(8)

Operating the inverse linear operator $L^{-1}(\cdot)$ on both the sides of (7) yields

$$y = y(0) + xy'(0) + L^{-1}g(x) - L^{-1}Ry - L^{-1}Ny$$  

(9)

Next, we decompose the solution $y$ and the nonlinear function $Ny$ by an infinite series as

$$y = \sum_{n=0}^{\infty} y_n; \quad Ny = \sum_{n=0}^{\infty} A_n$$  

(10)

where $A_n$ are Adomian polynomials that can be constructed for various classes of nonlinear functions with the formula given by Adomian and Rach [11],

$$A_n = \frac{1}{n!} \frac{d^n}{d\gamma^n} \left[ N \left( \sum_{k=0}^{\infty} y_k \gamma^k \right) \right]_{\gamma=0}; \quad n = 0; 1; 2; \ldots.$$  

(11)

Substituting the series (10) into (9) gives

$$\sum_{n=0}^{\infty} y_n = y(0) + xy'(0) + L^{-1}g(x) - L^{-1}R \sum_{n=0}^{\infty} y_n - L^{-1} \sum_{n=0}^{\infty} A_n;$$  

(12)

The various components $y_n$ of the solution $y$ can be easily determined by using the recursive relation

$$\begin{align*}
y_0 &= y(0) + xy'(0) + L^{-1}g(x); \\
y_{k+1} &= -L^{-1}Ry_k - L^{-1}A_k; \quad k \geq 0;
\end{align*}$$  

(13)

Having determined the components $y_n$, $n \geq 0$, the solution $y$ in a series form follows immediately. However, the $n$-term partial sum

$$y_n(x) = \sum_{m=0}^{n-1} y_m(x);$$  

(14)

may be used to give the approximate solution. It is important to note that we can not apply above standard Adomian decomposition method (13) to solve two-point boundary value problem as the zeroth component is not independent from unknown constant. So, there is need of modification in zeroth component so that the scheme can be applied.

Many researches established the convergence of Adomian decomposition method [24–27]. The first proof of convergence of ADM was given by Cherruault [25]. Cherruault and Adomian [26] proposed a new convergence proof of Adomian decomposition method based on the properties of convergent series. Recently, Hosseini and Nasabzadeh [27] introduced a simple technique to determine the rate of convergence of ADM.

2.2 Modified Adomian decomposition method (MADM)

In this subsection, we propose two modified Adomian decomposition method for a large class of Sturm-Liouville problems (SLPs).

Consider again the class of singular Sturm-Liouville two-point boundary value problems given by (1)-(2), that is

$$-(p(x)y'(x))' + q(x)y(x) = w(x)y(x); \quad 0 < x \leq 1;$$

$$y(0) = 0; \quad \text{or } (y'(0) = 0) \text{ ay}(1) + by'(1) = 0;$$  

(15)
which can be rewritten as

\[ L y(x) = R(x; y; ) \]  \hspace{1cm} (16) \]

where \( L y(x) = (p(x)y'(x))' \) is the linear differential operator, \( R(x; y; ) = q(x)y(x) - w(x)y(x) \) is linear remainder operator. A two-fold integral operator \( L^{-1}(\cdot) \) regarded as the inverse operator of \( L(\cdot) \) is proposed as

\[ L^{-1}(\cdot) = \int_{0}^{x} \int_{0}^{\cdot} \frac{1}{p(x)} (\cdot) \, dx \, ds; \]  \hspace{1cm} (17) \]

To establish the modified recursive scheme for problem (1), we operate \( L^{-1}(\cdot) \) on the left hand side of (16), yields

\[ L^{-1}[(p(x)y'(x))'] = \int_{0}^{x} \int_{0}^{\cdot} \frac{1}{p(x)} (p(x)y'(s) - p(0)y'(0)) \, dx; \]
\[ = \int_{0}^{x} \frac{1}{p(x)} (p(x)y'(0)) \, dx; \]
\[ = y(x) - y(0) - c_2 \int_{0}^{x} \frac{1}{p(x)} \, dx; \]

\[ L^{-1}[(p(x)y'(x))'] = y(x) - c_1 - c_2 \int_{0}^{x} \frac{1}{p(x)} \, dx; \]  \hspace{1cm} (18) \]

where \( c_1 = y(0); c_2 = p(0)y'(0). \)

Operating the inverse linear operator \( L^{-1}(\cdot) \) on both the sides of (16) and using (18), we have

\[ y(x) = c_1 + c_2 \int_{0}^{x} \frac{1}{p(x)} \, dx + \int_{0}^{x} \int_{0}^{\cdot} \frac{1}{p(x)} (R(x; y; ) \, dx \, dx; \]  \hspace{1cm} (19) \]

For simplicity, we set \( h(x) = \int_{0}^{x} \frac{1}{p(x)} \, dx; \)

Thus equation (19) may be written in operator form as

\[ y(x) = c_1 + c_2 h(x) + L^{-1} q(x)y(x) - L^{-1} w(x)y(x); \]  \hspace{1cm} (20) \]

Since the problem (15) is linear, i.e., does not involve any nonlinear term, therefore, we only decompose the solution \( y(x) \) by an infinite series

\[ y(x) = \sum_{n=0}^{\infty} y_n(x; ); \]  \hspace{1cm} (21) \]

Substituting the series given by (21) into (20), we obtain

\[ \sum_{n=0}^{\infty} y_n(x; ) = c_1 + c_2 h(x) + L^{-1} q(x) y_0(x; ) - L^{-1} w(x) \sum_{n=0}^{\infty} y_n(x; ); \]  \hspace{1cm} (22) \]

Upon matching both sides of (22), and we have the following recursive scheme

\[ \begin{cases} y_0 = c_1 + c_2 h(x); \\ y_1(x; ) = L^{-1} q(x) y_0(x; ) - L^{-1} w(x) y_0(x; ); \\ \vdots \\ y_{n+1}(x; ) = L^{-1} q(x) y_n(x; ) - L^{-1} w(x) y_n(x; ); \quad n \geq 0; \end{cases} \]  \hspace{1cm} (23) \]
Note that in recursive scheme (23), the zeroth component has to be determined in order to determine all other components \( y_n(x; \lambda); n \geq 1 \). But the zeroth component, i.e., \( y_0 = c_1 + c_2 h(x) \) contains two unknown constants \( c_1 \) and \( c_2 \). So, the zeroth component cannot be directly used. Therefore, there is a need to identify zeroth component which do not contain any constant. To do so, we use first boundary condition \( y'(0) = 0 \) or \( y(0) = 0 \).

If \( y(0) = 0 \), then the zeroth component reduce to \( y_0 = h(x); c_1 = 0; c_2 \neq 0 \), and the recursive scheme (23) becomes

\[
\begin{align*}
  y_0 &= h(x); \; (c_1 = 0; \; c_2 \neq 0); \\
  y_{n+1}(x; \lambda) &= L^{-1} q(x) y_n(x; \lambda) - L^{-1} w(x) y_n(x; \lambda); \; n \geq 0 \\
\end{align*}
\]

(24)

thus the recursive scheme (24) is known a modified scheme.

Similarly, if \( y'(0) = 0 \), then the zeroth component becomes \( y_0 = 1; \; c_1 \neq 0; \; c_2 = 0 \), the recursive (23) reduces as follows

\[
\begin{align*}
  y_0 &= 1; \; (c_1 \neq 0; \; c_2 = 0); \\
  y_{n+1}(x; \lambda) &= L^{-1} q(x) y_n(x; \lambda) - L^{-1} w(x) y_n(x; \lambda); \; n \geq 0 \\
\end{align*}
\]

(25)

Thus we have established two modified recursive schemes (24) and (25), now we can determine any component \( y_n(x; \lambda) \) and the series solution by adding the components, i.e., \( y_0; y_1; \cdots; y_{n-1} \) as

\[
y_n(x; \lambda) = \sum_{m=0}^{n-1} y_m(x; \lambda); \]

(26)

For \( n \geq 2 \), the approximate series \( n(x; \lambda) \) may be assumed solution and must satisfy both the boundary conditions. In order to estimate the eigenvalues, we utilize the other boundary condition in (26), and we get

\[
\tilde{n}(1; \lambda) \equiv a_n(1; \lambda) + b_n'(1; \lambda) = 0; \; n \geq 2;
\]

(27)

where \( \tilde{n}(1; \lambda) \) is polynomial of \( \lambda \). Then we solve equation \{ \( \tilde{n}(1; \lambda) = 0 \; : \; n = 2; 3; \cdots \) \} for \( \lambda \), and real roots may be assumed the eigenvalues of the problem (1)-(2). In general, we obtain \( j = (n_j), \; j = 1; 2; \cdots; \) where \( (n_j) \) is the \( j \)th estimated eigenvalue corresponding to \( n \), if the following condition is satisfied

\[
| (n_j) - (n_{j-1}) | < \epsilon;
\]

(28)

where \( (n_{j-1}) \) is the \( j \)th estimated eigenvalue corresponding to \( n - 1 \) and \( \epsilon \) is a very small real number that will be prescribed in advanced. If \( | (n_j) - (n_{j-1}) | < \epsilon \) is satisfied, then \( (n_j) \) is the \( j \)th estimated eigenvalue. Otherwise, repeat above steps unless (28) is satisfied.

By substituting the approximate eigenvalue \( j \) into (26), the corresponding approximate eigenfunction \( n(x; j) \) can be obtained as

\[
n(x; j) = \sum_{m=0}^{n-1} y_m(x; j);
\]

(29)

The \( j \)th normalized eigenfunction is defined as

\[
\tilde{n}(x; j) = \frac{n(x; j)}{\int_0^1 | n(x; j) | dx};
\]

(30)

3 Numerical illustrations and discussion

In this section, we will present five second order Sturm-Liouville problems using the method outlined in the previous subsection 2.2. We are interested in approximating an eigenvalue and corresponding eigenfunction of such boundary value problems. The approximate eigenvalues are denoted by \( j; j = 1; 2; \cdots \) and the exact eigenvalues are denoted by \( j; j = 1; 2; \cdots \).
Example 1 Consider singular Sturm-Liouville two-point boundary value problem

\[ \begin{align*}
-\left( x^{1/2}y' \right)' &= x^{-1/2}y(x); \quad 0 < x \leq 1; \\
y'(0) &= 0; \quad y(1) = 0;
\end{align*} \tag{31} \]

where the analytical eigenfunction is \( y(x) = \cos(2\sqrt{x}) \) and corresponding eigenvalue is \( \frac{\pi^2(2k+1)^2}{16} \), where \( k \) is nonnegative integer.

We first consider \( (x^{1/2}y')' = 0 \); the homogeneous part of problem (31) and its solution say \( y = c_1 + c_2\sqrt{x} \). Then by using given condition \( y'(0) = 0 \), we obtain the zeroth component as \( y_0(x) = 1; \quad c_i \neq 0 \). By applying modified scheme (25), the problem (31) becomes as

\[ \begin{align*}
y_0(x; \lambda) &= 1; \\
y_{n+1}(x; \lambda) &= \begin{array}{l}
\int_0^x x^{-1/2} \int_0^x x^{-1/2} y_n(x; \lambda) dx \; dx; \quad n \geq 0;
\end{array} \tag{32}
\]

By adding the solution components \( y_0; y_1; \ldots; y_{n-1} \), we have \( n \)-term truncated series as

\[ n(x; \lambda) = \sum_{m=0}^{n-1} y_m(x; \lambda); \quad n \geq 1; \tag{33} \]

In order to obtain first eigenvalue of the problem (31), we follow the procedure given by (27), that is we solve equation \{ \( \lambda \)

\[ \begin{align*}
\approx_3(1) &= 1 - 2 + 0.666667^2 - 0.0888889^3 = 0; \quad \text{root is } \lambda_1^{(3)} = 0.61615107; \\
\approx_4(1) &= 1 - 2 + 0.666667^2 - 0.0888889^3 + 0.00634921^4 = 0; \quad \text{root is } \lambda_1^{(4)} = 0.61686970; \\
\approx_5(1) &= 1 - 2 + 0.666667^2 - 0.0888889^3 + 0.00634921^4 - 0.000282187^5 = 0; \\
\text{root is } \lambda_1^{(5)} &= 0.61684991; \\
\approx_6(1) &= 1 - 2 + 0.666667^2 - 0.0888889^3 + 0.00634921^4 - 0.000282187^5 + 8.5511196 \times 10^{-6} = 0; \quad \text{root is } \lambda_1^{(6)} = 0.61685028;
\end{align*} \]

By (28), i.e., the condition \( |\lambda_1^{(6)} - \lambda_1^{(5)}| < 0.00001 \) is satisfied, we have \( \lambda_1 = \lambda_1^{(6)} = 0.61685028 \), as the first approximate eigenvalue. The corresponding approximate normalized eigenfunction is obtained using (29)–(30) given by

\[ \tilde{\psi}_1(x; \lambda) = 2.16136736 \cos(1.57079632\sqrt{x}) \]

The first exact normalized eigenfunction is \( \tilde{\psi}_1(x) = 2.16136736 \cos(1.57079632\sqrt{x}) \) corresponding to the first eigenvalue \( \lambda_1 = 0.61685027 \). We plot exact \( \tilde{\psi}_1 \) and approximate \( \tilde{\psi}_6 \) normalized eigenfunction in Figure 1.

Similarly, by equations (27)–(30) the second eigenvalue \( \lambda_2 \) can be found as

\[ \begin{align*}
\approx_6(2) &= 0; \quad \text{root is } \lambda_2^{(6)} = 5.490861953; \\
\approx_7(2) &= 0; \quad \text{root is } \lambda_2^{(7)} = 5.557913211; \\
\approx_8(2) &= 0; \quad \text{root is } \lambda_2^{(8)} = 5.551196465; \\
\approx_9(2) &= 0; \quad \text{root is } \lambda_2^{(9)} = 5.551679415;
\end{align*} \]

Since \( |\lambda_2^{(9)} - \lambda_2^{(8)}| < 0.00048 \) is satisfied then the second eigenvalue will be \( \lambda_2 = 5.551679415 \) and the corresponding approximate normalized eigenfunction is

\[ \tilde{\psi}_2(x; \lambda_2) = 1.61998392(1 - 11.10335883x + 20.54742955x^2 - 15.2069889x^3 \\
+ 6.03138373x^4 - 1.48819150x^5 + 0.25036248x^6 - 0.03054796x^7 \\
+ 0.0028654x^8 - 0.00020512x^9); \]
The second exact normalized eigenfunction is \( \tilde{y}_2(x) = 161997880 \cos(471238898/\sqrt{x}) \) corresponding to eigenvalue \( \mu_2 = 5516524746 \). Again, we plot the second approximate eigenfunctions almost lie on the exact eigenfunctions.

In similar manner, we can find any eigenvalue, i.e., \( j; j = 1; 2; \cdots \) and corresponding eigenfunction of the problem by using proposed modified recursive scheme (24). In order to compare our results, we list first few exact eigenvalues as \( \mu_1 = 61685027, \mu_2 = 5516524746, \mu_3 = 15442125687, \mu_4 = 3022566347, \) and so on. In Table 1, we have also listed first nine approximate eigenvalues of the problem (31). The value of \( j \) in Table 1 shows that the results agree reasonably well with those exact values \( j; j = 1; 2; 3; 4 \).

**Example 2** Consider the following Sturm-Liouville boundary value problem

\[
-\left( x^\alpha y' \right)' = x^\alpha y; \quad 0 < x \leq 1; \\
\int_0^1 x^{-\alpha} y dx = 0; \quad y'(0) = 0; \quad y(1) = 0; \quad n \geq 0,
\]

for every \( \alpha > 0 \) the problem (35) is called singular. Here endpoint 0 is singular and 1 is a regular point.

Solving \( (x^\alpha y')' = 0 \) and using the initial condition \( y'(0) = 0 \), we have the zeroth component, i.e., \( y_0(x) = 1 \). Applying the modified recursive scheme (25), the problem (35) becomes

\[
y_0(x; ) = 1; \\
y_{n+1}(x; ) = -\int_0^x x^{-\alpha} \int_0^x x^\alpha y_n(x; ) dx dx; \quad n \geq 0,
\]

For \( \alpha = 0.5 \), we follow the methodology given by (27)–(28), to obtain the eigenvalues, i.e., \( j; j = 1; 2; \cdots \) of the problem (35), and first nine eigenvalues are presented in Table 2. In particular, the first eigenvalue is obtained as

\[
\text{as the condition } | -0.1 - 0.000002 | < 0.00002 \text{ is satisfied so the first approximate eigenvalue will be } \mu_1 = 40252388 \text{ and corresponding normalized eigenfunction is obtained using the equations (29)–(30) as}
\]

\[
\tilde{\psi}_1(x; ) = 1; \quad \tilde{\psi}_2(x; ) = \int_0^x x^\alpha \tilde{\psi}_1(x; ) dx; \quad \tilde{\psi}_n(x; ) = \int_0^x x^\alpha \tilde{\psi}_{n-1}(x; ) dx; \quad n \geq 1
\]

As the condition \( | -0.1 - 0.000002 | < 0.00002 \) is satisfied so the first approximate eigenvalue will be \( \mu_1 = 40252388 \) and corresponding normalized eigenfunction is obtained using the equations (29)–(30) as

\[
\tilde{\psi}_1(x; ) = 1; \quad \tilde{\psi}_2(x; ) = \int_0^x x^\alpha \tilde{\psi}_1(x; ) dx; \quad \tilde{\psi}_n(x; ) = \int_0^x x^\alpha \tilde{\psi}_{n-1}(x; ) dx; \quad n \geq 1
\]

In particular, for \( \alpha = 0.5 \) the problem (35) is known as order zero Bessel’s equation [1, 28]. To obtain the eigenvalues, i.e., \( j; j = 1; 2; \cdots \) numerically, we follow the procedure outlined in Example 1, and numerical results are presented in Table 3. To compare our approximate results, the exact eigenvalues are obtained by solving the modified recursive scheme (25), the problem (35) becomes

\[
y_0(x; ) = 1; \\
y_{n+1}(x; ) = -\int_0^x x^{-\alpha} \int_0^x x^\alpha y_n(x; ) dx dx; \quad n \geq 0,
\]

For \( \alpha = 0.5 \), we follow the methodology given by (27)–(28), to obtain the eigenvalues, i.e., \( j; j = 1; 2; \cdots \) of the problem (35), and first nine eigenvalues are presented in Table 2. In particular, the first eigenvalue is obtained as

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\tilde{\psi}_1(x; ) = 1; \quad \tilde{\psi}_2(x; ) = \int_0^x x^\alpha \tilde{\psi}_1(x; ) dx; \quad \tilde{\psi}_n(x; ) = \int_0^x x^\alpha \tilde{\psi}_{n-1}(x; ) dx; \quad n \geq 1
\]

As the condition \( | -0.1 - 0.000002 | < 0.00002 \) is satisfied so the first approximate eigenvalue will be \( \mu_1 = 40252388 \) and corresponding normalized eigenfunction is obtained using the equations (29)–(30) as

\[
\text{In particular, for } \alpha = 0.5 \text{ the problem (35) is known as order zero Bessel’s equation [1, 28]. To obtain the eigenvalues, i.e., } j; j = 1; 2; \cdots \text{ numerically, we follow the procedure outlined in Example 1, and numerical results are presented in Table 3. To compare our approximate results, the exact eigenvalues are obtained by solving the modified recursive scheme (25), the problem (35) becomes}
\]

\[
y_0(x; ) = 1; \\
y_{n+1}(x; ) = -\int_0^x x^{-\alpha} \int_0^x x^\alpha y_n(x; ) dx dx; \quad n \geq 0,
\]

where \( p(x) = 1, w(x) = \frac{1}{x} \) and \( q(x) = 0 \) and \( \int_0^x \infty dx = \infty \).

In this problem, we apply modified algorithm (24) and the problem (38) is converted to following recursive scheme as

\[
y_0(x) = x; \\
y_{n+1}(x; ) = -\int_0^x \frac{1}{x} y_n(x; ) dx; \quad n \geq 0
\]

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Now we can find any eigenvalue, i.e., $j$ of the problem (38) by using (26)- (28). The first few approximate eigenvalues are given in Table 4. The exact eigenvalues can be obtained by solving $\int_1^2 (2 \sqrt{\lambda}) = 0$, for and eigenfunction is $y(x) = \sqrt{\lambda} J_1(2 \sqrt{\lambda})$. Here are first few exact eigenvalues $1 = 19158529$, $2 = 35077933$, $3 = 50867340$, $4 = 66618459$, $5 = 82353150$. We compare approximate eigenvalues $j$ given in Table 4 with those exact eigenvalues $j$ obtained above. In order to compare whether eigenfunction corresponding to first two eigenvalues are converge to exact eigenfunction, we plot first approximate $\psi_6$ and first exact $\psi_1$ in Figure 3, and second exact $\psi_2$ and approximate $\psi_{10}$ in Figure 4, corresponding to 1st and 2nd eigenvalues, respectively.

**Example 4** The Boyed equation considered by Baily et al. [10]
\[
-\psi''(x) = \psi(x) + \frac{1}{x} \psi(x); \quad 0 < x \leq 1
\]
\[
\psi(0) = 0; \quad \psi(1) = 0;
\]
where $p(x) = 1$, $w(x) = 1$, and $q(x) = \frac{1}{x}$. Thus $q$ is discontinuous at $x = 0$ and also $\int_0^x q(x) dx = \infty$ the problem (40) is called singular.

The problem (40) is reduced into the following recursive scheme by using the modified algorithm (24) as
\[
y_0(x; \lambda) = x;
y_{n+1}(x; \lambda) = - \int_0^x \int_0^x y_n(x; \lambda) dx \lambda dx - \int_0^x \int_0^x \frac{1}{x} \psi_n(x; \lambda) dx \lambda dx; \quad n \geq 0;
\]
(41)

Now we are in the position that we can find any eigenvalue of the problem (40). To obtain these eigenvalues the same procedure is followed as given in (26)-(28). However, same problem was solved by Baily et al. [10] using famous code SLEIGEN2 and Transcendental equation. The approximate eigenvalues have been obtained and compared with some known results in Table 5.

**Example 5** Consider regular Sturm-Liouville two-point boundary value problem
\[
-\psi''(x) = \psi(x); \quad 0 < x \leq 1;
y'(0) = 0; \quad \psi(1) = 0;
\]
(42)

This problem was solved by Hassan [4] using the differential transformation method. The first few eigenvalues and corresponding eigenfunctions were obtained and the numerical results were compared with exact solution.

In order to convert the problem (42) into the following recursive scheme the modified Adomian method (25) is applied, and we obtain
\[
y_0(x; \lambda) = 1;
y_{n+1}(x; \lambda) = - \int_0^x \int_0^x y_n(x; \lambda) dx \lambda dx; \quad n \geq 0;
\]
(43)

The $n$-term approximate series can be obtained by adding $y_0; y_1; \ldots; y_{n-1}$, i.e.,
\[
y_n(x; \lambda) = \sum_{m=0}^{n-1} y_m(x; \lambda); \quad n \geq 2;
\]
(44)

In order to calculate eigenvalues and the corresponding normalized eigenfunctions, we turn back to subsection 2.2, and apply the procedure given by (26)-(30), and we easily obtain first eigenvalue as $\lambda_1 = 246740112$ and normalized eigenfunction is
\[
\sim_6(x; 1) = 1.5707963(1 - 1.2337005x^2 + 0.2536695x^4) - 0.0208634x^6 + 0.0009192x^8 - 0.0000252x^{10} + 4.7108750 \times 10^{-7}x^{12};
\]
(45)

By analytical method, we have exact eigenfunction $\psi(x) = \cos(x\sqrt{\lambda})$ and eigenvalue $\lambda_1 = \frac{\pi^2}{4}(2k+1)^2$, where $k$ is nonnegative integer. The first eigenvalue is $\lambda_1 = 246740112$ and normalized eigenfunction is $\bar{\psi}_1(x) = 1.5707963 \cos(1.5707963x)$.

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To demonstrate graphically, we plot the approximate $\widehat{\psi}_6$ and the exact $\hat{y}_1$ in Figure 5. It can be concluded from this figure that both the approximate and exact eigenfunctions are very close to each other.

Similarly, the second approximate eigenvalue $\lambda_2 = 22:20660468$ and normalized eigenfunction is given as

$$\psi_{10}(x, \lambda_2) = 1 - 11.1033032\times 10^{-6} - 15.2094667\times 10^{-6}$$

Again, we have the second exact eigenvalue is $\mu_2 = 22:2066099$ and corresponding normalized eigenfunction is $\hat{y}_2(x) = 1.5707961 \cos(4.7123884x)$. Then both the exact $\hat{y}_2$ and approximate $\lambda_6$ have been plotted in Figure 6. It should be noted that the approximate eigenfunction converge to exact eigenfunction. In similar manner as we did in Example 1, here we list first few exact eigenvalues as $\mu_1 = 2.4674011$, $\mu_2 = 22.2066099$, $\mu_3 = 61.6850275$, $\mu_4 = 120.9026539$, and so on.

By same procedure given in subsection 2.2, we find first few eigenvalue, i.e., $\lambda_j$, $j = 1; 2; \ldots$. In Table 6, we have also listed first nine approximate eigenvalues of the problem (42). The value of $\lambda_j$ in Table 6 shows that the results agree reasonably well with those exact values $\mu_j$, $j = 1; 2; 3; 4$.

Table 1: List of first nine approximated eigenvalues of Example 1

<table>
<thead>
<tr>
<th>j</th>
<th>$\lambda_j$</th>
<th>j</th>
<th>$\lambda_j$</th>
<th>j</th>
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<tbody>
<tr>
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<td>2</td>
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<td>49.9648722</td>
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<td>3</td>
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<td>74.6388832</td>
<td>9</td>
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</table>

Table 2: List of first nine approximated eigenvalues of Example 2, when $\alpha = 0.5$

<table>
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<tr>
<th>j</th>
<th>$\lambda_j$</th>
<th>j</th>
<th>$\lambda_j$</th>
<th>j</th>
<th>$\lambda_j$</th>
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<tbody>
<tr>
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<td>129.8794760</td>
<td>7</td>
<td>433.3703639</td>
</tr>
<tr>
<td>2</td>
<td>26.2457718</td>
<td>5</td>
<td>211.3040143</td>
<td>8</td>
<td>574.0122856</td>
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<tr>
<td>3</td>
<td>68.1937576</td>
<td>6</td>
<td>312.4676174</td>
<td>9</td>
<td>734.3936458</td>
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</tbody>
</table>

Table 3: List of first nine approximated eigenvalues of Example 2, when $\alpha = 1$

<table>
<thead>
<tr>
<th>j</th>
<th>$\lambda_j$</th>
<th>j</th>
<th>$\lambda_j$</th>
<th>j</th>
<th>$\lambda_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>139.0402844</td>
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<td>593.0428696</td>
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<td>74.8870067</td>
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<td>755.8920214</td>
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Table 4: List of first nine approximated eigenvalues of Example 3

<table>
<thead>
<tr>
<th>j</th>
<th>λ_j</th>
<th>j</th>
<th>λ_j</th>
<th>j</th>
<th>λ_j</th>
</tr>
</thead>
<tbody>
<tr>
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<td>6.6618459</td>
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<td>11.380422</td>
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<tr>
<td>2</td>
<td>3.5077933</td>
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<td>8.2353150</td>
<td>8</td>
<td>12.9518360</td>
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<tr>
<td>3</td>
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<td>9.8079292</td>
<td>9</td>
<td>14.5235964</td>
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</tbody>
</table>

Table 5: Comparison of numerical results with known results of Example 4

<table>
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</thead>
<tbody>
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<td>7.37399</td>
<td>7.3740</td>
</tr>
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<tr>
<td>3</td>
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</tr>
<tr>
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<td>154.0986237</td>
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</tr>
<tr>
<td>5</td>
<td>242.7055594</td>
<td>242.70555</td>
<td>242.705</td>
</tr>
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</table>

Figure 3: Comparison of the 1st exact $\tilde{\varphi}_1$ and the 1st approximate $\tilde{\varphi}_1$ normalized eigenfunction of Example 3

Figure 4: Comparison of the 2nd exact $\tilde{\varphi}_2$ and the 2nd approximate $\tilde{\varphi}_{10}$ normalized eigenfunction of Example 3

Figure 5: Comparison of the 1st exact $\tilde{\varphi}_1$ and 1st approximate $\tilde{\varphi}_5$ normalized eigenfunction of Example 5

Figure 6: Same as in Figure 7, but for the 2nd normalized eigenfunction of Example 5

Table 6: List of first nine approximated eigenvalues of Example 5

<table>
<thead>
<tr>
<th>j</th>
<th>λ_j</th>
<th>j</th>
<th>λ_j</th>
<th>j</th>
<th>λ_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>120.9026539</td>
<td>7</td>
<td>416.9907859</td>
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<tr>
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<td>22.2066099</td>
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<td>199.8594891</td>
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<td>555.1652476</td>
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<tr>
<td>3</td>
<td>61.6850275</td>
<td>6</td>
<td>298.5553331</td>
<td>9</td>
<td>713.0790150</td>
</tr>
</tbody>
</table>

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4 Conclusions

Adomian decomposition method has been known to be a powerful technique for solving many functional equations. In this paper, we have applied modified Adomian decomposition method to compute eigenvalues of singular Sturm-Liouville two-point boundary value problems (1). To demonstrate the computational efficiency of the method we have examined the solutions of somewhat difficult singular Sturm-Liouville problems. Comparison of the results obtained by proposed method with analytical/numerical solutions confirms that the method is considerably accurate and computationally well-situated. The results agree reasonably well with those of solutions obtained from exact solution. We have also verified graphically and numerically. The accuracy of the numerical results indicates that the method is well suited for the solution of this type of problems. This method provides a reliable technique that requires less work if compared with the differential transform method [2–4] and even with the special code developed by Baiy et al. [10].

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References


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