Abstract: The option pricing problem is one of the central contents in modern financial market. In this paper, we present a new stock model for uncertain financial market based on canonical uncertain process. Firstly, Liu’s stock model and its European option price formulas are introduced. Secondly, a general stock model for uncertain financial market is formulated by the way of uncertain differential equation. American call and put option price formulas of the generalized stock model are obtained.

Keywords: Canonical process, uncertain differential equation, uncertain stock model, European option Pricing Formula, American option Pricing Formula.

1 Introduction

Brownian motion was introduced to finance by Bachelier [13]. Samuelson [14] proposed the argument that geometric Brownian motion is a good model for stock prices. Black-Scholes [9] and independently, Metron [15] used the geometric Brownian motion to construct a theory for determining the stock options price. Their work won the 1997 Nobel Prize in economics. Stochastic financial mathematics was founded based on the assumption that stock price follows geometric Brownian motion. The Black-Scholes formula has become an indispensable tool in today’s daily financial market practice. It is widely applied in describing dynamic random phenomena.


The rest of this paper is structured as follows. Firstly, we introduce some useful concepts of uncertain process, uncertain differential and uncertain equation. Secondly, a general stock model for uncertain markets is formulated. Some American option pricing formulas on the proposed uncertain stock model are investigated.

2 Text

2.1 Uncertain Process

Definition 1 (Liu [2]) Given an index set $T$ and an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, an uncertain process is a function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, for each $t \in T$ and Borel set $B$ of real numbers, the set

$$\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\}$$

is an event.

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That is, an uncertain process \( X_t(\gamma) \) is a function of two variables \( t \) and \( \gamma \) such that the function \( X_t(\gamma) \) is an uncertain variable for each \( t \). In other words, an uncertain process is an sequence of uncertain variables indexed by time \( t \).

**Definition 2** (Liu[1]) An uncertain process \( X_t \) is said to have independent increments if

\[
X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}
\]

are independent uncertain variables for any times \( t_0 < t_1 < \cdots < t_k \). An uncertain process \( X_t \) is said to have stationary increments if, for any given \( s > 0 \), \( X_{t+s} - X_t \) are identically distributed uncertain variables for all \( t \).

**Definition 3** (Liu[1]) An uncertain process \( C_t \) is said to be a canonical process if

(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous,
(ii) \( C_t \) has stationary and independent increments,
(iii) every increment \( C_{s+t} - C_s \) is a normal uncertain variable with expected value 0 and variance \( t^2 \).

Note that almost all sample paths of canonical process are Lipschitz continuous function. However, almost all sample paths of Brownian motion are continuous but non-Lipschitz function. If we say Brownian motion describes the irregular movement of pollen with infinite speed, then we may say the canonical process describes the irregular movement of pollen with finite speed. We can verify the canonical process has an uncertainty distribution

\[
\Phi(x) = (1 + \exp(-\frac{\pi x}{\sqrt{3t}}))^{-1}
\]

**Definition 4** (Liu[1]) Let \( C_t \) be a canonical process. Then for any real numbers \( e \) and \( \sigma \)

\[
G_t = \exp(et + \sigma C_t)
\]

is called a geometric canonical process, where \( e \) is called the log-drift and \( \sigma \) is called the log-diffusion.

The geometric canonical process is expected to model stock prices in a fuzzy environment whose uncertainty distribution

\[
\Phi(x) = (1 + \exp(\frac{\pi |\ln x - et|}{\sqrt{3\sigma t}}))^{-1}
\]

### 2.2 Uncertain Differential Equation

**Definition 5** (Liu[1]) Suppose \( C_t \) is a canonical process, and \( f \) and \( g \) are some given function. Then

\[
dx_t = f(t, X_t)dt + g(t, X_t)dC_t
\]

is called an uncertain differential equation. A solution is an uncertain process \( X_t \) that satisfied (4) identically in \( t \).

Let \( u_t \) and \( v_t \) be two integrable uncertain process. Then the uncertain differential equation \( dX_t = u_t X_t dt + v_t X_t dC_t \) has a solution

\[
X_t = X_0 \exp(\int_0^t u_s ds + \int_0^t v_s dC_s)
\]

Let \( u_{1t}, u_{2t}, v_{1t}, v_{2t} \) be integrable uncertain process. Chen and Liu[16] proved the linear uncertain differential equation

\[
dx_t = (u_{1t}X_t + u_{2t})dt + (v_{1t}X_t + v_{2t})dC_t
\]

has a solution

\[
X_t = U_t(X_0 + \int_0^t \frac{u_{2s}}{U_s} ds + \int_0^t \frac{v_{2s}}{U_s} dC_s)
\]

where

\[
U_t = \exp(\int_0^t u_{1s} ds + \int_0^t v_{1s} dC_s)
\]

For example, let \( m, a, \sigma \) be real numbers. Consider the uncertain differential equation

\[
dx_t = (m - aX_t)dt + \sigma dC_t
\]

has a solution

\[
X_t = m + \exp(-at)(X_0 - \frac{m}{a}) + \sigma \exp(-at) \int_0^t \exp(as) dC_t
\]
provided that \(a \neq 0\). Note that \(X_t\) is a normal uncertain variable, i.e.

\[
X_t \sim N\left(\frac{m}{a} + \exp(-at)(X_0 - \frac{m}{a}), \frac{\sigma}{a} - \exp(-at)\frac{\sigma^2}{a}\right) 
\]  

(10)

Let \(\alpha\) and \(\beta\) be real numbers with \(\beta \neq 1\). Consider the uncertain differential equation

\[
dX_t = \alpha X_t dt + X_t^\beta dC_t 
\]

has a solution

\[
X_t = \exp(\alpha t)\left(X_0^{1-\beta} + (1 - \beta) \int_0^t \exp((\beta - 1)\alpha s) dC_s\right)^{\frac{1}{1-\beta}}
\]  

(12)

### 2.3 Uncertain Stock Model

Uncertain theory was first introduced into finance by Liu in 2009. Liu \[8\] presented an alternative assumption that stock price follows geometric canonical process and presented a basic stock model for uncertain financial market in which the bond price \(X_t\) and the stock price \(Y_t\) follow

\[
\begin{align*}
\frac{dX_t}{X_t} = r dt + \sigma dC_t \\
\frac{dY_t}{Y_t} = \gamma dt + \sigma dC_t
\end{align*}
\]

(13)

where \(r\) is the riskless interest rate, \(\gamma\) is the log-drift, \(\sigma\) is the log-diffusion, and \(C_t\) is a canonical process. It is just a fuzzy counterpart of Black-Scholes stock model\[9\]. In this model, the market is comprised of a riskless cash bond and a risky tradable stock.

A European call option is a contract that gives the holder the right to buy a stock at an expiration time \(T\) for a strike price \(K\). The payoff from a European call option is \(\max(Y_T - K, 0)\). Considering the time value of money resulted from the bond, the present value of this payoff is \(\exp(-rT)(Y_T - K)^+\). Hence the American call option price should be the expected value of the payoff.

**Definition 6** Assume a European call option has a strike price \(K\) and an expiration time \(T\). Then this option has price

\[
f_c = \exp(-rT)E[\max(Y_T - K, 0)]
\]

Assume a European call option for the stock model (13) has a strike price \(K\) and an expiration time \(T\). Then the European call option price is

\[
f_c = \exp(-rT)Y_0 \int_{Y_T}^{+\infty} \left(1 + \exp\left(\frac{\pi \ln y - \gamma T}{\sqrt{3} \sigma T}\right)\right)^{-1} dy
\]

(14)

It is clear that the European call option price is a decreasing function of interest rate \(r\). That is, the European call option will devalue if the interest rate is raised; and the European call option will appreciate in value if the interest rate is reduced. In addition, the European call option price is also a decreasing function of strike price \(K\). For example, the interest rate \(r = 0.08\), the log-drift \(\gamma = 0.06\), the log-diffusion \(\sigma = 0.32\), the initial price \(Y_0 = 20\), the strike price \(K = 25\) and the expiration time \(T = 2\). The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European call option price \(f_c = 6.9028\).

A European put option is a contract that gives the holder the right to sell a stock at an expiration time \(T\) for a strike price \(K\). The payoff from a European put option is \(\max(K - Y_T, 0)\). Considering the time value of money resulted from the bond, the present value of this payoff is \(\exp(-rT)(K - Y_T)^+\). Hence the American put option price should be the expected value of the payoff.

**Definition 7** Assume a European put option has a strike price \(K\) and an expiration time \(T\). Then this option has price

\[
f_p = \exp(-rT)E[\max(K - Y_T, 0)]
\]

Assume a European put option for the stock model (13) has a strike price \(K\) and an expiration time \(T\). Then the European put option price is

\[
f_p = \exp(-rT)Y_0 \int_{K}^{Y_T} \left(1 + \exp\left(\frac{\pi(eT - \ln y)}{\sqrt{3} \sigma T}\right)\right)^{-1} dy
\]

(15)

For example, the interest rate \(r = 0.08\), the log-drift \(\gamma = 0.06\), the log-diffusion \(\sigma = 0.32\), the initial price \(Y_0 = 20\), the strike price \(K = 25\) and the expiration time \(T = 2\). The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields
the European call option price \( f_p = 4.4074 \).

**2.4 A General Stock Model**

In this section, our goal is to show a general stock model. Let \( X_t \) be the bond price, and \( Y_t \) the stock price, then we characterize the price dynamics as follow

\[
\begin{align*}
    dX_t &= r_t X_t \, dt \\
    dY_t &= (m_t - \alpha Y_t) \, dt + \sigma_t Y_t^\beta \, dC_t
\end{align*}
\]

(16)

Where \( r_t \) is the interest rate function, \( \alpha, \beta \) are constants, \( m_t, \sigma_t \) are deterministic functions of time \( t \).

Now, we discuss some option pricing formulas. An option is a financial instrument which gives the holder a right without being under obligation to trade the underlying asset at or by expiry date for a certain prescribed price known as exercise or strike price.

Let us denote by \( Y_T \) the price of the underlying asset at the expiry time \( T \) and \( K \) the strike or exercise price. American options are financial instruments that give its holder the right to trade an underlying asset at any time \( t \leq T \) for prescribed price \( K \) without being obliged to do so. It is clear that the payoff from an American call option is the supremum of \( (Y_t - K)^+ \) over the time interval \([0, T]\). Considering the time value of money resulted from the bond, the present value of this payoff is

\[
\sup_{0 \leq t \leq T} \exp(-rt)(Y_t - K)^+
\]

Hence the American call option price should be the expected value of the payoff.

**Definition 7** Let \( X_t \) be the bond price and \( Y_t \) the stock price. American call option price \( f_c \) for the general stock model (16) is defined as

\[
f_c = E[ \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s \, ds)(Y_t - K)^+] \]

(17)

Where \( K \) is the strike price at exercise time \( t \).

**Theorem 1** (American Call Option Pricing Formula) Let \( X_t \) be the bond price and \( Y_t \) the stock price. Suppose that \( X_t \) and \( Y_t \) satisfying the price dynamics described by the general stock model (16). Then the American call option pricing formula is given by

\[
f_c = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s \, ds) \left\{ \exp(-\alpha t) Y_0 + \int_0^t m_u \exp(\alpha u) \, du \right\} + \int_0^t \sigma_u \exp(\alpha u) Y_u^\beta \, dC_u \geq s \}
\]

(18)

**Proof** Firstly, it follows from (6)

\[
\begin{align*}
    d(\exp(\alpha t)Y_t) &= \alpha \exp(\alpha t)Y_t \, dt + \exp(\alpha t) \, dY_t \\
    &= \alpha \exp(\alpha t)Y_t \, dt + \exp(\alpha t)(m_t - \alpha Y_t) \, dt + \exp(\alpha t)\sigma_t Y_t^\beta \, dC_t \\
    &= m_t \exp(\alpha t)Y_t \, dt + \sigma_t \exp(\alpha t) Y_t^\beta \, dC_t
\end{align*}
\]

Integration of both sides of above equation yields

\[
\exp(\alpha t)Y_t - Y_0 = \int_0^t m_u \exp(\alpha u) \, ds + \int_0^t \sigma_s \exp(\alpha s) Y_s^\beta \, dC_s
\]

This means

\[
Y_t = \exp(-\alpha t) \{ Y_0 + \int_0^t m_u \exp(\alpha u)ds + \int_0^t \sigma_s \exp(\alpha s) Y_s^\beta dC_s \}
\]
Secondly, according to the definition of expected value of uncertain variable, we have

\[ f_c = \sup_{0 \leq t \leq T} E[\exp(-\int_0^t r_s ds)(Y_t - K)^+] \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)E(Y_t - K)^+] \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ Y_t - K \}^+ \geq s \} ds \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ Y_t - K \} \geq s \} ds \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ Y_t \geq K \} ds \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ Y_t \geq s \} ds \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ \exp(-\alpha t)|Y_0 + \int_0^t m_u \exp(\alpha u)du \} \leq s \} ds \]
\[ + \int_0^t \sigma_u \exp(\alpha u)Y_u^0 dC_u \leq s \} ds \]

An American put option is a contract that gives the holder the right to sell the underlying asset at time \( t \leq T \) as long as the seller wishes to do so. It is clear that the payoff from an American put option is the supremum of \( (K - Y_t)^+ \) over the time interval \([0, T]\). Considering the time value of money resulted from the bond, the present value of this payoff is

\[ \sup_{0 \leq t \leq T} \exp(-rt)(K - Y_t)^+ \]

Hence the American put option price should be the expected value of the payoff.

**Definition 8** Let \( X_t \) be the bond price and \( Y_t \) the stock price. American put option price \( f \) for the general stock model \((16)\) is defined as

\[ f_p = E[ \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)(K - Y_t)^+] \]

(19)

where \( K \) is the strike price at exercise time \( t \).

**Theorem 2** (American Put Option Pricing Formula) Let \( X_t \) be the bond price and \( Y_t \) the stock price. Suppose that \( X_t \) and \( Y_t \) satisfying the price dynamics described by the general stock model \((16)\). Then the American put option pricing formula is given by

\[ f_p = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ \exp(-\alpha t)|Y_0 + \int_0^t m_u \exp(\alpha u)du \} \leq s \} ds \]
\[ + \int_0^t \sigma_u \exp(\alpha u)Y_u^0 dC_u \leq s \} ds \]

(20)

**Proof** According to the definition of expected value of uncertain variable, we have

\[ f_p = \sup_{0 \leq t \leq T} E[\exp(-\int_0^t r_s ds)(K - Y_t)^+] \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)E(K - Y_t)^+] \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ (K - Y_t)^+ \geq s \} ds \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ K - Y_t \geq s \} ds \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ Y_t \leq K \} ds \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ Y_t \leq s \} ds \]
\[ = \sup_{0 \leq t \leq T} \exp(-\int_0^t r_s ds)\int_0^{+\infty} M\{ \exp(-\alpha t)|Y_0 + \int_0^t m_u \exp(\alpha u)du \} \leq s \} ds \]
\[ + \int_0^t \sigma_u \exp(\alpha u)Y_u^0 dC_u \leq s \} ds \]

This yields the desired result and completes the proof.
3 Conclusions

In this paper, we propose a generalized stock model for financial market and investigate the option pricing problems based on it. American call and put option price formula for the generalized stock model are defined and computed. The proposed model can be further extended in many straightforward ways. Based on the results, some potential applications of uncertain stock models will be an interesting topic of further research.

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