Adaptive Generalized Function Projective Synchronization of Uncertain Hyperchaotic Systems

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Abstract: This paper investigates the adaptive generalized function projective synchronization (GFPS) of hyperchaotic systems with unknown parameters. Based on the Lyapunov stability theorem and adaptive control method, adaptive controllers and parameters update laws for the GFPS are presented. The corresponding numerical simulations on hyperchaotic Lorenz system and hyperchaotic Chen system are given to demonstrate the validity and effectiveness of the proposed method.

Keywords: generalized synchronization; function projective synchronization; hyperchaotic system

1 Introduction

In nonlinear science, chaos synchronization [22] is an important topic, which has gained a lot of attention in various fields including secure communication [1, 11], chemical reactions [17], biological systems [9], etc. Various types of the chaos synchronization have been revealed, such as complete synchronization [3, 29], phase synchronization [6, 23], anti–synchronization [10, 20], generalized synchronization [8, 18], lag synchronization [4, 10], projective synchronization [19, 25], etc. Also, papers [7, 27], suggest alternative ways for chaotic systems.

In recent years, projective synchronization, which has been first reported by Mainieri and Rehacek [19] in partially linear chaotic systems, is the most noticeable one, because it can obtain faster communication with its proportional feature [2]. Recently, Li [13, 14] considered a new type of projective synchronization method, called modified projective synchronization (MPS), where the response of the synchronized dynamical state can synchronize up to a constant scaling matrix. Then, the concept of function projective synchronization (FPS) was introduced by some researchers [5, 24], where the responses of the synchronized dynamical states can synchronize up to a scaling function. Up to now, there have only been a few papers investigating the FPS method. More recently, Yu and Li [28] discussed a new type of synchronization phenomenon, adaptive generalized function projective synchronization (GFPS) of two entirely different chaotic systems with fully unknown parameters. Particularly, GFPS is an extension of many existing projective synchronization schemes, such as MPS, FPS and so forth. The responses of synchronized dynamical states can synchronize up to a function matrix in the GFPS schemes.

In this paper, we apply GFPS method for hyperchaotic Lorenz system and hyperchaotic Chen system with unknown parameters. Most of the works mentioned so far involved mainly with low-dimensional chaotic systems with only one positive Lyapunov exponent. Hyperchaotic systems possessing at least two positive Lyapunov exponents have more complex behaviour and abundant dynamics than chaotic systems and are more suitable for some engineering applications such as secure communication. Hence how to realize synchronization of hyperchaotic systems is an interesting and challenging work.

2 Adaptive GFPS scheme of hyperchaotic systems

The drive system and the response system are defined as

\[ \dot{X}(t) = F(X), \]  
(1)  
\[ \dot{Y}(t) = G(Y) + U(X, Y, t), \]  
(2)
where $X = (x_1, x_2, \ldots, x_n)^T$, $Y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$ are the state vectors of the systems (1) and (2), respectively; $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are two continuous vector functions and $U : \{\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}\} \to \mathbb{R}^n$ is a controller which be designed. Let the error vector between systems (1) and (2) for GFPS, is

$$E = Y - \Lambda(X)X,$$

where $\Lambda(X) = \text{diag}\{H_1(X), H_2(X), \ldots, H_n(X)\}$, $H_i(X) : \mathbb{R}^n \to \mathbb{R}$ are continuous functions, and $E = (e_1, e_2, \ldots, e_n)^T$, $e_i = y_i - H_i(X)x_i$, $(i = 1, 2, \ldots, n)$.

**Definition 1** It is said that the system (1) and system (2) have the property of GFPS, if there exists a vector function $U(X, Y, t)$ such that

$$\lim_{t \to \infty} \| E \| = \lim_{t \to \infty} \| Y - \Lambda(X)X \| = 0,$$

where $\| . \|$ represents a vector norm induced by the matrix norm.

**Remark 1** The function matrix $\Lambda(X)$ is called a scaling matrix, and $H_1(X), H_2(X), \ldots, H_n(X)$ are called scaling function factors. In particular, if $H_1(X) = H_2(X) = \ldots = H_n(X)$, the GFPS is simplified to the FPS.

**Remark 2** If $\Lambda = \sigma I$, $\sigma \in \mathbb{R}$, the GFPS problem will be reduced to projective synchronization, where $I$ is an $n \times n$ identity matrix. In particular, if $\sigma = 1$ and $\sigma = -1$ the problem is further simplified to complete synchronization and anti-phase synchronization, respectively. Furthermore, if $\Lambda = \text{diag}\{a_1, a_2, \ldots, a_n\}$, the modified projective synchronization will appear, i.e. the MPS is also the special case of the proposed scheme.

**Remark 3** If the scaling matrix $\Lambda = 0$, the synchronization problem will be turned into a chaos control problem.

To investigate the GFPS of two different chaotic or hyperchaotic systems with unknown parameters, the drive and response systems (1), (2) can be rewritten as

$$\dot{X}(t) = F_1(X) + F_2(X)\xi,$$

$$\dot{Y}(t) = G_1(Y) + G_2(Y)\eta + U(X, Y, t),$$

respectively, where $F_1, G_1 : \mathbb{R}^n \to \mathbb{R}^n$ are continuous vector functions, $F_2, G_2 : \mathbb{R}^n \to R^{n \times m}, G_2 : \mathbb{R}^n \to \mathbb{R}^{n \times l}$ are continuous matrix functions, and $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^l$ are unknown parameter vectors of systems (3) and (4).

**Remark 4** For a system with unknown parameters in this paper it is only considered that its parameters cannot be known in advance, but it has a certain structure.

**Theorem 5** For the given continuous differential scaling matrix function $\Lambda(X)$ and any initial values $X(0)$, $Y(0)$, the GFPS between systems (3) and (4) will be obtained and the unknown parameters $\xi$, $\eta$ will be estimated if the controller and the parameter update laws are designed as below:

$$U(X, Y, t) = J(F_1(X) + F_2(X)\xi) - G_1(Y) - G_2(Y)\eta - E \quad (5)$$

and

$$\dot{\xi} = -\dot{\xi} = \dot{\xi} = F_2^T(X)J^T E,$$

$$\dot{\eta} = -\dot{\eta} = \dot{\eta} = G_2^T(Y)E,$$

where $\dot{\xi}$ and $\dot{\eta}$ are the estimated vectors of unknown parameters $\xi$ and $\eta$, respectively; $\dot{\xi} = \xi - \dot{\xi}$, $\dot{\eta} = \eta - \dot{\eta}$ and $J = \frac{d\Lambda(X)X}{dX}$ represents the Jacobian matrix of vector $\Lambda(X)X$.

**Proof.** From Eqs. (3) and (4), the error system is

$$\dot{E} = \dot{Y} - J\dot{X},$$

$$\quad = G_1(Y) + G_2(Y)\eta - J(F_1(X) + F_2(X)\xi) + U(X, Y, t). \quad (8)$$

Choose the following Lyapunov function

$$V(t) = \frac{1}{2}E^TE + \frac{1}{2}\dot{\xi}^T\dot{\xi} + \frac{1}{2}\dot{\eta}^T\dot{\eta}. \quad (9)$$
Thus, the error system is

\[ \dot{e} = E^T \dot{e} + \xi^T \dot{\xi} + \eta^T \dot{\eta} \]

where \( E \) is the error matrix, \( \xi \) is the error vector, and \( \eta \) is the error parameter vector. The hyperchaotic Lorenz system \([15]\) is described by

\[\begin{align*}
\dot{x}_1 &= a_1(x_2 - x_1) + x_4, \\
\dot{x}_2 &= c_1x_1 - x_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - b_1x_3, \\
\dot{x}_4 &= -x_2x_3 + d_1x_4,
\end{align*}\]

where \( x_i, \quad i = 1, \ldots, 4 \) are the state variables, and \( a_1, b_1, c_1 \) and \( d_1 \) are unknown parameters to be identified. When \( a_1 = 10, b_1 = \frac{8}{3}, c_1 = 28 \) and \( d_1 = -1 \), system (10) has two positive Lyapunov exponents, i.e., \( \lambda_1 = 0.3381 \) and \( \lambda_2 = 0.1586 \).

The differential equations of Chen hyperchaotic system \([16, 26]\), as the response system, is given by

\[\begin{align*}
\dot{y}_1 &= a_2(y_2 - y_1) + y_4 + u_1, \\
\dot{y}_2 &= b_2y_1 - y_3 + c_2y_2 + u_2, \\
\dot{y}_3 &= y_1y_2 - d_2y_3 + u_3, \\
\dot{y}_4 &= y_2y_3 + f_2y_4 + u_4,
\end{align*}\]

where \( y_i, \quad i = 1, \ldots, 4 \) are state variables, and \( a_2, b_2, c_2, d_2 \) and \( f_2 \) are unknown parameters to be estimated, and \( u_i, \quad i = 1, \ldots, 4 \) are the control laws to be designed. If \( a_2 = 35, b_2 = 7, c_2 = 12, d_2 = 3 \) and \( 0 \leq f_2 \leq 0.085 \), then the system (11) is chaotic; while if \( a_2 = 35, b_2 = 7, c_2 = 12, d_2 = 3 \) and \( 0.085 \leq f_2 \leq 0.798 \), then the system (11) is hyperchaotic; and if \( a_2 = 35, b_2 = 7, c_2 = 12, d_2 = 3 \) and \( 0.798 \leq f_2 \leq 0.9 \), system (11) is periodic.

According to the GFPS scheme presented in the previous section, without loss of generality, we choose the scaling function matrix

\[ A(X) = diag\{h_{11}x_1 + h_{12}, h_{21}x_2 + h_{22}, h_{31}x_3 + h_{32}, h_{41}x_4 + h_{42}\}, \]

where \( h_{ij} (i = 1, 2, 3, 4; \quad j = 1, 2) \) are constant numbers. The error vector can be defined as

\[ e_1 = y_1 - (h_{11}x_1 + h_{12})x_1, \]
\[ e_2 = y_2 - (h_{21}x_2 + h_{22})x_2, \]
\[ e_3 = y_3 - (h_{31}x_3 + h_{32})x_3, \]
\[ e_4 = y_4 - (h_{41}x_4 + h_{42})x_4. \]

Thus, the error system is

\[\begin{align*}
\dot{e}_1 &= a_2(y_2 - y_1) + y_4 - (2h_{11}x_1 + h_{12})(a_1(x_2 - x_1) + x_4) + u_1, \\
\dot{e}_2 &= b_2y_1 - y_3 + c_2y_2 - (2h_{21}x_2 + h_{22})(c_1x_1 - x_2 - x_1x_3) + u_2, \\
\dot{e}_3 &= y_1y_2 - d_2y_3 - (2h_{31}x_3 + h_{32})(x_1x_2 - b_1x_3) + u_3, \\
\dot{e}_4 &= y_2y_3 + f_2y_4 - (2h_{41}x_4 + h_{42})(-x_2x_3 + d_1x_4) + u_4.
\end{align*}\]
According to the Theorem 5, we get the controller
\begin{align*}
u_1 &= -\hat{a}_1 y_2 - y_1 - y_4 + (2h_{12} x_1 + h_{12})(\hat{a}_1 x_2 - x_1) + x_4 - e_1, \\
u_2 &= -\hat{b}_2 y_1 + y_1 y_2 - \hat{c}_2 y_2 + (2h_{22} x_2 + h_{22})(\hat{c}_2 x_1 - x_2 x_3) - e_2, \\
u_3 &= -y_1 y_2 + \hat{d}_2 y_3 + (2h_{31} x_3 + h_{32})(x_1 x_2 - \hat{b}_1 x_3) - e_3, \\
u_4 &= -y_2 y_3 - \hat{f}_2 y_4 + (2h_{42} x_4 + h_{42})(-x_2 x_3 + d_1 x_4) - e_4
\end{align*}

and the parameters estimation update laws
\begin{align*}
\dot{\hat{a}}_1 &= -\hat{a}_1 - (2h_{11} x_1 + h_{12})(x_2 - x_1) e_1 + \hat{a}_1, \\
\dot{\hat{b}}_1 &= -\hat{b}_1 - (2h_{31} x_3 + h_{32}) x_3 e_3 + \hat{b}_1, \\
\dot{\hat{c}}_1 &= -\hat{c}_1 - (2h_{21} x_2 + h_{22}) x_1 e_2 + \hat{c}_1, \\
\dot{\hat{d}}_1 &= -\hat{d}_1 - (2h_{41} x_4 + h_{42}) x_4 e_4 + \hat{d}_1, \\
\dot{\hat{a}}_2 &= -\hat{a}_2 - (y_2 - y_1) e_1 + \hat{a}_2, \\
\dot{\hat{b}}_2 &= -\hat{b}_2 + y_1 e_2 + \hat{b}_2, \\
\dot{\hat{c}}_2 &= -\hat{c}_2 + y_2 e_2 + \hat{c}_2, \\
\dot{\hat{d}}_2 &= -\hat{d}_2 - y_3 e_3 + \hat{d}_2, \\
\dot{\hat{f}}_2 &= -\hat{f}_2 + y_4 e_4 + \hat{f}_2,
\end{align*}
in which \(\hat{a}_1 = a_1 - \hat{a}_1, \hat{b}_1 = b_1 - \hat{b}_1, \hat{c}_1 = c_1 - \hat{c}_1, \hat{d}_1 = d_1 - \hat{d}_1, \hat{a}_2 = a_2 - \hat{a}_2, \hat{b}_2 = b_2 - \hat{b}_2, \hat{c}_2 = c_2 - \hat{c}_2, \hat{d}_2 = d_2 - \hat{d}_2, \) and \(\hat{f}_2 = f_2 - \hat{f}_2\) are estimate variables of the unknown parameters, and \(\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2\) and \(\hat{f}_2\) are the corresponding parameter errors. According to the results of Theorem 5, the GFPS between uncertain Lorenz and Chen hyperchaotic systems is obtained under the control of the controller (12), and the unknown parameters are estimated using the parameter update laws (13).

**Remark 6** During the GFPS design in this section, the scaling functions are only chosen in the form \(H_i(x) = h_{1i} x_i + h_{2i}\). In fact, they can be chosen as many other elementary functions arbitrarily.

## 4 Numerical simulations

In this section, numerical simulations are presented to verify the effectiveness of the proposed synchronization controller. Fourth-order Runge-Kutta method is used to solve the systems with time step size 0.001. We chose the parameters of drive system and response system as \(a_1 = 10, b_1 = \frac{8}{3}, c_1 = 28, d_1 = -1, a_2 = 35, b_2 = 7, c_2 = 12, d_2 = 3\) and \(f_2 = 0.5\) so that they exhibit hyperchaotic behavior if no control is applied.

Firstly, we choose the scaling function factors \(H_1(X) = 3x_1 - 1, H_2(X) = x_2 - 1, H_3(X) = 2x_3 - 3\) and \(H_4(X) = -x_4 + 2\). The initial states of the drive system and response system are
\begin{align*}
(x_1(0), x_2(0), x_3(0), x_4(0)) &= (1, 2, 1, 3), \\
(y_1(0), y_2(0), y_3(0), y_4(0)) &= (1, 3, -2, -1), \\
(\hat{a}_1(0), \hat{b}_1(0), \hat{c}_1(0), \hat{d}_1(0)) &= (8, 2, 10, 1), \\
(\hat{a}_2(0), \hat{b}_2(0), \hat{c}_2(0), \hat{d}_2(0), \hat{f}_2(0)) &= (21, 12, 3, -1, -2).
\end{align*}
The dynamics of the GFPS errors are plotted in Figure 1 display the trajectories of \(e_1, e_2, e_3\) and \(e_4\) of the error system tended to zero. Figures 2 and 3 show that the estimated values of the unknown parameters converge to chaotic values. From the numerical results, we can see that the GFPS between Lorenz and Chen hyperchaotic systems is gained, and the unknown parameters are also identified.

With the scaling function factors \(H_i(X) = -1 (i = 1, 2, 3, 4)\) and the initial states
\begin{align*}
(x_1(0), x_2(0), x_3(0), x_4(0)) &= (1, 2, 2, -1), \\
(y_1(0), y_2(0), y_3(0), y_4(0)) &= (-2, -1, -3, -3), \\
(\hat{a}_1(0), \hat{b}_1(0), \hat{c}_1(0), \hat{d}_1(0)) &= (12, -1, 30, 3), \\
(\hat{a}_2(0), \hat{b}_2(0), \hat{c}_2(0), \hat{d}_2(0), \hat{f}_2(0)) &= (40, 10, 9, 5, -1),
\end{align*}

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Figure 1: Time evolutions of the error system between uncertain Lorenz and Chen hyperchaotic systems for GFPS. (a) $e_1$; (b) $e_2$; (c) $e_3$; and (d) $e_4$.

Figure 2: The estimation of the unknown parameters for Lorenz hyperchaotic system for GFPS.
Figures 4 and 5 show an anti–phase synchronization of the uncertain Lorenz and Chen hyperchaotic systems and the estimation of corresponding unknown parameters.

And finally, when $H_i(X) = 0$ ($i = 1, 2, 3, 4$), the GFPS problem is simplified to a control problem for the uncertain Chen hyperchaotic system. The simulation results are shown in Figures 6 and 7, in which the initial states are

$$(y_1(0), y_2(0), y_3(0), y_4(0)) = (1, 2, -1, 3),$$

$$(\hat{a}_2(0), \hat{b}_2(0), \hat{c}_2(0), \hat{d}_2(0), \hat{f}_2(0)) = (15, 5, 17, 1, 2).$$

![Figure 6](image1.png)  ![Figure 7](image2.png)

Figure 3: The estimation of the unknown parameters for Chen hyperchaotic system for GFPS.

5 Conclusions

In this paper, we have investigated the generalized function projective synchronization of two different uncertain hyperchaotic systems with unknown parameters. On the basis of the Lyapunov stability theorem and adaptive control theory, the proposed scheme guarantees the GFPS between the Lorenz and Chen hyperchaotic systems. All the theoretical results are verified by numerical simulations to demonstrate the effectiveness of the proposed synchronization schemes.

References


Figure 4: Time evolutions of the error system between uncertain Lorenz and Chen hyperchaotic systems for Anti-phase synchronization. (a) $e_1$; (b) $e_2$; (c) $e_3$; and (d) $e_4$.

Figure 5: The estimation of the unknown parameters for Anti-phase synchronization. (a) for Lorenz hyperchaotic system and (b) for Chen hyperchaotic system.
Figure 6: The control for the uncertain Chen hyperchaotic system. (a) $y_1$; (b) $y_2$; (c) $y_3$; and (d) $y_4$.

Figure 7: The estimation of the unknown parameters for Chen hyperchaotic system.


