Strong Convergence of Independent Fuzzy Variables Sequence

Zhigang Wang¹ *, Fanji Tian²

¹ Department of Applied Mathematics, Hainan University, Haikou, 570228, China
² Institute of Mathematics and Computer Science, Hubei University, Wuhan, 430062, China

(Received 9 April 2012, accepted 29 September 2012)

Abstract: The emphasis in this paper is mainly on strong convergence of independent fuzzy variable sequence. Firstly, two main types convergence concepts of fuzzy sequence: convergence almost surely (a.s.), convergence in credibility are proposed. Secondly, several sufficient and necessary conditions of convergence almost surely are presented. In the end, utilizing truncated method, we verify Kolmogorov inequality, Borel-Cantelli lemma and Three-series theorem of fuzzy variables sequence. Strong convergence of independent fuzzy variables sequence sum are obtained.

Keywords: fuzzy variable; convergence almost surely; convergence in credibility; strong convergence; fuzzy variables sequence.

1 Introduction

Probability theory is a branch of mathematics for studying the behavior of random phenomena based on the normality, nonnegativity and countable additivity axioms. Classical probability measure has been widely applied in both theory and practice. However, the additivity axiom of classical measure theory has been challenged by many mathematicians. The earliest challenge was from the theory of capacities by Choquet [1] in which monotonicity and continuity axioms were assumed. Sugeno [2] generalized classical measure theory to fuzzy measure theory by replacing additivity axiom with monotonicity and semi-continuity axioms. In order to deal with general uncertainty, self-duality and countable subadditivity are much more important than continuity and semi-continuity. For this reason, Liu [3] founded an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality and countable subadditivity axioms. Limit theory plays an important role on credibility theory. The emphasis in this paper is mainly on strong convergence of independent fuzzy variable sequence. By truncated method, we verify Kolmogorov inequality, Borel-Cantelli lemma and Three-series theorem of fuzzy variables sequence. Strong convergence of independent fuzzy variables sequence sum are obtained.

2 Credibility measure

In this section, we review some preliminary concepts within the framework of credibility theory.

Definition 1 [3] Let Θ be a nonempty set, and P (Θ) the power set of Θ. A set function Cr(·) called a credibility measure if it satisfies the following axioms:

(1) (Normality) Cr(·) = 1;
(2) (Monotonicity) Cr{A} ≤ Cr{B} whenever A ⊂ B;
(3) (Self-Duality) Cr{A} + Cr{A'} = 1 for any event A;
(4) (Maximality) Cr{∪i Ai} = supi Cr{Ai} for any events {Ai} with supi Cr{Ai} < 0.5. The triplet (Θ, P (Θ), Cr) is called a credibility space.

* Corresponding author. wzhigang@hainu.edu.cn.
The law of contradiction tells us that a proposition cannot be both true and false at the same time, and the law of excluded middle tells us that a proposition is either true or false. Self-duality is in fact a generalization of the law of contradiction and law of excluded middle. In other words, a mathematical system without self-duality assumption will be inconsistent with the laws. This is the main reason why self-duality axiom is assumed. Credibility theory has been discussed by many authors from different angles, for example, Liu Y., K., and Liu B. \[4\], Liu B, and Liu YK \[5\], Li X, and Liu B \[6\] J. Peng \[7\].

**Definition 2** \[3\] A fuzzy variable is a function from a credibility space \((\Theta, P, Cr)\) to the set of real numbers.

**Definition 3** \[3\] Let \(X\) be a fuzzy variable defined on the credibility space \((\Theta, P, Cr)\). Then its membership function is derived from the credibility measure by

\[
\mu(x) = (2Cr(\xi = x)) \land 1, x \in \mathcal{R}
\]

**Definition 4** \[8\] The fuzzy variables \(X_1, X_2, \ldots, X_n\) are said to be independent if

\[
Cr\{\bigcap_{i=1}^{n}\{X_i \in B_i\}\} = \min_{1 \leq i \leq n} Cr\{X_i \in B_i\}
\]

for any sets \(B_1, B_2, \ldots, B_n\) of \(\mathcal{R}\).

It is easy to prove that the fuzzy variables \(X_1, X_2, \ldots, X_n\) are independent if and only if

\[
Cr\{\bigcup_{i=1}^{n}\{X_i \in B_i\}\} = \max_{1 \leq i \leq n} Cr\{X_i \in B_i\}
\]

for any sets \(B_1, B_2, \ldots, B_n\) of \(\mathcal{R}\).

**Definition 5** \[9\] Suppose that \(X_1, X_2, \ldots\) are fuzzy variables defined on the credibility space \((\Theta, P, Cr)\). The sequence \(\{X_n\}\) is said to be convergent a.s. to \(X\) if and only if there exists an event \(A\) with \(Cr\{A\} = 1\) such that

\[
\lim_{n \to \infty} |X_n(\theta) - X(\theta)| = 0
\]

for every \(\theta \in A\); in this case we write \(X_n \to X, a.s.\).

**Definition 6** \[9\] Suppose that \(X_1, X_2, \ldots\) are fuzzy variables defined on the credibility space \((\Theta, P, Cr)\). We said the sequence \(\{X_i\}\) converges in credibility to \(X\) if

\[
\lim_{n \to \infty} Cr\{|X_n - X| \geq \epsilon\} = 0
\]

for every \(\epsilon > 0\).

**Theorem 1** \[9\] (Chebyshev inequality) Let \(X\) be a fuzzy variable whose variance \(Var(X)\) exists. Then for any given number \(t > 0\), we have

\[
Cr\{|X - E(X)| \geq t\} \leq \frac{Var(X)}{t^2}
\]

### 3 Convergence almost surely and convergence in credibility

**Theorem 2** \[10\] Suppose that \(X_1, X_2, \ldots\) are fuzzy variables defined on the credibility space \((\Theta, P, Cr)\). If the sequence \(\{X_n\}\) converges in credibility to \(X\), then \(\{X_n\}\) converges a.s. to \(X\).

**Proof.** If \(\{X_n\}\) does not converge a.s. to \(X\), then there exists an element \(\theta^* \in \Theta\) with \(Cr\{\theta^*\} > 0\) such that \(X_n(\theta^*)\) does not converge to \(X(\theta^*)\) as \(n \to \infty\). In other words, there exists a small number \(\epsilon > 0\) and a subsequence \(\{X_{n_k}(\theta^*)\}\) such that \(|X_{n_k}(\theta^*) - X(\theta^*)| \geq \epsilon\) for any \(k\). Since credibility measure is an increasing set function, we have

\[
Cr\{|X_{n_k} - X| \geq \epsilon\} \geq Cr\{\theta^*\} > 0
\]

for any \(k\). It follows that \(\{X_n\}\) does not converge in credibility to \(X\). A contradiction proves the theorem. ■
Example 1 Convergence a.s. does not imply convergence in credibility. For example, take \((\Theta, \mathcal{P}, Cr)\) to be \(\{\theta_1, \theta_2, \ldots\}\) with \(Cr\{\theta_j\} = \frac{1}{2^n + 1}\) for \(j = 1, 2, \ldots\). The fuzzy variables are defined by

\[ X_n(\theta_j) = \begin{cases} n, & j = n \\ 0, & \text{otherwise} \end{cases} \]

for \(n = 1, 2, \ldots\) and \(X = 0\). Then the sequence \(\{X_n\}\) convergence a.s. to \(X\). However, for any small number \(\epsilon > 0\), we have

\[ Cr\{|X_n - X| \geq \epsilon\} = \frac{n}{2n + 1} \to \frac{1}{2} \]

That is, the sequence \(\{X_n\}\) does not converge in credibility to \(X\).

Theorem 3 Suppose that \(\{X_n, n \geq 1\}\) is fuzzy variables sequence defined on the credibility space \((\Theta, \mathcal{P}, Cr)\). Then \(\{X_n\}\) converges a.s. to \(X\) if and only if

\[ Cr\{|X_n - X| \geq \epsilon \text{ i.o.}\} = 0 \quad \forall \epsilon > 0 \]

Where the symbol i.o indicates that \(|X_n - X| \geq \epsilon\) occurrence of infinite.

Proof. It follows from definition 2.5, \(X_n \to X\) a.s. that there exists an event \(A\) with \(Cr\{A\} = 1\) such that \(\lim_{n \to \infty} |X_n(\theta) - X(\theta)| = 0\). In other words, it is

\[ Cr\{\bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} |X_{n+k} - X| \geq \epsilon\} = 0 \quad \forall \epsilon > 0 \]

That is

\[ Cr\{\lim \sup_{n \to \infty} |X_{n+k} - X| \geq \epsilon\} = 0 \quad \forall \epsilon > 0 \]

If and only if

\[ Cr\{|X_n - X| \geq \epsilon \text{ i.o.}\} = 0 \quad \forall \epsilon > 0 \]

Theorem 3.2 is proved. ■

The following theorem present us one criterion of convergence a.s.

Theorem 4 Suppose that \(\{X_n\}\) is a fuzzy variables sequence and \(X\) is a fuzzy variable such that

\[ \lim_{n \to \infty} Cr\bigcup_{m=n}^{\infty} \{|X_m - X| \geq \epsilon\} = 0 \quad \forall \epsilon > 0 \]

Then \(X_n \to X\) a.s.

Proof. For \(\forall n < m\), the argument

\[ \{|X_n - X| \geq \epsilon\} \subset \bigcup_{m=n}^{\infty} \{|X_m - X| \geq \epsilon\} \]

holds trivially. It follows from the monotonicity of credibility measure that

\[ 0 \leq \lim_{n \to \infty} Cr\{|X_n - X| \geq \epsilon\} \leq \lim_{n \to \infty} Cr\bigcup_{m=n}^{\infty} \{|X_m - X| \geq \epsilon\} \to 0 \quad \forall \epsilon > 0 \]

That means \(X_n \to X\) in credibility measure. It follows from theorem 3.1 that sequence \(\{X_n\}\) converges a.s. to \(X\). ■

Corollary 5 Suppose that \(\{X_n\}\) is fuzzy variables sequence such that

\[ \sum_{n=1}^{\infty} Cr\{|X_n - X| \geq \epsilon\} < \infty \]

Then \(X_n \to X\) a.s.

Corollary 6 Suppose that \(\{X_n\}\) is fuzzy variables sequence such that

\[ \sum_{n=1}^{\infty} E\{|X_n - X|^2\} < \infty \]

Then \(X_n \to X\) a.s.
Convergence of independent fuzzy variables sequence sum

Lemma 7 (Kolmogorov inequality) Suppose that \( \{X_k; 1 \leq k \leq n\} \) is a independent fuzzy variables sequence such that \( EX_k = 0 \) and \( EX_k^2 < \infty \) for any \( k \in N \). Let \( S_k = \sum_{j=1}^{k} X_j \), then for \( \forall \varepsilon > 0 \), we have

\[
Cr\left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} EX_k^2
\]

Furthermore, if \( |X_k| \leq c < \infty \) for any \( 1 \leq k \leq n \), we have

\[
Cr\left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon \right\} \geq 1 - \frac{(\varepsilon + c)^2}{\sum_{k=1}^{n} EX_k^2}
\]

Proof. Set \( A_0 = \Omega \), \( A_k = \{\max_{1 \leq j \leq k} |S_j| < \varepsilon \} \), \( k = 1, 2, \cdots, n \), \( B_1 = \{|S_1| \geq \varepsilon \} \), \( B_k = A_{k-1} - A_k \) = \( \{|S_j| < \varepsilon, 1 \leq j \leq k-1; |S_k| \geq \varepsilon \} \), \( k = 1, 2, \cdots, n \).

Since \( B_k \in \sigma(X_1, X_2, \cdots, n) \), we obtain immediately that \( S_k I_{B_k} \) and \( S_n - S_k \) are independent. We have

\[
ES^2_n I_{B_k} = ES^2_k I_{B_k} + E(S_n - S_k)^2 I_{B_k} \geq ES^2_k I_{B_k} \geq \varepsilon^2 Cr(B_k)
\]

It follows from \( A^c_n = \bigcup_{k=1}^{n} B_k \) that

\[
ES^2_n I_{A^c_n} \geq \varepsilon^2 Cr(A^c_n)
\]

That is

\[
\sum_{k=1}^{n} EX_k^2 = ES^2_n \geq ES^2_n I_{A^c_n} \geq \varepsilon^2 Cr(A^c_n)
\]

Which proves the equality (1).

The equality (2) holds trivially if \( c = \infty \). Now we assume \( C < \infty \). It is easy to prove the equality \( |S_k| \leq |S_{k-1}| + |X_k| \leq \varepsilon + c \) in \( B_k \)

\[
ES^2_n I_{B_k} = ES^2_k I_{B_k} + E(S_n - S_k)^2 I_{B_k} \leq (\varepsilon + c)^2 Cr(B_k) + \sum_{j=k+1}^{n} EX_k^2 Cr(B_k) \leq [(\varepsilon + c)^2 + \sum_{k=1}^{n} EX_k^2] Cr(B_k)
\]

That is

\[
ES^2_n I_{A^c_n} \leq [(\varepsilon + c)^2 + \sum_{k=1}^{n} EX_k^2] Cr(A^c_n)
\]

In addition, we have

\[
ES^2_n I_{A^c_n} = ES^2_n - ES^2_n I_{A_n} \geq \sum_{k=1}^{n} EX_k^2 - \varepsilon^2 Cr(A_n) = \sum_{k=1}^{n} EX_k^2 - \varepsilon^2 + \varepsilon^2 Cr(A^c_n)
\]

It follow from the equality (4) and (5) that

\[
Cr(A^c_n) \geq \frac{\sum_{k=1}^{n} EX_k^2 - \varepsilon^2}{(\varepsilon + c)^2 + \sum_{k=1}^{n} EX_k^2 - \varepsilon^2} \geq 1 - \frac{(\varepsilon + c)^2}{\sum_{k=1}^{n} EX_k^2}
\]

Which proves the inequality (2).

Lemma 8 (Borel-Cantelli Lemma) Suppose that \( \{A_n, n \geq 1\} \) is fuzzy events sequence

(1) If the inequality \( \sum_{n=1}^{\infty} Cr(A_n) < \infty \) holds, then \( Cr(\limsup_{n \to \infty} A_n) = 0 \);

(2) For \( \forall \varepsilon \in N \), if the equality \( \sup_{k \geq n} Cr(A_k) = 1 \) holds, then \( Cr(\limsup_{n \to \infty} A_n) = 1 \).

Proof. (1) It follows from countably subadditive of credibility measure and

\[
\sum_{n=1}^{\infty} Cr(A_n) < \infty \quad \text{that}
\]

\[
0 \leq Cr(\limsup_{n \to \infty} A_n) = Cr\left\{ \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right\} \leq Cr\left\{ \bigcup_{k=n}^{\infty} A_k \right\} \leq \sum_{k=n}^{\infty} Cr\{A_k\} \to 0 \quad n \to \infty
\]
\(0 \leq 1 - Cr(\limsup_{n \to \infty} A_n) = 1 - Cr(\bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_k)\)

\[= Cr(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k)^c = Cr(\bigcup_{n=1}^{\infty} (\bigcap_{k=n}^{\infty} A_k^c))\]

\[\leq \sum_{n=1}^{\infty} Cr(\bigcap_{k=n}^{\infty} A_k^c) = \sum_{n=1}^{\infty} [1 - \sup_{k \geq n} Cr(A_k)] = 0\]

The proof is complete. ■

**Theorem 9** Suppose that \(\{X_n, n \geq 1\}\) is a fuzzy variables sequence such that expected value is zero. If \(\sum_{n=1}^{\infty} EX_n^2 < \infty\), then \(\sum_{n=1}^{\infty} X_n\) a.s. converges.

**Proof.** Set \(S_n = \sum_{k=1}^{n} X_k\). For any given number \(\varepsilon > 0\) and positive integer \(m \geq n \to \infty\), we have

\[Cr\{|S_m - S_n| \geq \varepsilon\} \leq \frac{1}{\varepsilon^2} \sum_{k=n+1}^{m} EX_k^2 \to 0\]

In other words, \(\{S_n\}\) is a Cauchy sequence which converges in credibility measure. It follows from theorem 3.1 that \(\{S_n\}\) a.s. converges. ■

**Theorem 10** Suppose that \(\{X_n, n \geq 1\}\) is a independent fuzzy variables sequence such that \(|X_n| \leq c < \infty\) a.s.

(1) If \(EX_n = 0\) and \(\sum_{n=1}^{\infty} EX_n^2 = \infty\) for any \(n \geq 1\), then \(\sum_{n=1}^{\infty} X_n\) a.s. diverges;

(2) If \(\sum_{n=1}^{\infty} X_n\) a.s. converges, then \(\sum_{n=1}^{\infty} EX_n\) and \(\sum_{n=1}^{\infty} Var X_n\) converges.

**Proof.** (1) Since \(|X_n| \leq c < \infty\) a.s., \(EX_n = 0\) and \(\sum_{n=1}^{\infty} EX_n^2 = \infty\), it follows from lemma 4.1 that

\[Cr\{\max_{1 \leq k \leq m} |X_{n+1} + \cdots + X_{n+k}| \geq \varepsilon\} \geq 1 - \frac{(\varepsilon + c)^2}{\sum_{k=n+1}^{n+m} EX_k^2} \to 1 \quad (m \to \infty)\]

For any positive integer \(n\)

\[Cr\{\sup_{k \geq 1} |X_{n+1} + \cdots + X_{n+k}| \geq \varepsilon\} = 1\]

That is \(\sum_{n=1}^{\infty} X_n\) a.s. diverges.

(2) Set a sequence \(\{X'_n\}\) such that \(\{X_n\}\) and \(\{X'_n\}\) independent identically distributed. Let \(X''_n = X_n - X'_n\), then \(\{X''_n\}\) is an independent fuzzy sequence such that \(|X''_n| \leq 2c, EX''_n = 0, Var X''_n = 2Var X_n\).

Since \(\sum_{n=1}^{\infty} X_n\) a.s. converges, then \(\sum_{n=1}^{\infty} X'_n\) a.s. and \(\sum_{n=1}^{\infty} X''_n\) a.s. converges. It follows from (1) that

\[\sum_{n=1}^{\infty} Var X''_n < \infty\]

Thus we obtain \(\sum_{n=1}^{\infty} Var X_n < \infty\). It follows from theorem 4.1 that \(\sum_{n=1}^{\infty} (X_n - EX_n)\) a.s. converges. That is \(\sum_{n=1}^{\infty} X_n\) a.s. converges. We prove the theorem 4.2. ■

**Theorem 11** (Three-series theorem) Suppose that \(\{X_n\}\) is an independent fuzzy variables sequence. Let \(X^n_c = X_nI(X_n \leq c)\). The necessary condition of the series \(\sum_{n=1}^{\infty} X_n\) a.s. converges is for any \(c \in (0, +\infty)\)

(i) \(\sum_{n=1}^{\infty} Cr\{|X_n| > c\} < \infty\);

(ii) \(\sum_{n=1}^{\infty} EX_n^c\) converges;

(iii) \(\sum_{n=1}^{\infty} Var X_n < \infty\).

The sufficient condition of the series \(\sum_{n=1}^{\infty} X_n\) a.s. converges is that for only one \(c \in (0, +\infty)\), the above three series converges.

**Proof.** Since \(\sum_{n=1}^{\infty} X_n\) a.s. converges, then \(X_n \to 0\) a.s. It follows from theorem 3.2 that, for any \(c > 0\), let \(A_n = \{|X_n| \geq c\},\) then \(Cr\{A_n, i.o.\} = 0\). It follows from Borel – Cantelli lemma that condition (i) is verified. Thus

\[\sum_{n=1}^{\infty} Cr\{X_n \neq X^n_c\} = \sum_{n=1}^{\infty} Cr\{|X_n| \geq c\} < \infty\]

That is

\[Cr\{\{X_n \neq X^n_c\}, i.o.\} = 0\]

It follows from \(\sum_{n=1}^{\infty} X_n\) a.s. convergence that \(\sum_{n=1}^{\infty} X^n_c\) a.s. converges. It follows from theorem 4.2 again that (ii) and (iii) holds trivially.

Conversely, it follows from condition (i) that \(Cr\{\{X_n \neq X^n_c\}, i.o.\} = 0\). Addition to \(\omega\) sets which credibility measure is zero, series \(\sum_{n=1}^{\infty} X_n\) and \(\sum_{n=1}^{\infty} X^n_c\) have the same convergence and divergence. It follows from theorem 4.1, condition (ii) and (iii) that \(\sum_{n=1}^{\infty} X_n\) a.s. converges. The theorem is proved. ■
5 Conclusion

The main contribution of the present paper is to obtain several sufficient and necessity conditions of fuzzy variables convergence a.s. By truncated method, Kolmogorov inequality, Borel-Cantelli lemma and Three-series theorem of fuzzy variables sequence are verified. Strong convergence of independent fuzzy variables sequence sum are obtained.

Acknowledgements

This work is supported by National Natural Science Foundation of China (No.11261015), Hainan Natural Science Foundation (No.112005), and institution of higher learning scientific research Program of Hainan Provincial Department of Education (No.Hjsk2012-29), China.

References