An Extended Elliptic Equation Expansion Method and Its Application in The ZK-MEW Equation

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Abstract: In this paper, an extended elliptic equation expansion method is presented for constructing exact traveling wave solutions of nonlinear partial differential equations. The main idea of this method is to take full advantage of the solutions of the elliptic equation to construct exact traveling wave solutions of nonlinear partial differential equations. The ZK-MEW equation is chosen to illustrate the application of the extended elliptic function expansion method. Consequently, more new exact traveling wave solutions are derived that are not obtained by the previously known methods.

Keywords: An extended elliptic equation expansion method; nonlinear partial differential equations; the ZK-MEW equation; travelling wave solutions

1 Introduction

The investigation of exact travelling wave solutions to nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena. Up to now, many powerful methods have been developed such as inverse scattering transformation [1], Bäcklund transformation [2], Darboux transformation [3], Hirota bilinear method [4], homogeneous balance method [5], Jacobi elliptic function expansion method [6], and Ma’s transformed rational function method [7] and so on [8-12].

With the rapid development of computer algebraic system like Maple or mathematica, many powerful algebraic methods have been proposed and applied to many nonlinear evolution equations. For example, sine-cosine method [13], the Fu’s method [14], the tanh function expansion method [15-17], Riccati equation expansion method [18-20], the generalized Riccati equation method [21], auxiliary ordinary differential equation method [22-23], modified extended tanh function method [24-26], the Fan’s unified algebraic method [27-28], the extended Fan’s sub-equation method [29] and so on. Generally speaking, the methods have a common characteristic: the possible solutions are constructed from the known functions or the solutions of some simple and solvable nonlinear ordinary differential equations such as: sine-cosine function, tanh function, Jacobi elliptic function or the Riccati equation, the coupled Riccati equation, sinh-Gordon equation. The purpose of this paper is to propose the extended elliptic equation expansion method for solving NLPDEs. The main idea of this method is to take full advantage of the solutions of the elliptic equation to construct exact travelling wave solutions of nonlinear evolution equations.

The rest of this paper is organized as follows: In Section 2, we present the extended elliptic equation expansion method and its algorithm. In Section 3, five families of travelling wave solutions of the ZK-MEW equation are obtained as an example. Finally, some conclusions and discussions are provided in Section 4.

2 An extended elliptic equation method and its algorithm

In the following we would like to outline the main steps of our extended method algorithm:
Step 1. Reduce the nonlinear PDE system to the nonlinear ODE system
For a given nonlinear evolution equation system with some physical fields $u(x, y, t)$ in three variables $x, y, t$.

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, u_{xy}, \cdots) = 0$$

where $F = (F_1, F_2, \cdots, F_p)$, $u = (u_1, u_2, \cdots, u_q)$, $p$ and $q$ are positive integers.

By using the following wave transformation

$$u_i(x, y, t) = u_i(\xi), \quad \xi = kx + ly - \lambda t, \quad i = 1, 2, \cdots, q$$

where $k, l$ and $\lambda$ are constants to be determined later. Then the nonlinear evolution Eq.(1) is reduced to a nonlinear ordinary differential equation (ODE) system:

$$G(u, u', u'', \cdots) = 0$$

where $G = (G_1, G_2, \cdots, G_p)$.

Step 2. Set the series formal solutions
To seek the traveling wave solutions of (3), in [30] Xu assumes that (3) has the solutions in the form of

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{n_i} a_{ij} \varphi^j(\xi), \quad i = 1, 2, \cdots, q,$$

with the new variable $\varphi(\xi)$ satisfying the elliptic equation

$$\varphi'^2 = A \varphi(\varphi - \alpha)(\varphi - \beta)(\varphi - \gamma),$$

where $A$, $\alpha$, $\beta$ and $\gamma$ are constants and $n_i$ the integer to be determined later.

In order to get more exact solutions of (3), here we assume that (3) has the following formal solutions

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{n_i} \left( a_{ij} \varphi^j(\xi) + b_{ij} \varphi^{j-1}(\xi) \varphi'(\xi) + \frac{c_{ij}}{\varphi}(\xi) + d_{ij} \varphi'(\xi) \right), \quad i = 1, 2, \cdots, q$$

where the variable $\varphi(\xi)$ satisfies the elliptic equation (5).

Step 3. Determine the truncation expansion terms in (6)

The underlying mechanism for obtained exact solutions to occur is that differ effects that act to change wave forms in many nonlinear equations, i.e., dispersion, dissipation, and nonlinearity, either separately or various combination, are able to balance out. We define the degree of $u(\xi)$ as $D[u_i(\xi)] = n_i$ which gives rise to the degrees of other expressions as

$$D \frac{d^4 u_i}{d\xi^4} = n_i + q, \quad D \left[ u_i^p \left( \frac{d^3 u_i}{d\xi^3} \right)^4 \right] = n_i p + s(n_i + q), \quad i = 1, 2, \cdots, q$$

Therefore we can get the value of $n_i$ in (6) by balancing the highest-order contributions from such terms in (1) or (3). If $n_i$ is a nonnegative integer, we first make the transformation $u_i = w_i^{n_i}$.

Step 4. Lead to the set of algebraic equations

We substitute (6) along with (5) into (3) and then set to zero the coefficients of like powers of $\varphi^j \varphi'^j$ ($i = 0, 1, 2, \cdots; j = 0, 1$) to obtain a set of nonlinear algebraic equations with respect to the unknowns $a_{i0}$, $a_{ij}$, $b_{ij}$, $c_{ij}$ and $d_{ij}$ ($i = 1, 2, \cdots; j = 1, 2, \cdots, n_i$).

Step 5. Solve the algebraic equations

By solving the over-determined system of nonlinear algebraic equations by use of symbolic computation system Maple, we can get these unknowns $a_{i0}$, $a_{ij}$, $b_{ij}$, $c_{ij}$ and $d_{ij}$ ($i = 1, 2, \cdots; j = 1, 2, \cdots, n_i$).

Step 6. Solve for (5)

Motivated by [30], we obtain some special solutions of (5) which are now listed in the following

Case 1. If $A > 0$ and $\alpha < 0 < \beta < \gamma$, (5) has the solution

$$\varphi(\xi) = \frac{\alpha \text{sn}^2(\alpha_1 \xi)}{\alpha - \beta \text{cn}^2(\alpha_1 \xi)}, \quad \alpha_1 = \frac{1}{2} \sqrt{A \gamma (\beta - \alpha)}, \quad m = \sqrt{\frac{\beta (\gamma - \alpha)}{\gamma (\beta - \alpha)}}$$

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Case 2. If $A > 0$ and $\alpha < \beta < 0 < \gamma$, (5) has the solution
$$\varphi(\xi) = \frac{\alpha \gamma \text{sn}^2(\alpha_1 \xi)}{\gamma - \alpha \text{cn}^2(\alpha_1 \xi)}, \alpha_1 = \frac{1}{2} \sqrt{A \beta (\alpha - \gamma)}, m = \sqrt{\frac{\alpha (\gamma - \beta)}{\beta (\gamma - \alpha)}}$$ (9)

Case 3. If $A < 0$ and $0 < \alpha < \beta < \gamma$, (5) has the solution
$$\varphi(\xi) = \frac{\alpha (\gamma - \beta) \text{sn}^2(\alpha_1 \xi) - \beta (\gamma - \alpha)}{(\gamma - \beta) \text{sn}^2(\alpha_1 \xi) - (\gamma - \alpha)}, \alpha_1 = \frac{1}{2} \sqrt{A \beta (\alpha - \gamma)}, m = \sqrt{\frac{\alpha (\gamma - \beta)}{\beta (\gamma - \alpha)}}$$ (10)

Case 4. If $A < 0$ and $\alpha < 0 < \beta < \gamma$, (5) has the solution
$$\varphi(\xi) = \frac{\beta \gamma \text{sn}^2(\alpha_1 \xi)}{\gamma + (\beta - \gamma) \text{sn}^2(\alpha_1 \xi)}, \alpha_1 = \frac{1}{2} \sqrt{-A \gamma (\beta - \alpha)}, m = \sqrt{\frac{\alpha (\beta - \gamma)}{\gamma (\beta - \alpha)}}$$ (11)

Case 5. If $A < 0$ and $\alpha < \beta < 0 < \gamma$, (5) has the solution
$$\varphi(\xi) = \frac{\beta \gamma \text{sn}^2(\alpha_1 \xi)}{\beta - \gamma \text{cn}^2(\alpha_1 \xi)}, \alpha_1 = \frac{1}{2} \sqrt{A \alpha (\gamma - \beta)}, m = \sqrt{\frac{\gamma (\alpha - \beta)}{\alpha (\gamma - \beta)}}$$ (12)

Step 7. Obtain the exact solutions of (3).
By inserting each solution of the above set of algebraic equations into (6) and making use of the solutions (8)-(12), some new types of elliptic function solutions of (3) can be obtained.

Remark 1 Compared with the method proposed by Xu[30], our ansatz is more general than the ansatz in [30]. When $b_i = c_i = d_i = 0$ in Eq. (6), Eq. (6) becomes the ansatz proposed by Xu.

3 ZK-MEW equation
As known, the Zakharov-Kuznetsov (ZK) equation are given by
$$(u_t + a u u_x + u_{xxx})_x + u_{yy} = 0$$ (13)
and
$$u_t + a u u_x + (\nabla^2 u)_x = 0$$ (14)
where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the isotropic Laplacian. The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [31]. In [31], the ZK equation is solved by the sine-cosine and the tanh-function methods. In [32], the numbers of solitary waves, periodic waves, and kink waves of the modified Zakharov-Kuznetsov equation are obtained.

The regularized long wave (RLW) equation given by
$$u_t + u_x + \frac{1}{2} (u^2)_x + u_{xxt} = 0, -\infty < x < +\infty, t > 0$$ (15)
appears in many physical applications and has been studied in [33]. Gardner et al. [34] solved the equal width equation by a Petrov Galerkin method using quadratic B-spline spatial finite elements.

The modified equal width (MEW) equation given by
$$u_t + 3 u^2 u_x - \beta u_{xxt} = 0$$ (16)
has been discussed in [33]. The MEW equation is related to the RLW equation. This equation has solitary waves with both positive and negative amplitudes. The two-dimensional ZK-MEW equation which first appeared in [35] is given by
$$u_t + a (u^3)_x + (bu_{xt} + ru_{yy})_x = 0$$ (17)
where $u = u(x, y, t), a, b, r$ are constants. In [35], some exact solutions of the ZK-MEW equation (1.5) was obtained by using the tanh and sine-cosine methods. To more detailed description for ZK-MEW equation (17) the reader can find
in paper [35]. Recently in [36], an extended tanh method is used to establish exact travelling wave solution of the ZK-MEW equation (17). In [37], some new solutions of Jacobi elliptic function type of ZK-MEW equation (17) was given by using an extended Jacobi elliptic function method. In [38], He’s Exp-function method is used to obtain a generalized soliton solution of ZK-MEW equation. A good understanding of all solutions of (17) is very helpful for physical scientists and engineers to apply the model in the research of a uniform magnetic field. Therefore, finding more types of exact solutions of Eq.(17) is of fundamental interest in magnetic field. There is an amount of paper devoted to this paper[39-42]. According to the above method, to seek travelling wave solutions of Eq.(17), we make the following transformation

\[ u(x, y, t) = u(\xi), \xi = kx + ly - \lambda t \]  

where \( k, l, \lambda \) are constants to be determined later, and thus Eq.(17) becomes

\[ u'' = \frac{\lambda}{k(r^2 - bk\lambda)}u - \frac{a}{r^2 - bk\lambda}u^3 \]  

(19)

According to Step 1 in Section 2, if \( \frac{a}{r^2 - bk\lambda} \neq 0 \), by balancing \( u'' \) and \( u^3 \) in Eq.(19), we suppose that Eq.(19) has the following formal solutions:

\[ u = a_0 + a_1 \varphi + b_1 \varphi' + \frac{c_1}{\varphi} + d_1 \frac{\varphi'}{\varphi} \]  

(20)

where \( \varphi \) satisfies (5), where \( a_0, a_1, b_1, c_1 \) and \( d_1 \) are constants to be determined later.

With the aid of Maple, substituting (20) along with (5) into (19), yields a set of algebraic equations for \( \varphi^i \varphi'^j \) (\( i = 0, 1, 2, \cdots, j = 0, 1 \)). Setting the coefficients of these terms \( \varphi^i \varphi'^j \) (\( i = 0, 1, 2, \cdots, j = 0, 1 \)) to zero yields a set of over-determined algebraic equations with respect to \( a_0, a_1, b_1, c_1 \) and \( d_1 \).

By use of the Maple, solving the over-determined algebraic equations, we can get the following results:

Case 1.

\[ \begin{cases} a_0 = \pm \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, & a_1 = \pm \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha - \beta + \gamma}}, \quad b_1 = 0, \quad c_1 = 0, \\ d_1 = 0, & \lambda = -\frac{\sqrt{\alpha}}{2\sqrt{\beta}}(3\beta^2 - 2\alpha \beta - 2\beta \gamma + 3\alpha^2 - 2\alpha \gamma + 3\gamma^2) \end{cases} \]  

(21)

Case 2.

\[ a_0 = 0, \quad a_1 = 0, \quad b_1 = 0, \quad c_1 = 0, \quad d_1 = \pm \sqrt{-\frac{2A(r^2 - bk\lambda)}{2a}} \]  

(22)

Case 3.

\[ \begin{cases} a_0 = 0, \quad a_1 = \pm \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, \quad b_1 = 0, \quad c_1 = 0, \quad d_1 = \pm \sqrt{-\frac{r^2 - bk\lambda}{2a}}, \\ \lambda = -\frac{\sqrt{\alpha}}{2\sqrt{\beta}}(3\beta^2 - 2\alpha \beta - 2\beta \gamma + 3\alpha^2 - 2\alpha \gamma + 3\gamma^2) \end{cases} \]  

(23)

From (20), (8)-(12) and Case 1-3, we obtain the following solutions for Eq.(17):

Family 1. If \( A > 0 \) and \( \alpha < 0 < \beta < \gamma \)

\[ \begin{cases} u_{11} = \pm \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, \quad & \alpha = \frac{\alpha_1 \beta \gamma}{\alpha_1 \beta \gamma}, \quad & \alpha_1 = \frac{1}{2} \sqrt{A}(\beta - \alpha), \\ m = \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, \quad & \beta = \frac{\alpha_1 \beta \gamma}{\alpha_1 \beta \gamma}, \quad & \beta_1 = \frac{1}{2} \sqrt{A}(\beta - \alpha), \\ u_{12} = \pm \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, \quad & \gamma = \frac{\alpha_1 \beta \gamma}{\alpha_1 \beta \gamma}, \quad & \gamma_1 = \frac{1}{2} \sqrt{A}(\beta - \alpha) \end{cases} \]  

(24)

\[ \begin{cases} u_{13} = \pm \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, \quad & \alpha = \frac{\alpha_1 \beta \gamma}{\alpha_1 \beta \gamma}, \quad & \alpha_1 = \frac{1}{2} \sqrt{A}(\beta - \alpha), \\ m = \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, \quad & \beta = \frac{\alpha_1 \beta \gamma}{\alpha_1 \beta \gamma}, \quad & \beta_1 = \frac{1}{2} \sqrt{A}(\beta - \alpha) \end{cases} \]  

(25)

where \( \frac{r^2 - bk\lambda}{\alpha + \beta + \gamma} < 0, \xi = kx + ly - \lambda t \).

Family 2. If \( A > 0 \) and \( \alpha < \beta < 0 < \gamma \)

\[ \begin{cases} u_{21} = \pm \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, \quad & \alpha = \frac{\alpha_1 \beta \gamma}{\alpha_1 \beta \gamma}, \quad & \alpha_1 = \frac{1}{2} \sqrt{A}(\beta - \alpha), \\ m = \sqrt{-\frac{2A(r^2 - bk\lambda)}{\alpha + \beta + \gamma}}, \quad & \beta = \frac{\alpha_1 \beta \gamma}{\alpha_1 \beta \gamma}, \quad & \beta_1 = \frac{1}{2} \sqrt{A}(\beta - \alpha) \end{cases} \]  

(27)
\[ u_{22} = \pm \sqrt{\frac{2(r^2 - bk\lambda)}{a} \frac{2\alpha_1 (\gamma - \alpha)\cos(\alpha_1 \xi)\sin(\alpha_1 \xi)}{\sin(\alpha_1 \xi) [\gamma - \alpha \cos^2(\alpha_1 \xi)]}} , \]
\[ \alpha_1 = \frac{1}{2} \sqrt{\frac{A\beta (\alpha - \gamma)}{\beta (\gamma - \alpha)}}, \]
\[ m = \sqrt{\frac{\alpha (\gamma - \beta)}{\beta (\gamma - \alpha)}} \] (28)

\[ \{ u_{23} = \pm \sqrt{\frac{2(r^2 - bk\lambda)}{a} \frac{2\alpha_1 (\gamma - \beta)\sin^2(\alpha_1 \xi)}{\gamma - \alpha \cos^2(\alpha_1 \xi)}}, \]
\[ \alpha_1 = \frac{1}{2} \sqrt{A\beta (\alpha - \gamma)}, \] m = \[ \sqrt{\frac{\alpha (\gamma - \beta)}{\beta (\gamma - \alpha)}}, \]
\[ \frac{\lambda}{k(r^2 - bk\lambda)} = -\frac{1}{2} A(\alpha \beta + \beta \gamma + \alpha \gamma) \] (29)

where \( \frac{(r^2 - bk\lambda)}{a} < 0, \xi = kx + ly - \lambda t. \)

**Family 3.** if \( \alpha < 0 \) and \( 0 < \beta < \gamma < \)

\[ \{ u_{31} = \pm \frac{1}{\sqrt{2}} \sqrt{\frac{2A(r^2 - bk\lambda)}{a} (\alpha + \beta + \gamma) \pm \sqrt{\frac{2A(r^2 - bk\lambda)}{a} \frac{2\alpha_1 (\gamma - \beta)\sin^2(\alpha_1 \xi) - \beta (\gamma - \alpha)}{\gamma - \alpha \cos^2(\alpha_1 \xi)}}}, \]
\[ \alpha_1 = \frac{1}{2} \sqrt{A\beta (\alpha - \gamma)}, \] m = \[ \sqrt{\frac{\alpha (\gamma - \beta)}{\beta (\gamma - \alpha)}}, \]
\[ \frac{\lambda}{k(r^2 - bk\lambda)} = -\frac{1}{2} A(\alpha \beta + \beta \gamma + \alpha \gamma) \] (30)

where \( \frac{(r^2 - bk\lambda)}{a} > 0, \xi = kx + ly - \lambda t. \)

**Family 4.** if \( \alpha < 0 \) and \( 0 < \beta < \gamma < \)

\[ \{ u_{41} = \pm \frac{1}{\sqrt{2}} \sqrt{\frac{2A(r^2 - bk\lambda)}{a} (\alpha + \beta + \gamma) \pm \sqrt{\frac{2A(r^2 - bk\lambda)}{a} \frac{2\alpha_1 (\gamma - \beta)\sin^2(\alpha_1 \xi) - \beta (\gamma - \alpha)}{\gamma - \alpha \cos^2(\alpha_1 \xi)}}}, \]
\[ \alpha_1 = \frac{1}{2} \sqrt{A\beta (\alpha - \gamma)}, \] m = \[ \sqrt{\frac{\alpha (\gamma - \beta)}{\beta (\gamma - \alpha)}}, \]
\[ \frac{\lambda}{k(r^2 - bk\lambda)} = -\frac{1}{2} A(\alpha \beta + \beta \gamma + \alpha \gamma) \] (33)

where \( \frac{(r^2 - bk\lambda)}{a} > 0, \xi = kx + ly - \lambda t. \)

**Family 5.** if \( \alpha < 0 \) and \( 0 < \beta < \gamma < \)

\[ \{ u_{51} = \pm \frac{1}{\sqrt{2}} \sqrt{\frac{2A(r^2 - bk\lambda)}{a} (\alpha + \beta + \gamma) \pm \sqrt{\frac{2A(r^2 - bk\lambda)}{a} \frac{2\alpha_1 (\gamma - \alpha)\sin(\alpha_1 \xi)\cos(\alpha_1 \xi)}{\sin(\alpha_1 \xi) [\beta - \gamma \cos^2(\alpha_1 \xi)]}}}, \]
\[ m = \sqrt{\frac{\alpha (\gamma - \beta)}{\beta (\gamma - \alpha)}}, \]
\[ \frac{\lambda}{k(r^2 - bk\lambda)} = -\frac{1}{2} A(\alpha \beta + \beta \gamma + \alpha \gamma) \] (36)

\[ \{ u_{52} = \pm \sqrt{\frac{2(r^2 - bk\lambda)}{a} \frac{2\alpha_1 (\gamma - \beta)\sin(\alpha_1 \xi)\cos(\alpha_1 \xi)}{\sin(\alpha_1 \xi) [\beta - \gamma \cos^2(\alpha_1 \xi)]}}}, \]
\[ \alpha_1 = \frac{1}{2} \sqrt{A\alpha (\gamma - \beta)}, \] m = \[ \sqrt{\frac{\alpha (\gamma - \beta)}{\beta (\gamma - \alpha)}}, \]
\[ \frac{\lambda}{k(r^2 - bk\lambda)} = -\frac{1}{2} A(\alpha \beta + \beta \gamma + \alpha \gamma) \] (37)

where \( \frac{(r^2 - bk\lambda)}{a} > 0, \xi = kx + ly - \lambda t. \)

**Remark 2** Up to now we have obtained some new formal solutions which cannot be obtained by Xu’s method, such as the solutions of (25)–(26), (28)–(29), (31)–(32), (34)–(35) and (37)–(38). When \( b_1 = c_1 = d_1 = 0 \), the form of the solutions of (24), (27), (30), (33) and (36) becomes the form which can be obtained by Xu’s method. This further shows our method is more general than Xu’s method.

**Remark 3** Some kinds of solutions derived by the generalized transformation are single soliton solutions and Jacobi elliptic doubly periodic wave solutions. There is much current interest in the formation of so-called “hot spots” or “blow up” of solutions. It speaks that these singular solutions will model this physical phenomena.
4 Conclusion and discussions

In summary, based on a more general ansatz than one presented by Xu [30], we have presented the extended method used to find more formal solutions of nonlinear evolution equations. As a result, some new exact travelling wave solutions of the ZK-MEW equation are obtained which may be useful for describing certain nonlinear physical phenomena. The method which we have proposed in this letter is standard, direct and computerized method, which allow us to do complicated and tedious algebraic calculation. It is shown that the algorithm can be also applied to other NLPDEs in mathematical physics. In addition, a natural problem is that whether the elliptic equation can be further extended to generate more types of solutions of NLPDEs in nonlinear science. Hence, the further study of the extended elliptic equation expansion method is needed.

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