Some Global Bifurcations in Piecewise Maps

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Abstract: A simple two-dimensional piecewise linear map is discussed. Main attention is paid to border-collision bifurcations and attractors. We present a theoretical framework for analyzing such a map by considering critical sets and we describe the creation of non-connected and multiply-connected basins.

Keywords: Piecewise linear maps; Border-collision bifurcation; basin bifurcations

1 Introduction

Many problems in engineering and applied science lead us to consider piecewise-linear maps [8]. It is known that for piecewise smooth dynamical systems we can in general distinguish between two types of bifurcations: one includes the bifurcations which occur in smooth dynamical systems (either local, related to the eigenvalues crossing the unit circle, or global, such as, for example homoclinic bifurcations), while the other is the called border-collision bifurcation [4]. This bifurcation occurs when a trajectory collides with one of the boundaries separating regions in which the function changes its definition. A simple type of border-collision bifurcation consists in the direct transition from one periodic orbit into another with the same period. However, more complicated phenomena are also possible, including period-multiplying bifurcations, multiple-choice bifurcations, and direct transition from periodicity to chaos [8, 9]. The term border-collision bifurcation was used for the first time in [5] and it is now widely used in this context, i.e. for piecewise smooth maps, although the study and description of such border-collision bifurcations began several years before those papers.

The present paper is the continuation of a former work presented in [6], where we study a behavior under iteration of a two-dimensional piecewise system with one line of discontinuity. In the present article we present in more detail mathematical properties. Especially, we examine bifurcation phenomena for 2D piecewise map and state explicitly which bifurcation does occur depending on the parameters. Effects of border-collision bifurcation are known, it enables us to explain some complex phenomena such as the appearance of chaotic attractor starting from a fixed point. We also find new and interesting results on the basin structure. Two basin bifurcations specific to the endomorphisms of the type \((Z_0 - Z_2)\) as “simply connected basin ↔ multiply connected basin” and “simply connected basin ↔ nonconnected basin” are evidenced for our invertible planar map.

In Aharonov et al. [3] the dynamics of a piecewise linear area preserving map of the plane \((x, y) \rightarrow (1 - y - |x|, x)\) was described in detail. This system contains an absolute value with

\[
J(f, g) = \left( \frac{\partial f}{\partial x} \right) \left( \frac{\partial g}{\partial y} \right) - \left( \frac{\partial f}{\partial y} \right) \left( \frac{\partial g}{\partial x} \right) = 1
\]

This conservative map is characterized by the island structure and exhibits chaotic behavior, but it does not contain any parameter.

We consider the piecewise linear map \(T : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) depending on three real parameters \(a, b, c\) given by

\[
T : (x, y) \mapsto \begin{cases} T_1(x, y), & \text{if } (x, y) \in R_1 \\ T_2(x, y), & \text{if } (x, y) \in R_2 \end{cases}
\]

where

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The map \( T \) is invertible, its inverse is given by
\[
T^{-1}(x', y') = \begin{cases} 
    T_1^{-1} : x = y', \ y = \frac{1 - ax' - by'}{b} \ , & \text{if } y' > 0 \\
    T_2^{-1} : x = y', \ y = \frac{1 + cx' + ay'}{b} \ , & \text{if } y' \leq 0
\end{cases}
\] (2)

The image of this line by \( T \) is called critical line of rank one, denoted as in the continuous case by \( LC \)
\[
LC = T(LC_{-1}) = \{(x, y) : y = 0\}
\]

It is the equivalent of the critical curve \( LC \) related to a two-dimensional continuous noninvertible map \[1\]. In the continuous case \( LC \) is characterized by two equivalent properties: \( LC \) is the set of points having two coincident rank-one preimages \[2\]. And then it locally separates two regions \( Z_i \) and \( Z_j \), the points of each one have a different number \( i \neq j \) of rank-one preimages. In the piecewise continuous case only this second property remains but with the following difference: crossing through \( LC \) rank-one preimages appear, or disappear by one unity not by pair as in continuous case \[1\].

Its image \( LC_i = T^i(LC) \), \( i = 1 \ldots \), which is a curve made up by a finite number of linear segments, is also called critical line of higher rank.

As state above, invertibility is controlled by the parameter \( b \). For \( b \neq 0 \) a point below the line \( LC \) has a unique rank-1 preimage by \( T_1^{-1} \) (given a point in \( R_1 \)), while a point above the line \( LC \) has a unique rank-1 preimage by \( T_2^{-1} \) (given a point in \( R_2 \)).

This paper is organized as follows: Section 2 describes some characteristics of the piecewise-linear map, their dependence on the parameters, and the stability of the fixed points. The qualitative behavior and bifurcations of this map are examined by using a qualitative theory and revisited bifurcation theory. In Section 3, bistability regime occurring in the parameter plane is discussed. In particular, we investigate the phenomenon of coexistence of fixed points at both sides of the border, and describe a mechanism of appearance of a sequence of border-collision bifurcations under specified parameter variations. The main results are given in Section 4. Furthermore, some global bifurcations responsible of creation of complex basins are considered. Finally, conclusion is presented in Section 5.

2 Existence and stability of fixed points

In this section, we consider basic concepts for piecewise linear maps and describe our settings on Eq. (1) in order to do an analytical investigation for border collision bifurcations. Since Eq. (1) consists of two affine-submaps, the phase plane is divided into two halves.

By simple computation, the map \( T \) has the following two fixed points:

1. \( P = \left( \frac{1}{1 + a + b}, \frac{1}{1 + a + b} \right) \) is a fixed point of \( T_1 \), exists for \( 1 + a + b > 0 \).
Figure 1: Triangle of stability of the fixed points

(ii) \( Q = \left( \frac{1 + c}{1 - a + b}, \frac{1 + c}{1 - a + b} \right) \) is a fixed point of \( T_2 \), exists for \( \frac{1 + c}{1 - a + b} \leq 0 \) and \( 1 - a + b \neq 0 \).

\( P \) is the fixed point of the map \( T \) if \( \frac{1}{1 + a + b} > 0 \), otherwise it is a so-called virtual fixed point which we denote by \( \bar{P} \).

Similarly, \( Q \) is the fixed point of \( T \) if \( \frac{1}{1 - a + b} \leq 0 \), otherwise it is a virtual fixed point denoted by \( \bar{Q} \). If the parameter \( c \) varies through \(-1\), the fixed point \( Q \) (actual or/and virtual) crosses the border line \( x = 0 \), so that the collision with it occurs at the value \( c = -1 \), at which \( Q \) merges with the origin \((0,0)\).

The stability of the fixed point is determined by the eigenvalues of the Jacobian matrix evaluated at this fixed point. From a straightforward computation of eigenvalues \( \lambda_{1,2} \) of the map \( T_1 \) for the fixed point \( P \), we have

\[
\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}
\]

We may derive conditions for having the triangle of stability of \( P \), denoted by \( S(P) \), and defined as follows:

\[
S(P) = \{(a, b) : 1 + b + a > 0, 1 - a + b > 0, 1 - b > 0\}
\]  

The stability of the fixed point \( Q \) is defined by the eigenvalues \( \mu_{1,2} \) of the Jacobian matrix of \( T_2 \) given by

\[
\mu_{1,2} = \frac{a \pm \sqrt{a^2 - 4b}}{2}
\]

and the triangle of stability \( S(Q) \) is defined as in (3). The stability region of the fixed points is shown in Fig. 1. Studying the map \( T \) numerically we get an interesting two-dimensional bifurcation diagram in \((a, b)\)-parameter plane for \( c = -1 \) (Fig. 2.a) and for \( c = 4 \) (Fig. 2.b). Different color zones indicate the existence of a stable cycle of given order, each color is attributed to an attractor of a certain order, the black zone (that corresponds to the color 15) indicates the existence of cycles of order superior to 14 or any other attractor unidentified such closed invariant curves and the chaotic attractors. This structure shows that the map \( T \) has stable fixed points in a region of the form of triangle.

Figure 3 presents a 1D bifurcation diagram in two cases: in the first case for parameter values \( a = 0.7, c = -1 \) and \( b \in [-0.27, -0.34] \), we observe a direct transition from a stable period-1 focus cycle to chaos via the homoclinic bifurcation. In the second case for parameter values \( a = 0.84, c = -1 \) and \( b \in [-0.4, -0.6] \), at which it is possible to observe a bifurcation sequence of cycles of different periods until chaos via border-collision bifurcations.

To describe a mechanism causing the occurrence of border collision bifurcations, we start our consideration by examining these results involving the interplay of dangerous border collision bifurcations, homoclinic bifurcations and multistability.

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3 Border collision bifurcation and multistability

In this section we explore the global dynamic behaviors of the system (1) when the values of the parameters are in the region of coexisting of several attractors (region of bistability). The different attractors give different possible long-run behaviors according to the starting condition of the system. In a situation of multistability we have a coexistence of several attractors, each one with its own basin of attraction.

In all our numerical explorations we will assume $a = -1.3$, $b = 0.92$, and we will vary $c$ ($-1.9 < c < -0.6$). Figure 4.a. shows the coexisting of the two fixed points and two attracting cycles: $P((1.61, 1.61) \in R_1)$ is a stable focus (its basin of attraction is shown in red) and $Q = (-0.27, -0.27) \in (R_2)$, $Q$ is a stable focus (its basin of attraction is shown in gray). The two attracting cycles are: an attracting cycle of period 6, and an attracting cycle of period 11 (these basins of attraction are shown respectively in green and blue clear). The color blue dark represents the whole of initial conditions which generate a divergent succession of points. Any point (initial condition of the system) chosen in the red zone (resp. gray) generates iterated sequences which converge towards the fixed points $P$ (resp. $Q$) ; on the other hand any point chosen in the green zone (resp. blue clear) generates iterated sequences which converge towards a cycle of period 6 (resp. cycle of period 11).

By fixing the values of the parameters $a$, $b$ and while varying the parameter $c$, we obtain a various changes of the structure of the basin of attraction, which also illustrate the existence of multistability.
For \( c = -1 \), as shown in Fig. 4.b, the fixed point \( Q = (0, 0) \) belongs to the critical curve \( LC_{-1} \), i.e. the fixed point has a contact with the border of a region of definition of the map. This is a case of border-collapse bifurcation. After this bifurcation the fixed point disappears and the structure of the basin of attraction changes, we have a coexistence of three basins of attraction: the basin of the fixed point \( P(1.61, 1.61) \in (R_1) \) (which remains the same because it does not depend on \( c \)) as well as the basins of cycles 6 and 11. From Fig. 5, we can also observe that another border-collapse bifurcation is emerging for \( c = -0.7 \), where a point of 11-cycle belongs to \( LC_{-1} \), after this bifurcation we have a disappearance of this cycle, only two attractors survive, a fixed point and a cycle of period 6. It is worth to note that the map here is invertible \( (b \neq 0) \), the total basin is simply connected, in spite of its structure which appears very complex because of the phenomenon of multistability and the existence of several attractors having different basins intermingled in a very complex way.

Figure 4: (a) Coexistence of the two fixed points \( P \) and \( Q \) with two attracting cycles of period 6 and 11. (b) the fixed point \( Q \) belongs to the critical curve \( LC_{-1} \).

Figure 5: (a) Border-collapse bifurcation when a point of 11-cycle belongs to \( LC_{-1} \). (b) The 11-cycle disappears and we have a coexistence of two basins.
4 Basin Bifurcations

By virtue of the interplay of contact bifurcation with heteroclinic and/or homoclinic bifurcations, we examine the transition to chaos, and basin bifurcations in such a map $T$ (invertible because $b \neq 0$) and we discuss the case when we are close of the conservative case ($|b| = 1$).

Let us take the values of the parameters in the green zone of parameter plane $(a, b)$ for $c = 4$ (see Fig. 2.b), where we have a 2-cycle. Indeed for $a = 1.28$, $b = -0.82$, $c = 4$, the phase portrait of the transformation $T$ in Fig. 6, shows the existence of only one attractor: a cycle of period 2, of type stable focus which points are represented in dark green, its immediate basin $D_0$ is simply connected of clear color green. The two fixed points here are saddles: $P = (4.46, 4.46) \in (R_1)$ with the eigenvalues $|S_1| = 0.93 < 1$, $|S_2| = 3.49 > 1$, and $Q = (\frac{5}{11}, \frac{5}{11}) \in (R_2)$ with the eigenvalues $|S_1| = 0.93 < 1$, $|S_2| = 1.74 > 1$. We show on the figure the stable and unstable manifolds of the saddle fixed point $Q$. By fixing the parameters $a$ and $c$ and while varying the parameter $b$, we shall see a route from simply connected to multiply connected and nonconnected basin, and can observe the transition from a stable fixed point to a chaotic attractor, via a heteroclinic bifurcation connecting a saddle fixed point with the points of 2-cycle. The immediate basin $D_0$ is simply connected as long as the set $D_0 \cap LC$ is connected. The contact bifurcation of the boundary of the immediate basin with the critical curve $LC$, which occurs as $b$ increases, will lead to a multiply connected immediate basin (as described in [1]). This can be seen in Fig. 6, at $b = -0.85$. The attracting set is still a 2-cycle, and its immediate basin of attraction $D_0$ is multiply connected, i.e. connected with holes (or lakes) $H_i$, as $D_0 \cap LC$ is nonconnected. The lakes $H_1^{j_1, i_1}, \ldots, H_k^{j_k, i_k}$ are preimages of $H_0$ by the two inverse determinations $T_{-1}^{i_1}, T_{-1}^{i_2}, \ldots, i_1, i_2, \ldots$ is connected. The contact bifurcation of the boundary of the immediate basin $D_0$ between the values $b = -0.82$ and $b = -0.85$. In this first analysis, we observe that the holes have a triangular form, forms that we do not find for the continuous maps.

In the same moment another bifurcation arises together with the first basin bifurcation and is of type “simply connected ↔ multiply connected” of the immediate basin $D_0$ between the values $b = -0.82$ and $b = -0.85$. In this first analysis, we observe that the holes have a triangular form, forms that we do not find for the continuous maps.

At $b = -0.89$ (Fig. 7.a) the chaotic attractor has a contact with its basin boundary. This value of $b$ is the value of contact bifurcation. Since for the chaotic attractor no longer exists, it disappears at $b = -0.9$ and then the basin of the 2-cycle is nonconnected (Fig. 7.b). By varying the parameter $b$, other heteroclinic bifurcations arise for $Q$ with the points of 2-cycle accompanied of appearance of other $k$-cycles (Fig. 8). Therefore, we have the appearance of an infinity of the centers for $b = -1$ (Fig. 9).
Figure 7: (a) Appearance of chaotic attractor by a heteroclinic bifurcation. (b) Contact bifurcation of the chaotic attractor with the boundary of its attraction basin.

Figure 8: Appearance of other cycles coexisting with the 2-cycle by varying the parameter $b$. 
5 Conclusion

In the present paper we have considered a two-dimensional piecewise linear map, with only one set of discontinuity. We focus in our analysis on the global bifurcations of attracting sets associated with coexistence phenomena of attractors and intricate structures of the basins of attraction. We show different border collision bifurcations causing transitions to chaos, some other types of basin bifurcations specific to endomorphism may happened in the piecewise case.

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