An Existence Theorem of Impulsive Differential Equations with Nonlocal Conditions

Shu Wen¹ *, Shaochun Ji²

¹ Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai’an, Jiangsu 223003, P. R. China

Abstract: This paper is concerned with the existence of mild solutions for a class of functional-differential equations with impulsive conditions and nonlocal conditions in Banach spaces. By introducing a new measure of noncompactness in the space of piecewise continuous functions, we generalize and extend some existing results in this area.

Key Words: Functional-differential equation; impulsive conditions; nonlocal conditions; mild solution; measure of noncompactness

1 Introduction

In this paper, we shall consider the existence of mild solutions of impulsive differential equations with nonlocal conditions:

\[ u'(t) = Au(t) + f(t, u(t)), \quad \text{a.e. } t \in [0, b], \quad t \neq t_i, \quad (1.1) \]
\[ \Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i)), \quad i = 1, \ldots, p, \quad (1.2) \]
\[ u(0) = g(u), \quad (1.3) \]

where \( A \) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e. \( C_0 \)-semigroup) \( \{T(t)\}_{t \geq 0} \) on a Banach space \( X \); \( f : [0, b] \times X \rightarrow X; 0 < t_1 < \cdots < t_s < t_{p+1} = b; I_i : X \rightarrow X, i = 1, \ldots, p \) are impulsive functions; \( g : PC([0, b]; X) \rightarrow X \).

The theory of impulsive differential and partial differential equations has become an important area of investigation because of its wide use in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulsive evolution equations. We refer readers to the monographs of Benchohra et al.[1], Lakshmikantham et al.[2], and the papers [3–7].

On the other hand, the study of abstract nonlocal semilinear initial value problem was first discussed by Byszewski [8] and the importance of the problem consists in the fact that it is more general and has better effect than the classical initial conditions. Aizicovici et al.[9, 10] discussed the case when \( A \) generates a compact semigroup, and the existence of integral solutions to the associated nonlocal problem is shown. Ntouyas et al.[11], Liang et al.[12], Fan et al.[13] study the case when operator semigroup \( T(t) \) is compact and \( f, g \) satisfy appropriate conditions such as compactness conditions and Lipschitz conditions. Cardinali et al.[14], Xue [15] and Dong et al.[16] studied some semilinear equations under the conditions in respect of the measure of noncompactness.

Since it is quite difficult to determine whether an operator semigroup is compact (see Pazy[17]), unlike Refs. [6, 10, 12], we do not assume that \( A \) generates a compact semigroup. It allow us to discuss some differential equations which contain a linear operator that generates a noncompact semigroup. We give a simple example.

Let \( X = L^2(\mathbb{R}^+ \cup \mathbb{R}^+) \). The ordinary differential operator \( A = d/dx \) with \( D(A) = H^1(-\infty, +\infty) \), generates a parallel-translation semigroup \( T(t) \) defined by \( T(t)u(s) = u(t + s) \), for every \( u \in X \). The \( C_0 \)-semigroup \( T(t) \) is not compact on \( X \).

The approach in this paper relies on Mönch’s fixed-point theorem and the measure of noncompactness. As we do not require the compactness of operator semigroup, our method is applicable to a wide class of (impulsive) differential equations in Banach spaces.

*Corresponding author. E-mail address: wenshu119@yeah.net
2 Preliminaries

Let \((X, \| \cdot \|)\) be a real Banach space. We denote by \(C([0, b]; X)\) the space of \(X\)-valued continuous functions on \([0, b]\) with the norm \(\| x \| = \sup \{ \| x(t) \|, t \in [0, b] \}\) and by \(L^1(0, b; X)\) the space of \(X\)-valued Bochner integrable functions on \([0, b]\) with the norm \(\| f \|_{L^1} = \int_0^b \| f(t) \| \, dt\). Let \(PC([0, b]; X) = \{ u : [0, b] \to X : u \text{ is continuous at } t \neq t_i \text{ and left continuous at } t = t_i, \text{ and the right limit } u(t_i^+) \text{ exists, } i = 1, \cdots, p \}\). It is easy to verify that \(PC([0, b]; X)\) is a Banach space with the norm \(\| u \|_{PC} = \sup \{ \| u(t) \|, t \in [0, b] \}\).

**Definition 2.1** Let \(E^+\) be the positive cone of an ordered Banach space \((E, \leq)\). A function \(\Phi\) defined on the set of all bounded subsets of the Banach space \(X\) with values in \(E^+\) is called a measure of noncompactness (MNC) on \(X\) if \(\Phi(\overline{\partial} \Omega) = \Phi(\Omega)\) for all bounded subsets \(\Omega \subset X\), where \(\overline{\partial} \Omega\) stands for the closed convex hull of \(\Omega\).

The MNC \(\Phi\) is said:
1. **monotone** if for all bounded subsets \(\Omega_1, \Omega_2\) of \(X\) we have: \((\Omega_1 \subset \Omega_2) \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2))\);
2. **nonsingular** if \(\Phi(\{ a \}) = \Phi(\Omega)\) for every \(a \in X, \Omega \subset X\);
3. **regular** if \(\Phi(\Omega) = 0\) if and only if \(\Omega\) is relatively compact in \(X\).

One of the most important examples of MNC is the noncompactness measure of Hausdorff \(\beta\) defined on each bounded subset \(\Omega\) of \(X\) by
\[
\beta(\Omega) = \inf \{ \varepsilon > 0 : \Omega \text{ has a finite }\varepsilon \text{-net in } X \}.
\]

It is well known that MNC \(\beta\) enjoys the above properties and other properties (see [18, 19]): for all bounded subset \(\Omega_1, \Omega_2\) of \(X\),
1. \(\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)\), where \(\Omega_1 + \Omega_2 = \{ x + y : x \in \Omega_1, y \in \Omega_2 \}\);
2. \(\beta(\Omega_1 \cup \Omega_2) \leq \max \{ \beta(\Omega_1), \beta(\Omega_2) \}\);
3. \(\beta(\lambda \Omega) \leq |\lambda| \beta(\Omega)\) for any \(\lambda \in \mathbb{R}\);
4. If the map \(Q : D(Q) \subseteq X \to Z\) is Lipschitz continuous with constant \(k\), then \(\beta_Z(Q \Omega) \leq k \beta(\Omega)\) for any bounded subset \(\Omega \subseteq D(Q)\), where \(Z\) is a Banach space.

**Definition 2.2** A function \(u(\cdot) \in PC([0, b]; X)\) is a mild solution of (1.1)-(1.3) if
\[
u(t) = T(t)g(u) + \int_0^t T(t - s)f(s, u(s)) \, ds + \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)),
\]
for all \(t \in [0, b]\).

**Definition 2.3** A countable set \(\{ f_n \}_{n=1}^{\infty} \subset L^1(0, b; X)\) is said to be semicompact if:
- the sequence \(\{ f_n(t) \}_{n=1}^{\infty}\) is relatively compact in \(X\) for a.a. \(t \in [0, b]\);
- there is a function \(\mu \in L^1([0, b]; \mathbb{R}^+)\) satisfying \(\sup_{n \geq 1} \| f_n(t) \| \leq \mu(t)\) for a.e. \(t \in [0, b]\).

**Definition 2.4** Let \(\{ T(t) \}_{t \geq 0}^\infty\) be a \(C_0\)-semigroup. We call the operator \(G : L^1(0, b; X) \to C([0, b]; X)\) defined by
\[
(Gf)(t) = \int_0^t T(t - s)f(s, u(s)) \, ds,
\]
(2.1)
as the Cauchy operator.

Now we give the following properties about Cauchy operator \(G\) (see Kamenskii [19] Theorem 4.2.2, Theorem 5.1.1, respectively).

**Lemma 2.1** (19) Let \(G\) be the Cauchy operator defined by (2.1), \(\{ f_n \}_{n=1}^{\infty}\) be a sequence of functions in \(L^1(0, b; X)\). Assume that there exist \(\mu, \eta \in L^1([0, b]; \mathbb{R}^+)\) satisfying
\[
\sup_{n \geq 1} \| f_n(t) \| \leq \mu(t) \text{ and } \beta(\{ f_n(t) \}_{n=1}^{\infty}) \leq \eta(t) \text{ a.e. } t \in [0, b].
\]
Then for all \(t \in [0, b]\), we have
\[
\beta((Gf_n(t))_{n=1}^{\infty}) \leq 2M \int_0^t \eta(s) \, ds,
\]
where \(M = \sup_{0 \leq t \leq b} \| T(t) \|\).
Lemma 2.2 ([19]) Let $G$ be the Cauchy operator defined by (2.1). If $\{f_n\}_{n=1}^{+\infty} \subset L^1([0,b];X)$ is semicompact, then the set $\{Gf_n\}_{n=1}^{+\infty}$ is relatively compact in $C([0,b];X)$ and moreover, if $f_n \to f_0$, then for all $t \in [0,b]$, $(Gf_n)(t) \to (Gf_0)(t)$ as $n \to +\infty$.

Let $t_{p+1} = b$, $J_0 = [0,t_1]$; $J_i = (t_{i}, t_{i+1}]$, $i = 1, \cdots, p$.

Lemma 2.3 ([20]) If $D \subseteq PC([0,b];X)$ is bounded, then $\beta(D(t)) \leq \beta(D)$ for all $t \in [0,b]$, where $D(t) = \{u(t) ; u \in W\} \subseteq X$. Furthermore, if $D$ is equicontinuous on each interval $J_i$ of $[0,b]$, $i = 1, 2, \cdots, p$, then $\beta(D) = \sup\{\beta(D(t))\}, t \in [0,b]$.

The following Mönch’s fixed-point theorem (see Theorem 2.2 in [21] and Theorem 4.16 in [22]) plays a key role in our proof.

Lemma 2.4 Let $D$ be a closed convex subset of a Banach space $X$ and $0 \in D$. Assume that $F : D \to D$ is a continuous map which satisfies Mönch’s condition, that is,

$$M \subseteq D \text{ is countable, } M \subseteq \sigma(\{0\} \bigcup F(M)) \Rightarrow \overline{M} \text{ is compact.}$$

Then there exists $x \in D$ with $x = F(x)$.

3 Main results

In this section, by using the technique of measure of noncompactness, we give the existence results for the impulsive problem (1.1)-(1.3) without the assumption of compactness on operator semigroup $T(t)$. Let $r$ be a finite positive constant and $W_r = \{u \in PC([0,b];X) : \|u(t)\| \leq r, \text{ for } t \in [0,b]\}$.

We give the following hypotheses:

$(H A)$ The linear operator $A : D(A) \subset X \to X$ generates a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$. And there exists a positive constant $M$ such that $M = \sup_{0 \leq t \leq b} \|T(t)\|$ (see Pazy [17]).

$(H g)$ $g : PC\{(0,b];X\} \to X$ is continuous and compact.

$(H I)$ $I_i : X \to X$ is continuous and compact for each $i = 1, 2, \cdots, p$.

$(H f_j)$ $f : [0,b] \times X \to X$, for a.e. $t \in [0,b]$, the function $f(t,\cdot) : X \to X$ is continuous and for all $x \in X$, the function $f(\cdot, x) : [0,b] \to X$ is measurable. Moreover, for any $r > 0$, there exists a function $\theta_r \in L^1(0,b;R^+)$ such that

$$\|f(t,u(t))\| \leq \theta_r(t),$$

for a.e. $t \in [0,b]$ and all $u \in W_r$.

$(H f_2)$ There exists a function $h \in L^1(0,b;R^+)$ such that for every bounded set $D \subset X$,

$$\beta(f(t,D)) \leq h(t)\beta(D),$$

for a.e. $t \in [0,b]$.

Theorem 3.1 Assume that the hypotheses $(H A)$, $(H g)$, $H I$, $H f_1$, $(H f_2)$ are satisfied, then the nonlocal impulsive problem (1.1)-(1.3) has at least one mild solution on $[0,b]$, provided that

$$M\left[ \sup_{u \in W_r} \|g(u)\| + \|\theta_r\|_{L^1} + \sum_{i=1}^{p} \|I_i(u(t_i))\| \right] \leq r.$$  \hspace{1cm} (3.2)

Remark 3.1 In Theorem 3.1 we assume $f$ and $I_i$ to satisfy a compactness condition, but not require any compactness restrictions on operator semigroup $T(t)$. Note that if $f$ is compact or Lipschitz continuous, then condition $(H f_2)$ is satisfied. Therefore, our work extends some previous results, where the compactness of $T(t)$ and $f$, or the Lipschitz continuity of $f$ are needed.

To prove the above theorem, we firstly introduce another measure of noncompactness (MNC) defined by

$$\Phi(\Omega) = \max_{E \in \Delta(\Omega)} \left(\alpha(E), \text{mod}_{C}(E)\right),$$  \hspace{1cm} (3.3)
for all bounded subsets \( \Omega \) of \( C([0, b]; X) \).

Where:
- \( \Delta(\Omega) \) stands for the set of countable subsets of \( \Omega \);
- \( \alpha \) is the real MNC defined by \( \alpha(E) = \sup_{t \in [0, b]} e^{-Lt}(E(t)) \), with \( E(t) = \{ x(t), x \in E \} \), \( L \) is a positive constant.
- \( \text{mod}_C(E) \) is the modulus of equicontinuity of the function set \( E \) given by the formula

\[
\text{mod}_C(E) = \lim_{\delta \to 0} \sup_{x \in E} \max_{t_1 - t_2 \leq \delta} \| x(t_1) - x(t_2) \|.
\]

\( \Phi(\Omega) \) is proved to be well-defined, monotone, nonsingular and regular (see [19] Example 2.1.4). This type of MNC is also used in Cardinali al.\([14]\), Fan et al.\([23]\) and Dong et al.\([16]\), where semilinear differential equations are discussed.

Now, we shall extend the MNC \( \Phi \) to the space of piecewise continuous functions. That is, we denote by \( \Phi^* \) the following measure of noncompactness in \( PC([0, b]; X) \) defined by

\[
\Phi^*(\Omega) = \max_{E \in \Delta(\Omega)} (\alpha(E), \text{mod}_C^*(E)),
\]

for all bounded subset \( \Omega \) of \( PC([0, b]; X) \). Here, \( \Delta(\Omega) \), \( \alpha \) are the same form with the above definition (3.3), \( L \) is a constant that we shall appropriately choose and

\[
\text{mod}_C^*(E) = \lim_{\delta \to 0} \sup_{\eta \in E} \max_{0 \leq p \leq p \in \mathbf{N}, t_1, t_2 \in J_i, |t_1 - t_2| \leq \delta} \max_{s, u \in \Omega} \| x(t_1) - x(t_2) \|,
\]

where \( J_0 = [0, t_1]; J_i = (t_i, t_{i+1}], i = 1, \cdots, p \). We can show that \( \Phi^* \) is well defined (i.e. there exists \( E_0 \in \Delta(\Omega) \) which achieve the maximum in (3.4)) and is a monotone, nonsingular and regular MNC. The proof is the same with Example 2.1.4 of [19] except for a slight and trivial modification.

**Proof of Theorem 3.1** Define a map \( Q : PC([0, b]; X) \to PC([0, b]; X) \) by \( (Q_u)(t) = (Q_1 u)(t) + (Q_2 u)(t) \) with

\[
(Q_1 u)(t) = T(t)g(u) + \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)),
\]

\[
(Q_2 u)(t) = \int_0^t T(t - s)f(s, u(s)) \, ds,
\]

for all \( t \in [0, b] \). It is easy to see that the fixed point of \( Q \) is the mild solution of nonlocal impulsive problem (1.1)-(1.3). Subsequently, we will prove that \( Q \) has a fixed point by using Mönch’s fixed-point theorem.

Step 1. The operator \( Q \) is continuous on \( PC([0, b]; X) \). For this purpose, we assume that \( u_n \to u \) in \( PC([0, b]; X) \). By hypothesis \((H_f_1)\) we have that

\[
f(s, u_n(s)) \to f(s, u(s)), \quad (n \to +\infty), \quad \text{for all } s \in [0, b].
\]

Then from the continuity of \( g, I_i \) and the dominated convergence theorem, we have

\[
\|Q u_n - Q u\|_{PC} \leq M\|g(u_n) - g(u)\| + M \int_0^b \|f(s, u_n(s)) - f(s, u(s))\| \, ds
\]

\[
+ \sum_{i=1}^p M\|I_i(u_n(t_i)) - I_i(u(t_i))\| \to 0, \quad \text{as } n \to +\infty,
\]

i.e., \( K \) is continuous.

Step 2. We claim that \( KW_r \subseteq W_r \). For any \( u \in K(W_r) \), we know that

\[
\|u(t)\| \leq M\|g(u)\| + \|\theta_t\|_{L^1} + \sum_{i=1}^p \|I_i(u(t_i))\|,
\]

for \( t \in [0, b] \). From (3.2), it follows that \( KW_r \subseteq W_r \).

Step 3. Mönch’s condition holds.

In this step we need the newly-defined MNC \( \Phi^* \). From Mönch’s condition, let \( D \subseteq B_{n_0} \) be countable and \( D \subseteq \overline{\cap(D)} \). From the regularity, it is enough to prove that \( \Phi^*(D) \equiv (0, 0) \).
Since $\Phi^*(Q(D))$ is a maximum, let $\{y_n\}_{n=1}^{\infty} \subseteq Q(D)$ be the countable set which achieves its maximum. Then there exists a set $\{u_n\}_{n=1}^{\infty} \subseteq D$ such that

$$y_n(t) = (Qu_n)(t) = (Q_1u_n)(t) + (Q_2u_n)(t)$$

(3.5)

for all $n \geq 1$, $t \in [0, b]$.

Now we give an estimation for $\alpha(\{y_n\}_{n=1}^{\infty})$. Noticing that $g(\cdot)$ and $I_1(\cdot)$ are compact operators, we get that

$$\beta((\{Q_1u_n(t)\}_{n=1}^{\infty}) \cup \alpha((\{Q_2u_n(t)\}_{n=1}^{\infty})) = 0,$$

(3.6)

for $t \in [0, b]$. From Lemma 2.1 and hypothesis $(H_f)$, we have that

$$\beta((\{Q_2u_n(t)\}_{n=1}^{\infty}) = \frac{\beta(\{\int_0^t T(t-s)f(s,u_n(s))ds\}_{n=1}^{\infty})}{\beta((\{Q_1u_n(t)\}_{n=1}^{\infty}))} \leq 2\int_0^t \beta(T(t-s)f(s,u_n(s)))ds$$

(3.7)

$$\leq 2M\int_0^t h(s)\beta((\{u_n(s)\}_{n=1}^{\infty}))ds$$

$$\leq 2M\int_0^t h(s)e^{Ls}\sup_{t \in [0,b]}(e^{-Lt}\beta((\{u_n(t)\}_{n=1}^{\infty})))ds$$

$$= 2M\int_0^t h(s)e^{Ls}ds \cdot \alpha((u_n)_{n=1}^{\infty}),$$

for $t \in [0, b]$.

Combining (3.6) and (3.7), we get that

$$\alpha((y_n)_{n=1}^{\infty}) = \sup_{t \in [0, b]} e^{-Lt}\beta((\{Q_1u_n(t)\}_{n=1}^{\infty}) \cup (\{Q_2u_n(t)\}_{n=1}^{\infty}))$$

$$\leq 2M\sup_{t \in [0, b]}\int_0^t h(s)e^{Ls}ds \cdot \alpha((u_n)_{n=1}^{\infty}))$$

$$= 2M\sup_{t \in [0, b]}\int_0^t h(s)e^{-Lt}ds \cdot \alpha((u_n)_{n=1}^{\infty}).$$

Let $q = 2M\sup_{t \in [0, b]}\int_0^t h(s)e^{-Lt}ds$. We can choose a proper constant $L > 0$ such that $q < 1$.

Therefore, from Mönch’s condition, we have that

$$\alpha((u_n)_{n=1}^{\infty}) \leq \alpha(D) \leq \alpha(G(D)) = \alpha((y_n)_{n=1}^{\infty}) \leq \alpha((u_n)_{n=1}^{\infty}))q.$$

As $q < 1$, it implies that

$$\alpha((u_n)_{n=1}^{\infty}) = \alpha(D) = \alpha((y_n)_{n=1}^{\infty}) = 0.$$

Coming back to the definition of $\alpha$, we can see that

$$\beta((u_n(t))_{n=1}^{\infty}) = \beta((y_n(t))_{n=1}^{\infty}) = 0,$$

(3.8)

for every $t \in [0, b]$.

From (3.8), we get that

$$\beta((f(t, u_n(t)))_{n=1}^{\infty}) \leq h(t)\beta((u_n(t))_{n=1}^{\infty})ds = 0,$$

i.e., $(f(t, u_n(t)))_{n=1}^{\infty}$ is relatively compact for a.a. $t \in [0, b]$ in $X$. Moreover, from the fact $(u_n)_{n=1}^{\infty} \subseteq W_r$, by $(H_f)$, it is easy to check that $(f(t, u_n(t)))_{n=1}^{\infty}$ is uniformly integrable for a.e. $t \in [0, b]$. So $\{f_n\}_{n=1}^{\infty} = \{f(\cdot, u_n(\cdot))\}_{n=1}^{\infty}$ is semicompact according to Definition 2.3. By applying Lemma 2.2, we have that set $\{G(f_n)\}_{n=1}^{\infty}$ is relatively compact in $C([0, b]; X)$, $G$ is the Cauchy operator. So is the set $Q_1((u_n)_{n=1}^{\infty})$.

On the other hand, by the strong continuity of $T(t)$ and the compactness of $g, I_1$, we can show that the set $Q_1((u_n)_{n=1}^{\infty})$
is relatively compact. In fact, for \( t_i \leq t < t + h \leq t_{i+1}, i = 1, 2, \ldots, p \), we have, using the semigroup properties,

\[
\| \sum_{0 < t_j < t + h} T(t + h - t_j)I_j(u_n(t_j)) - \sum_{0 < t_j < t} T(t + h - t_j)I_j(u_n(t_j)) \|
\leq \| \sum_{0 < t_j < t + h} T(t + h - t_j)I_j(u_n(t_j)) - \sum_{0 < t_j < t} T(t + h - t_j)I_j(u_n(t_j)) \|
+ \sum_{0 < t_j < t} T(t - t_j)I_j(u_n(t_j)) - \sum_{0 < t_j < t} T(t - t_j)I_j(u_n(t_j))
\leq \| \sum_{0 < t_j < t} T(t + h - t_j)I_j(u_n(t_j)) - \sum_{0 < t_j < t} T(t - t_j)I_j(u_n(t_j)) \|
+ \sum_{0 < t_j < t} \| T(h - I_j)(u_n(t_j)) \|\]

which follows that \( \{ \sum_{0 < t_j < t} T(t - t_j)I_j(u_n(t_j)) \}_{n=1}^{\infty} \) is equicontinuous on each \( I_j \) due to the compactness of \( I_i \) and the strong continuity of \( T(t) \).

Since \( I_i, i = 1, 2, \ldots, p \), are compact, we have that

\[
\beta(\{ \sum_{0 < t_j < t} T(t - t_j)I_j(u_n(t_i)) \}_{n=1}^{\infty}) = 0,
\]

for \( t \in [0, b] \). By Lemma 2.3, we find that

\[
\beta(\{ \sum_{0 < t_j < t} T(t - t_j)I_j(u_n(t_j)) \}_{n=1}^{\infty}) = \sup_{t \in [0, b]} \beta(\{ T(t - t_j)I_j(u_n(t_j)) \}_{n=1}^{\infty}) = 0,
\]

which implies that \( \{ \sum_{0 < t_j < t} T(t - t_j)I_j(u_n(t_j)) \}_{n=1}^{\infty} \) is relatively compact in \( PC([0, b]; X) \). Then the same idea can be used to prove the compactness of set \( \{ T(t)g(u_n) \}_{n=1}^{\infty} \) in \( C([0, b]; X) \) due to the famous Ascoli-Arzela Theorem. So the set \( Q_1(\{ u_n \}_{n=1}^{\infty}) \) is relatively compact.

Then the representation of \( y_n \) given by (3.5) yields that the set \( \{ y_n \}_{n=1}^{\infty} \) is also relatively compact in \( PC([0, b]; X) \). Since \( \Phi^* \) is a monotone, nonsingular, regular MNC, we, from Mönch’s condition, have that

\[
\Phi^*(D) \leq \Phi^*(\overline{\cap}_{n=1}^{\infty} Q(D)) = \Phi^*(\{ y_n \}_{n=1}^{\infty}) = (0, 0).
\]

Therefore, \( D \) is relatively compact in \( PC([0, b]; X) \).

Finally, due to Lemma 2.4, \( Q \) has at least a fixed point, which is just the mild solution of nonlocal impulsive problem (1.1)–(1.3). This completes the proof. \( \square \)

**Remark 3.2** The key to the proof of Theorem 3.1 lies in managing to introduce a new MNC \( \Phi^* \) in the space of piecewise continuous functions, which enables us to get rid of the compactness of operator semigroup \( T(t) \) and \( f \) by using the property of semicompact set. Moreover, since we no longer require that \( f \) is Lipschitz continuous or compact, our result is the generalization and extension in this area, see [12, 13, 20].

**Corollary 3.1** Assume that hypotheses \((HA), (Hg), (HI), Hf_1, (Hf_2)\) hold true for each \( r > 0 \). If

\[
\frac{\| g(u) \|}{\| u \|_{PC}} \to 0, \quad \| u \|_{PC} \to \infty,
\]

\[
\frac{\| \theta_r \|_{L^1}}{r} \to 0, \quad r \to \infty,
\]

\[
\sum_{s=1}^{p} \frac{\| I_s(x) \|}{\| x \|} \to 0, \quad \| x \| \to \infty,
\]

then the nonlocal impulsive problem (1.1)–(1.3) has at least one mild solution in \( PC([0, b]; X) \).
Acknowledgements

Research is supported by Foundation of Huaiyin Institute of Technology (HGB1004) and the second author is also supported by the the Postgraduate Innovation Project of Jiangsu Province (No. CXZZ12-0890).

References