Solitons and Periodic Wave Solutions for Coupled Nonlinear Equations

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Abstract: In this work we apply the tanh-coth method and the tan-cot method to study some nonlinear coupled equations. Four nonlinear coupled equations that appear in a variety of scientific applications are investigated. We derive soliton, singular solitons and periodic wave solutions for these coupled equations. The obtained results show that these four coupled equations reveal richness of explicit soliton and periodic solutions.

Keywords: Nonlinear coupled equation; tanh method; solitons; periodic solutions

1 Introduction

Many scientific fields such as plasma physics, nonlinear optics, fluid dynamics, solid state physics, and others give rise to coupled nonlinear evolution equations. The studies of these coupled equations have gained significant attention in searching for solitary and travelling wave solutions. A variety of powerful methods [1–10] have been developed to study these equations and to examine the physical properties of the obtained solutions. The inverse scattering method, Hirota’s bilinear method, the Hereman-Nuseir method, the tanh method [1–4, 11–15], the sine-cosine method [11], homogeneous balance method, and many others are examples of the methods used in the literature. The studies are always aim for closed form analytical solutions that may lead to insights through the physical structures of these solutions and to new developments. Several kinds of solutions are usually obtained, such as dark or bright soliton solutions, cnoidal waves, peakons, cuspons, compactons, stumpons, negatons, positions, periodic wave solutions, and many others.

The nonlinear phenomena appeared in a variety of science and engineering applications, such as plasma physics, fluid dynamics, elastic media, condensed matter physics, chemical physics, and many other applications. Mathematical modelling of many nonlinear phenomena leads to nonlinear evolution equations, integrable or non-integrable, and coupled nonlinear evolution equations as well. The study of coupled nonlinear evolution equations has gained importance and attention during the past few decades, due to its applications in numerous fields of mathematical physics and engineering. The coupled nonlinear evolution equations have a rich structure which merits further study to understand the equations fully.

In this work, we plan to study four coupled nonlinear equations that appear in a variety of science and engineering applications. The first two coupled equations are the coupled KdV equation and the coupled Boussinesq equation that read

\[ u_t + 6uu_x - 6vv_x + u_{xxx} = 0, \]
\[ v_t + 3uv_x + v_{xxx} = 0, \]  

and

\[ u_t + uu_x + v_x + \lambda u_{xxt} = 0, \]
\[ v_t + (uv)_x + \mu u_{xxx} = 0, \]  

respectively. The coupled equation (1) is also called the Hirots-Satsuma system. The coupled Boussinesq equation (2) describes the bi-directional wave propagation in various contexts and other phenomena.

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Moreover, we aim to study the generalized Hirota-Satsuma system given by
\begin{align*}
u_t + \frac{3}{2}(uv)_x + \frac{3}{2}(vw)_x + u_{xxx} &= 0, \\
v_t + 3(wu)_x + v_{xxx} &= 0, \\
w_t + 3(uv)_x + w_{xxx} &= 0.
\end{align*}
(3)

The final goal is to study the generalized Hirota-Satsuma-KP equation that reads
\begin{align*}
u_{tt} + \frac{3}{2}(uv)_x + \frac{9}{2}vw_x + u_{xxx} &= 0, \\
v_{tt} + 3(wu)_x + v_{xxx} &= 0, \\
w_{tt} + 3(uv)_x + w_{xxx} &= 0.
\end{align*}
(4)

It is obvious that Eq. (4) is derived from (3) by using the KP sense, where the restriction that the waves be strictly one dimensional, namely the $x$-direction of the generalized Hirota-Stasuma system (3) is relaxed.

In the literature, there is a variety of useful methods that are used to handle coupled nonlinear equations. A variety of solitary wave and periodic wave ansatze are used recently by many authors. In this work, we aim to investigate the four coupled nonlinear equations (1)–(4) by using the tanh method, the coth method, the tan method, and the cot method to determine soliton solutions, singular soliton solutions and periodic wave solutions.

2 The coupled nonlinear KdV equation

We first unite the independent variables $x$ and $t$ into one wave variable
\[ \xi = kx - ct, \]
(5)
to carry out
\begin{align*}
u_t + 6uu_x - 6vv_x + u_{xxx} &= 0, \\
v_t + 3wu_x + v_{xxx} &= 0,
\end{align*}
(6)
into a system of ODEs
\begin{align*}
-6uu' + 6kuu' - 6kvv' + k^3v''' &= 0, \\
-6vv' + 3kwv' + k^3v''' &= 0,
\end{align*}
(7)
(8)
In what follows we apply the proposed schemes to study (7)–(8).

2.1 The tanh method

In this section we will apply the tanh method [1–4, 11–15] and the coth method [11] independently. The tanh technique is based on the a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function. We then introduce a new independent variable
\[ Y = \tanh(\xi). \]
(9)
The solutions can be proposed as a finite power series in $Y$ in the form
\begin{align*}
u(\xi) &= \sum_{k=0}^{M_1} a_k Y^k, \\
v(\xi) &= \sum_{k=0}^{M_1} b_k Y^k,
\end{align*}
(10)
limiting them to solitary and shock wave profiles. The parameter $M_1$ is mostly a positive integer, that will be determined by using the balance method. If $M_1$ is not an integer, a transformation formula is usually used. Substituting (10) into the simplified ODE (6) gives an equation in powers of $Y$.

To determine $M_1$, we balance $uu'$ with $u'''$ in (7) gives $M_1 = 2$. Similarly we balance $uv'$ with $v'''$ in (8) to find that $M_1 = 2$. The tanh method suggests the use of the finite expansions
\begin{align*}
u(\xi) &= a_0 + a_1 Y + a_2 Y^2, \\
v(\xi) &= b_0 + b_1 Y + b_2 Y^2,
\end{align*}
(11)
where $a_i, b_i, 0 \leq i \leq 2$, are constants that will be determined.
Substituting (11) into (7) and (8), collecting the coefficients of each power of $Y$, and using any symbolic computation program such as Mathematica or Maple we obtain the following two sets of solutions

$$a_0 = \frac{c+2k^3}{3k}, a_1 = 0, a_2 = -2k^2$$

$$b_0 = 0, b_1 = \pm \frac{\sqrt{6k(4k^3+c)}}{3}, b_2 = 0,$$ (12)

and

$$a_0 = \frac{c+8k^3}{6k}, a_1 = 0, a_2 = -4k^2$$

$$b_0 = \pm \frac{\sqrt{2(c+8k^3)}}{6k}, b_1 = 0, b_2 = \pm 2\sqrt{2}k^2.$$ (13)

In view of these results, and using (11) we obtain two sets of soliton solutions, given by

$$u(x, t) = \frac{c+2k^3}{3k} - 2k^2 \tanh^2 (kx - ct),$$

$$v(x, t) = \pm \frac{\sqrt{6k(4k^3+c)}}{3} \tanh (kx - ct),$$ (14)

and

$$u(x, t) = \frac{c+8k^3}{6k} - 4k^2 \tanh^2 (kx - ct),$$

$$v(x, t) = \pm \frac{\sqrt{2(c+8k^3)}}{6k} \pm 2\sqrt{2}k^2 \tanh^2 (kx - ct).$$ (15)

### 2.2 The coth method

It is interesting to point out that if we use

$$u(\xi) = a_0 + a_1Y + a_2Y^2,$$

$$v(\xi) = b_0 + b_1Y + b_2Y^2,$$ (16)

where, in this case, $Y = \coth(\xi)$. Proceeding as before, obtain the singular soliton solutions

$$u(x, t) = \frac{c+2k^3}{3k} - 2k^2 \coth^2 (kx - ct),$$

$$v(x, t) = \pm \frac{\sqrt{6k(4k^3+c)}}{3} \coth (kx - ct),$$ (17)

and

$$u(x, t) = \frac{c+8k^3}{6k} - 4k^2 \coth^2 (kx - ct),$$

$$v(x, t) = \pm \frac{\sqrt{2(c+8k^3)}}{6k} \pm 2\sqrt{2}k^2 \coth^2 (kx - ct).$$ (18)

### 2.3 The tan method

We will apply now the tan method and the cot method [11] independently. The tan technique is based on the a priori assumption that the traveling wave solutions can be expressed in terms of the tan function similar to the tanh method. We then introduce a new independent variable

$$W = \tan(\xi).$$ (19)

The solutions can be proposed as a finite power series in $W$ in the form

$$u(\xi) = \sum_{k=0}^{M} a_k W^k.$$ (20)

The parameter $M$ is determined by using the balance technique that was used earlier. Proceeding as before, the tan method suggests the use of the finite expansion

$$u(\xi) = a_0 + a_1W + a_2W^2,$$

$$v(\xi) = b_0 + b_1W + b_2W^2.$$ (21)

Substituting (21) into (7) and (8), collecting the coefficients of each power of $W$, and proceeding as before we obtain the following two sets of solutions

$$a_0 = \frac{c-2k^3}{4k}, a_1 = 0, a_2 = -2k^2$$

$$b_0 = 0, b_1 = \pm \frac{\sqrt{-6k(4k^3+c)}}{3}, b_2 = 0, k(4k^3+c) < 0,$$ (22)
and

\[ a_0 = \frac{e^{-8k^3}}{6k}, a_1 = 0, a_2 = -4k^2 \]
\[ b_0 = \pm \sqrt{2} \frac{(e^{-8k^3})}{6k}, b_1 = 0, b_2 = \pm 2\sqrt{2} k^2. \]

(23)

In view of these results we obtain two sets of periodic solutions, given by

\[ u(x, t) = \frac{e^{-2k^3}}{6k} - 2k^2 \tan^2 (kx - ct), \]
\[ v(x, t) = \pm \sqrt{\frac{-6k}{3}} \frac{(4k^3 + c)}{6k} \tan (kx - ct), \]

and

\[ u(x, t) = \frac{e^{-8k^3}}{6k} - 4k^2 \tan^2 (kx - ct), \]
\[ v(x, t) = \pm \sqrt{2} \frac{(c - 8k^3)}{6k} \pm 2\sqrt{2} k^2 \tan^2 (kx - ct). \]

(24)

\[ (25) \]

2.4 The cot method

Proceeding as in the tan method, obtain the singular periodic solutions

\[ u(x, t) = \frac{e^{-2k^3}}{6k} - 2k^2 \cot^2 (kx - ct), \]
\[ v(x, t) = \pm \sqrt{\frac{-6k}{3}} \frac{(4k^3 + c)}{6k} \cot (kx - ct), \]

and

\[ u(x, t) = \frac{e^{-8k^3}}{6k} - 4k^2 \cot^2 (kx - ct), \]
\[ v(x, t) = \pm \sqrt{2} \frac{(c - 8k^3)}{6k} \pm 2\sqrt{2} k^2 \cot^2 (kx - ct). \]

(26)

\[ (27) \]

3 The coupled nonlinear Boussinesq equation

Using the wave variable

\[ \xi = kx - ct, \]

(28)

carries out the coupled nonlinear Boussinesq equation

\[ u_t + uu_x + u_x + \lambda u_{xxt} = 0, \]
\[ v_t + (uv)_x + \mu u_{xxx} = 0, \]

(29)

into a system of ODEs

\[ -cu' + kuu' + kv' - \lambda k^2 cv''' = 0, \]
\[ -cv' + k(uv)' + \mu k^3 u''' = 0. \]

(30)

\[ (31) \]

We will follow the approach presented earlier to study (29).

3.1 The tanh method

We now introduce a new independent variable

\[ Y = \tanh(\xi), \]

(32)

where the solutions can be proposed as a finite power series in \( Y \) in the form

\[ u(\xi) = \sum_{k=0}^{M} a_k Y^k, \]
\[ v(\xi) = \sum_{k=0}^{M_1} b_k Y^k, \]

(33)

limiting them to solitary and shock wave profiles.

To determine \( M \) and \( M_1 \), we balance \( u' \) with \( v''' \) in (30) and \( (uv)' \) with \( u''' \) in (31) to find that \( M = M_1 = 2 \). The tanh method suggests the use of the finite expansions

\[ u(\xi) = a_0 + a_1 Y + a_2 Y^2, \]
\[ v(\xi) = b_0 + b_1 Y + b_2 Y^2, \]

(34)

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where $a_i, b_i, 0 \leq i \leq 2$, are constants that will be determined.

Substituting (34) into (30) and (31), collecting the coefficients of each power of $Y$, we obtain the following set

\[
\begin{align*}
    a_0 &= \frac{2c^2(1-8\lambda k^2)+\mu k^2}{2\lambda c k}, \\
    a_1 &= 0, \\
    a_2 &= 12\lambda ck.
\end{align*}
\]

(35)

This in turn gives the soliton solutions

\[
\begin{align*}
    u(x, t) &= \frac{2c^2(1-8\lambda k^2)+\mu k^2}{2\lambda c k} + 12\lambda ck \tanh^2 (kx - ct), \\
    v(x, t) &= \frac{\mu k^2(\mu+16\lambda^2 c^2)}{4\lambda^2 c^2} - 6\mu k^2 \tanh^2 (kx - ct).
\end{align*}
\]

(36)

3.2 The coth method

In a like manner to the tanh method, we can use

\[
\begin{align*}
    u(\xi) &= a_0 + a_1 Y + a_2 Y^2, \\
    v(\xi) &= b_0 + b_1 Y + b_2 Y^2,
\end{align*}
\]

(37)

where, in this case, $Y = \coth(\xi)$. Proceeding as before, we obtain the singular soliton solutions

\[
\begin{align*}
    u(x, t) &= \frac{2c^2(1-8\lambda k^2)+\mu k^2}{2\lambda c k} + 12\lambda ck \coth^2 (kx - ct), \\
    v(x, t) &= \frac{\mu k^2(\mu+16\lambda^2 c^2)}{4\lambda^2 c^2} - 6\mu k^2 \coth^2 (kx - ct).
\end{align*}
\]

(38)

3.3 The tan method

We will apply now the tan method and the cot method independently. The tan technique is based on the a priori assumption that the traveling wave solutions can be expressed in terms of the tan function similar to the tanh method. We then introduce a new independent variable

\[ W = \tan(\xi). \]

(39)

The solutions can be proposed as a finite power series in $W$ in the form

\[
\begin{align*}
    u(\xi) &= \sum_{k=0}^{M} a_k W^k, \\
    v(\xi) &= \sum_{k=0}^{M_1} a_k W^k,
\end{align*}
\]

(40)

where $W = \tan(\xi)$. We can easily show that $M = M_1 = 2$. We therefore can set

\[
\begin{align*}
    u(\xi) &= a_0 + a_1 W + a_2 W^2, \\
    v(\xi) &= b_0 + b_1 W + b_2 W^2.
\end{align*}
\]

(41)

Proceeding as before we find

\[
\begin{align*}
    a_0 &= \frac{2c^2(1+8\lambda k^2)+\mu k^2}{2\lambda c k}, \\
    a_1 &= 0, \\
    a_2 &= 12\lambda ck.
\end{align*}
\]

(42)

Consequently, the periodic solutions

\[
\begin{align*}
    u(x, t) &= \frac{2c^2(1+8\lambda k^2)+\mu k^2}{2\lambda c k} + 12\lambda ck \tan^2 (kx - ct), \\
    v(x, t) &= \frac{\mu k^2(\mu-16\lambda^2 c^2)}{4\lambda^2 c^2} - 6\mu k^2 \tan^2 (kx - ct),
\end{align*}
\]

(43)

are readily obtained.

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3.4 The cot method

Proceeding as in the tan method, obtain the singular periodic solutions

\[ u(x,t) = \frac{c-2k^3}{\frac{1}{6k} + ct} \cot (kx - ct), \]
\[ v(x,t) = \pm \sqrt{\frac{1}{4k^3}} \cot (kx - ct), \]

(44)

and

\[ u(x,t) = \frac{2c^2\lambda(1+8k^2)+\mu k^2}{2\lambda k} + 12\lambda ck \cot (kx - ct), \]
\[ v(x,t) = \frac{\mu k^2(\nu-16\lambda^2c^2)}{4k^2c^2} - 6\mu k^2 \cot (kx - ct). \]

(45)

4 The generalized Hirota-Satsuma equation

Using the wave variable

\[ \xi = kx - ct, \]

(46)

carries out the generalized Hirota-Satsuma equaion

\[ u_t + \frac{3}{2}(uv)_x + \frac{3}{2}(v)w_x + u_{xxx} = 0, \]
\[ v_t + \frac{3}{2}(wu)_x + v_{xxx} = 0, \]
\[ w_t + \frac{3}{2}(uv)_x + w_{xxx} = 0, \]

(47)

into a system of ODEs

\[ -cu' + \frac{3}{2}k(uv)' + \frac{3}{2}k(vw)' + k^3u''' = 0, \]
\[ -cu' + 3k(wu)' + k^3v''' = 0, \]
\[ -cw' + 3k(uv)' + k^3w''' = 0. \]

(48)\( (49)\)\( (50)\)

In what follows we employ the same schemes used earlier to study (47).

4.1 The tanh method–the coth method

Proceeding as before, we find \( M = M_4 = 2. \) We apply the tanh method and the coth method independently. The tanh method, or the coth method, suggests the use of the finite expansions

\[ u(\xi) = a_0 + a_1 Y + a_2 Y^2, \]
\[ v(\xi) = b_0 + b_1 Y + b_2 Y^2, \]
\[ w(\xi) = c_0 + c_1 Y + c_2 Y^2, \]

(51)

where \( a_i, b_i, c_i, 0 \leq i \leq 2, \) are constants that will be determined.

Substituting (51) into (48) -(50), collecting the coefficients of each power of \( Y, \) we obtain the following four sets of solutions:

(i) The first set is given by

\[ a_0 = \frac{c+8k^3}{6k}, a_1 = 0, a_2 = -2k^2, \]
\[ b_0 = \frac{c+8k^3}{6k}, b_1 = 0, b_2 = -2k^2, \]
\[ c_0 = \frac{c+8k^3}{6k}, c_1 = 0, c_2 = -2k^2, \]

(52)

Consequently, we obtain the soliton solutions

\[ u(x,t) = v(x,t) = w(x,t), \]
\[ u(x,t) = \frac{c+8k^3}{6k} - 2k^2 \tanh^2 (kx - ct), \]
\[ u(x,t) = \frac{c+8k^3}{6k} - 2k^2 \coth^2 (kx - ct), \]

(53)\( (54)\)\( (55)\)

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(ii) The second set is given by

\[
\begin{align*}
  a_0 &= \frac{c+8k^3}{6k}, \quad a_1 = 0, \quad a_2 = 2k^2, \\
  b_0 &= \frac{c+8k^3}{6k}, \quad b_1 = 0, \quad b_2 = -2k^2, \\
  c_0 &= -\frac{c+8k^3}{6k}, \quad c_1 = 0, \quad c_2 = 2k^2,
\end{align*}
\]

(56)

In view of these results, and using (51) we obtain the soliton solutions

\[
\begin{align*}
  u(x,t) &= w(x,t) = -v(x,t),
\end{align*}
\]

(57)

where

\[
\begin{align*}
  u(x,t) &= \frac{c + 8k^3}{6k} - 2k^2 \tanh^2 (kx - ct), \\
  v(x,t) &= -\frac{c + 8k^3}{6k} - 2k^2 \coth^2 (kx - ct),
\end{align*}
\]

(58, 59)

(iii) The third set is given by

\[
\begin{align*}
  a_0 &= -\frac{c+8k^3}{6k}, \quad a_1 = 0, \quad a_2 = 2k^2, \\
  b_0 &= -\frac{c+8k^3}{6k}, \quad b_1 = 0, \quad b_2 = 4k^2, \\
  c_0 &= -\frac{c+8k^3}{4k}, \quad c_1 = 0, \quad c_2 = -4k^2,
\end{align*}
\]

(60)

Consequently, we obtain the soliton solutions

\[
\begin{align*}
  u(x,t) &= \frac{1}{2} v(x,t) = -\frac{1}{2} w(x,t),
\end{align*}
\]

(61)

where

\[
\begin{align*}
  u(x,t) &= -\frac{c + 8k^3}{6k} - 2k^2 \tanh^2 (kx - ct), \\
  v(x,t) &= -\frac{c + 8k^3}{6k} - 2k^2 \coth^2 (kx - ct),
\end{align*}
\]

(62, 63)

(iv) The fourth set is given by

\[
\begin{align*}
  a_0 &= \frac{c+8k^3}{6k}, \quad a_1 = 0, \quad a_2 = -2k^2, \\
  b_0 &= \frac{c+8k^3}{6k}, \quad b_1 = 0, \quad b_2 = 4k^2, \\
  c_0 &= \frac{c+8k^3}{3k}, \quad c_1 = 0, \quad c_2 = 4k^2,
\end{align*}
\]

(64)

Consequently, we obtain the soliton solutions

\[
\begin{align*}
  u(x,t) &= -\frac{1}{2} v(x,t) = -\frac{1}{2} w(x,t),
\end{align*}
\]

(65)

where

\[
\begin{align*}
  u(x,t) &= \frac{c + 8k^3}{6k} - 2k^2 \tanh^2 (kx - ct), \\
  v(x,t) &= \frac{c + 8k^3}{6k} - 2k^2 \coth^2 (kx - ct),
\end{align*}
\]

(66, 67)

4.2 The tan method and the cot method

Proceeding as before, we find \( M = M_1 = 2 \). We apply the tan method and the cot method independently. The tan method, or the cot method, suggests the use of the finite expansions

\[
\begin{align*}
  u(\xi) &= a_0 + a_1 Y + a_2 Y^2, \\
  v(\xi) &= b_0 + b_1 Y + b_2 Y^2, \\
  w(\xi) &= c_0 + c_1 Y + c_2 Y^2,
\end{align*}
\]

(68)

where \( a_i, b_i, c_i, 0 \leq i \leq 2 \), are constants that will be determined.
Substituting (68) into (48) – (50), collecting the coefficients of each power of \( Y \), we obtain the following four sets of solutions:

(i) The first set is given by
\[
\begin{align*}
a_0 &= e^{\frac{8k^3}{6k}}, a_1 = 0, a_2 = -2k^2, \\
b_0 &= e^{\frac{8k^3}{6k}}, b_1 = 0, b_2 = -2k^2, \\
c_0 &= e^{\frac{8k^3}{6k}}, c_1 = 0, c_2 = -2k^2,
\end{align*}
\]

Consequently, we obtain
\[
u(x, t) = w(x, t) = w(x, t),
\]

where the periodic solution is given by
\[
u(x, t) = \frac{c - 8k^3}{6k} - 2k^2 \tan^2 (kx - ct),
\]

and the singular solution is
\[
u(x, t) = \frac{c - 8k^3}{6k} - 2k^2 \cot^2 (kx - ct),
\]

(ii) The second set is given by
\[
\begin{align*}
a_0 &= e^{\frac{8k^3}{6k}}, a_1 = 0, a_2 = 2k^2, \\
b_0 &= e^{\frac{8k^3}{6k}}, b_1 = 0, b_2 = -2k^2, \\
c_0 &= e^{\frac{8k^3}{6k}}, c_1 = 0, c_2 = 2k^2,
\end{align*}
\]

In view of these results, and using (51) we obtain
\[
u(x, t) = w(x, t) = v(x, t),
\]

where the periodic solution is given by
\[
u(x, t) = \frac{c + 8k^3}{6k} - 2k^2 \tan^2 (kx - ct),
\]

and the singular solution is
\[
u(x, t) = \frac{c + 8k^3}{6k} - 2k^2 \cot^2 (kx - ct),
\]

(iii) The third set is given by
\[
\begin{align*}
a_0 &= e^{\frac{8k^3}{6k}}, a_1 = 0, a_2 = 2k^2, \\
b_0 &= e^{\frac{8k^3}{6k}}, b_1 = 0, b_2 = 4k^2, \\
c_0 &= e^{\frac{8k^3}{3k}}, c_1 = 0, c_2 = -4k^2,
\end{align*}
\]

Consequently, we obtain
\[
u(x, t) = \frac{1}{2} w(x, t) = \frac{1}{2} v(x, t),
\]

where the periodic solution is given by
\[
u(x, t) = \frac{c + 8k^3}{6k} - 2k^2 \tan^2 (kx - ct),
\]

and the singular solution is
\[
u(x, t) = \frac{c + 8k^3}{6k} - 2k^2 \cot^2 (kx - ct),
\]

(iv) The fourth set is given by
\[
\begin{align*}
a_0 &= e^{\frac{8k^3}{6k}}, a_1 = 0, a_2 = -2k^2, \\
b_0 &= e^{\frac{8k^3}{6k}}, b_1 = 0, b_2 = 4k^2, \\
c_0 &= e^{\frac{8k^3}{3k}}, c_1 = 0, c_2 = 4k^2,
\end{align*}
\]

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Consequently, we obtain
\[ u(x, t) = -\frac{1}{2} v(x, t) = -\frac{1}{2} w(x, t), \] (82)
where the periodic solution is given by
\[ u(x, t) = \frac{c + 8k^3}{6k} - 2k^2 \tan^2 (kx - ct), \] (83)
and and the singular solution is
\[ u(x, t) = \frac{c + 8k^3}{6k} - 2k^2 \cot^2 (kx - ct), \] (84)

5 The generalized Hirota-Satsuma-KP equation

Using the wave variable
\[ \xi = kx + ry - ct, \] (85)
carries out the generalized Hirota-Satsuma equation
\[
\begin{align*}
(u_t + \frac{3}{2}(uv)_x + \frac{3}{2}vw)_x + u_{xxx} + u_{yy} &= 0, \\
(v_t + 3(uw)_x + v_{xxx})_x + v_{yy} &= 0, \\
w_t + 3(uw)_x + w_{xxx}, + w_{yy} &= 0,
\end{align*}
\] (86)
into a system of ODEs
\[
\begin{align*}
-cku' + \frac{3}{2}k^2(uv)' + \frac{3}{2}k^2(vw)' + k^4u''' + r^2u' &= 0, \\
-ckv' + 3k^2(wu)' + k^4v''' + r^2v' &= 0, \\
-ckw' + 3k^2(uw)' + k^4w''' + r^2w' &= 0,
\end{align*}
\] (87)\(\text{to}(89)
obtained upon integrating once. This system will be handled in a like manner to the approach presented earlier.

5.1 The tanh method–the coth method

Proceeding as before, we find \( M = M_1 = 2 \). We apply the tanh method and the coth method independently. The tanh method, or the coth method, suggests the use of the finite expansions
\[
\begin{align*}
u(\xi) &= a_0 + a_1Y + a_2Y^2, \\
v(\xi) &= b_0 + b_1Y + b_2Y^2, \\
w(\xi) &= c_0 + c_1Y + c_2Y^2,
\end{align*}
\] (90)
where \( a_i, b_i, c_i, 0 \leq i \leq 2 \), are constants that will be determined.

Substituting (90) into (87) –(89), collecting the coefficients of each power of \( Y \), we obtain the following four sets of solutions:
(i) The first set is given by
\[
\begin{align*}
a_0 &= -\frac{r^2}{6k^2} - \frac{c + 8k^3}{6k}, a_1 = 0, a_2 = -2k^2, \\
b_0 &= -\frac{r^2}{6k^2} - \frac{c + 8k^3}{6k}, b_1 = 0, b_2 = -2k^2, \\
c_0 &= -\frac{r^2}{6k^2} - \frac{c + 8k^3}{6k}, c_1 = 0, c_2 = -2k^2.
\end{align*}
\] (91)
Consequently, we obtain the soliton solutions
\[ u(x, y, t) = v(x, y, t) = w(x, y, t), \] (92)
where
\[ u(x, y, t) = -\frac{r^2}{6k^2} - \frac{c + 8k^3}{6k} - 2k^2 \tanh^2 (kx + ry - ct), \] (93)
and
\[ u(x, y, t) = -\frac{r^2}{6k^2} - \frac{c + 8k^3}{6k} - 2k^2 \coth^2 (kx + ry - ct), \] (94)
(ii) The second set is given by
\[ a_0 = \frac{r^2}{6k^2} - \frac{c+8k^3}{6k}, \quad a_1 = 0, \quad a_2 = 2k^2, \]
\[ b_0 = -\frac{r^2}{6k^2} + \frac{c+8k^3}{6k}, \quad b_1 = 0, \quad b_2 = -2k^2, \]
\[ c_0 = -\frac{r^2}{6k^2} + \frac{c+8k^3}{6k}, \quad c_1 = 0, \quad c_2 = 2k^2, \]
(95)

In view of these results, and using (90) we obtain the soliton solutions
\[ u(x, y, t) = w(x, y, t) = -v(x, y, t), \]
(96)

where
\[ u(x, y, t) = \frac{r^2}{6k^2} - \frac{c+8k^3}{6k} + 2k^2 \tanh^2 (kx + ry - ct), \]
(97)
and
\[ u(x, y, t) = \frac{r^2}{6k^2} - \frac{c+8k^3}{6k} + 2k^2 \coth^2 (kx + ry - ct), \]
(98)

(iii) The third set is given by
\[ a_0 = \frac{r^2}{6k^2} - \frac{c+8k^3}{6k}, \quad a_1 = 0, \quad a_2 = 2k^2, \]
\[ b_0 = \frac{r^2}{6k^2} - \frac{c+8k^3}{6k}, \quad b_1 = 0, \quad b_2 = 4k^2, \]
\[ c_0 = -\frac{r^2}{6k^2} + \frac{c+8k^3}{6k}, \quad c_1 = 0, \quad c_2 = -4k^2, \]
(99)

Consequently, we obtain the soliton solutions
\[ u(x, y, t) = \frac{1}{2} v(x, y, t) = -\frac{1}{2} w(x, y, t), \]
(100)

where
\[ u(x, y, t) = \frac{r^2}{6k^2} - \frac{c+8k^3}{6k} + 2k^2 \tanh^2 (kx + ry - ct), \]
(101)
and
\[ u(x, y, t) = \frac{r^2}{6k^2} - \frac{c+8k^3}{6k} + 2k^2 \coth^2 (kx + ry - ct), \]
(102)

(iv) The fourth set is given by
\[ a_0 = -\frac{r^2}{6k^2} + \frac{c+8k^3}{6k}, \quad a_1 = 0, \quad a_2 = -2k^2, \]
\[ b_0 = \frac{r^2}{6k^2} - \frac{c+8k^3}{6k}, \quad b_1 = 0, \quad b_2 = 4k^2, \]
\[ c_0 = -\frac{r^2}{6k^2} + \frac{c+8k^3}{6k}, \quad c_1 = 0, \quad c_2 = 4k^2, \]
(103)

Consequently, we obtain the soliton solutions
\[ u(x, y, t) = \frac{1}{2} v(x, y, t) = -\frac{1}{2} w(x, y, t), \]
(104)

where
\[ u(x, y, t) = -\frac{r^2}{6k^2} + \frac{c+8k^3}{6k} - 2k^2 \tanh^2 (kx + ry - ct), \]
(105)
and
\[ u(x, y, t) = -\frac{r^2}{6k^2} + \frac{c+8k^3}{6k} - 2k^2 \coth^2 (kx + ry - ct), \]
(106)

5.2 The tan method and the cot method

Proceeding as before, we find \( M = M_1 = 2 \). We apply the tan method and the cot method independently. The tan method, or the cot method, admits the use of the finite expansions
\[ u(\xi) = a_0 + a_1 Y + a_2 Y^2, \]
\[ v(\xi) = b_0 + b_1 Y + b_2 Y^2, \]
\[ w(\xi) = c_0 + c_1 Y + c_2 Y^2, \]
(107)

where \( a_i, b_i, c_i, 0 \leq i \leq 2 \), are constants that will be determined.
Consequently, we obtain the soliton solutions

(i) The first set is given by

\[
\begin{align*}
    a_0 &= -\frac{r^2}{6k} + \frac{c - 8k^3}{6k},
    \quad a_1 = 0, a_2 = -2k^2, \\
    b_0 &= -\frac{r^2}{6k} + \frac{c - 8k^3}{6k},
    \quad b_1 = 0, b_2 = -2k^2, \\
    c_0 &= -\frac{r^2}{6k} + \frac{c - 8k^3}{6k},
    \quad c_1 = 0, c_2 = -2k^2,
\end{align*}
\]  

Consequently, we obtain the soliton solutions

\[
\begin{align*}
    u(x, y, t) &= v(x, y, t) = w(x, y, t),
\end{align*}
\]  

where

\[
\begin{align*}
    u(x, y, t) &= -\frac{r^2}{6k} + \frac{c - 8k^3}{6k} - 2k^2 \tan^2 (kx + ry - ct),
\end{align*}
\]  

and

\[
\begin{align*}
    u(x, y, t) &= -\frac{r^2}{6k} + \frac{c - 8k^3}{6k} - 2k^2 \cot^2 (kx + ry - ct),
\end{align*}
\]  

(ii) The second set is given by

\[
\begin{align*}
    a_0 &= \frac{r^2}{6k^2} - \frac{c - 8k^3}{6k},
    \quad a_1 = 0, a_2 = 2k^2, \\
    b_0 &= \frac{r^2}{6k^2} + \frac{c - 8k^3}{6k},
    \quad b_1 = 0, b_2 = -2k^2, \\
    c_0 &= \frac{r^2}{6k^2} - \frac{c - 8k^3}{6k},
    \quad c_1 = 0, c_2 = 2k^2,
\end{align*}
\]  

In view of these results, and using (90) we obtain the soliton solutions

\[
\begin{align*}
    u(x, y, t) &= w(x, y, t) = -v(x, y, t),
\end{align*}
\]  

where

\[
\begin{align*}
    u(x, y, t) &= \frac{r^2}{6k^2} - \frac{c - 8k^3}{6k} + 2k^2 \tan^2 (kx + ry - ct),
\end{align*}
\]  

and

\[
\begin{align*}
    u(x, y, t) &= \frac{r^2}{6k^2} - \frac{c - 8k^3}{6k} + 2k^2 \cot^2 (kx + ry - ct),
\end{align*}
\]  

(iii) The third set is given by

\[
\begin{align*}
    a_0 &= \frac{r^2}{6k^2} - \frac{c - 8k^3}{6k},
    \quad a_1 = 0, a_2 = 2k^2, \\
    b_0 &= \frac{r^2}{6k^2} - \frac{c - 8k^3}{6k},
    \quad b_1 = 0, b_2 = 4k^2, \\
    c_0 &= -\frac{r^2}{6k^2} + \frac{c - 8k^3}{6k},
    \quad c_1 = 0, c_2 = -4k^2,
\end{align*}
\]  

Consequently, we obtain the soliton solutions

\[
\begin{align*}
    u(x, y, t) &= \frac{1}{2} v(x, y, t) = -\frac{1}{2} w(x, y, t),
\end{align*}
\]  

where

\[
\begin{align*}
    u(x, y, t) &= \frac{r^2}{6k^2} - \frac{c + 8k^3}{6k} + 2k^2 \tan^2 (kx + ry - ct),
\end{align*}
\]  

and

\[
\begin{align*}
    u(x, y, t) &= \frac{r^2}{6k^2} - \frac{c + 8k^3}{6k} + 2k^2 \cot^2 (kx + ry - ct),
\end{align*}
\]  

(iv) The fourth set is given by

\[
\begin{align*}
    a_0 &= -\frac{r^2}{6k} + \frac{c - 8k^3}{6k},
    \quad a_1 = 0, a_2 = -2k^2, \\
    b_0 &= \frac{r^2}{6k^2} - \frac{c - 8k^3}{6k},
    \quad b_1 = 0, b_2 = 4k^2, \\
    c_0 &= \frac{r^2}{6k^2} - \frac{c - 8k^3}{6k},
    \quad c_1 = 0, c_2 = 4k^2,
\end{align*}
\]  

Consequently, we obtain the soliton solutions

\[
\begin{align*}
    u(x, y, t) &= -\frac{1}{2} v(x, y, t) = -\frac{1}{2} w(x, y, t),
\end{align*}
\]
where
\[
    u(x, y, t) = -\frac{r^2}{6k^2} + \frac{c - 8k^3}{6k} - 2k^2 \tan^2(kx + ry - ct),
\]
(122)
and
\[
    u(x, y, t) = -\frac{r^2}{6k^2} + \frac{c - 8k^3}{6k} - 2k^2 \cot^2(kx + ry - ct),
\]
(123)

6 Conclusions

In this work, we aimed to study four coupled nonlinear equations that appear in a variety of scientific fields. The tanh, coth, tan, and cot methods were employed to achieve the goals set for this work. An abundant sets of solutions, of a variety of distinct physical structures such as solitons, singular solitons and periodic solutions, were formally derived. The study highlights the power of these methods for the determination of exact solutions to several nonlinear evolution equations.

References


IJNS homepage: http://www.nonlinearscience.org.uk/