Persistence in a Generalized Prey-Predator Model with Prey Reserve

Debasis Mukherjee *
Department of Mathematics, Vivekananda College, Thakurpukur, Kolkata - 700 063, India
(Received 29 April 2011, accepted 15 March 2012)

Abstract: This paper deals with the study of a generalized prey-predator system with a reserve area. It is assumed that the habitat is divided into two regions, namely free zone and the other is reserved zone where predation is not allowed. The migration rate of prey population from free zone to reserved zone is predator density dependent. Predator functional response is considered as Holling type II. Boundedness and local stability are discussed. We obtain condition which influences the persistence of all the populations.

Keywords: Prey-predator; reserve zone; stability; persistence

1 Introduction

Predator-prey interactions is an important subject in ecology and mathematical ecology for which many problems still remain open [1]. Lotka-Volterra model was the first in this context to describe the interaction of species. After that many complex models are developed to study prey-predator systems. The hiding behavior of prey in particular has become a crucial part of prey-predator systems and its effects on stability have been focused in several models [2]. Some previous studies suggest that use of refuges by prey according to snapshot approach has a stabilizing effect on prey-predator dynamics [11,16] whereas other models exhibit no such simple pattern [5]. Dynamic nature of refuge has been studied in different models. Predator density dependent migration is not addressed too much in many models of prey refusal. It is observed that refuge has a stabilizing effect on the equilibrium for a simple Lotka-Volterra model. The main thrust of this article is to examine the role of predator density dependent migration in generalized prey-predator system. Actually in prey-predator interaction prey population are at the verge of extinction due to over predation, environmental pollution, mismanagement of natural resources so as to save these species, suitable measures such as restriction on harvesting, creating reserve zones/refuges should be implemented. Thus study of persistence is important from the biological point of view.

Biologically, persistence means the long term survival of all populations. In mathematical language, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of the non-negative cone. When the habitat is divided into two zones, namely free zone and reserved zone, predator density dependent migration of prey population plays a key role for the survival of the populations. The role of reserve zones/refuges in prey-predator system is discussed in [2,5-8,10,12-18].

The organization of this paper is as follows. In Sec. 2, we introduce our mathematical model. In Sec. 3, we analyze our model with regard to equilibria and their stabilities. Persistence criterion is developed. We conclude this section with an example to illustrate our results. A brief discussion is presented in Sec. 4.

2 Mathematical model

We consider a habitat where prey and predator are living together. The habitat is divided into two separate zones namely free zone and reserved zone. The predator species cannot enter into the reserved zone. But the prey species can move...
from reserved zone to unreserved zone and vice-versa. A mathematical model governing the system can be written as

\[ \begin{align*}
\frac{dx_1}{dt} &= x_1g_1(x_1) - m(y)x_1 + nx_2 - yp(x_1) \\
\frac{dx_2}{dt} &= x_2g_2(x_2) + m(y)x_1 - nx_2 \\
\frac{dy}{dt} &= y(-d + cp(x_1))
\end{align*} \]  

(1)

\(x_i(0) > 0, i = 1, 2, y(0) > 0\)

where \(x_1(t)\) is the density of prey species inside the unreserved area at time \(t\), \(x_2(t)\) is the density of prey species in the reserved area at time \(t\) where predation is prohibited. \(y(t)\) denotes the predator density at time \(t\). This model is applicable in a National Park where prey and predator cohabit. The prey species which are to be conserved can be safe from predators by making an artificial boundary or shelter that will separate the reserved zone from unreserved zone. The artificial boundary can be thought of as a fencing of suitable mesh size through which prey can enter but predators cannot. This situation is also found in fishery problem.

Model (1) is based on the following assumptions.

(A1) All the functions have second order derivatives continuous in their arguments on the interval \((0, \infty)\). This is sufficient to guarantee that solutions to initial value problems exist uniquely at least for some positive time.

(A2) \(g_i(0) > 0, g'_i(x_1) < 0, i = 1, 2\), there exists \(k_i > 0\), such that \(g_i(k_i) = 0, i = 1, 2\). For small values of prey population, it will grow. However, there exists a carrying capacity of the environment beyond which the prey population cannot increase even in the absence of predators.

(A3) \(m(0) > 0, m'(y) > 0\). The prey migration rate from free zone to reserved zone is assumed to be predator density dependent, which is supposed to be positive and to increase with \(y\). In other words, the more predators found on free zone, the more prey tends to leave the free zone.

(A4) \(p(0) = 0, p'(x_1) > 0, p(x_1)\) is the functional response of the predator which increases with \(x_1\) i.e. with the prey, and in the absence of prey there can be no predation.

(A5) \(c > 0\) is the conversion rate coefficient.

(A6) \(d > 0\) is the death rate of predator population.

(A7) \(n > 0\) is the prey migration rate from reserved zone to unreserved zone.

### 3 Boundary equilibria and stability

Computations of the boundary equilibria and their stabilities for system (1) provide information required to determine the persistence of system (1). To do so, we compute the variational matrix of system (1). The signs of the real parts of the eigenvalues of the matrix evaluated at a given equilibrium point determine its stability. The matrix is given by

\[ V(x_1, x_2, y) = \begin{pmatrix}
g_1(x_1) + x_1g'_1(x_1) - m(y) - yp'(x_1) & n & -m'(y)x_1 - p(x_1) \\
m(y) & g_2(x_2) + x_2g'_2(x_2) - n & m'(y)x_1 \\
0 & -d + cp(x_1) & 0
\end{pmatrix} \]

System (1) has at most three non-negative equilibria: \(E_0(0, 0, 0), E_1(\pi_1, \pi_2, 0), E_2(x^*_1, x^*_2, y^*)\). An interior planar equilibrium \(E_1(\pi_1, \pi_2, 0)\) occurring in the \(x_1 - x_2\) plane exists if and only if the algebraic system

\[ \begin{align*}
x_1g_1(x_1) - m(0)x_1 + nx_2 &= 0, \\
x_2g_2(x_2) + m(0)x_1 - nx_2 &= 0,
\end{align*} \]

has a positive solution \((\pi_1, \pi_2)\). The interior equilibrium occurring in the \(x_1 - x_2 - y\) plane is \(E_2(x^*_1, x^*_2, y^*)\). Hence \(x^*_1, x^*_2, y^*\) are obtained by solving

\[ \begin{align*}
x^*_1g_1(x^*_1) - m(y^*)x^*_1 + nx^*_2 - y^*p(x^*_1) &= 0, \\
x^*_2g_2(x^*_2) + m(y^*)x^*_1 - nx^*_2 &= 0, \\
-d + cp(x^*_1) &= 0.
\end{align*} \]

\[ (3) \]

\[ \text{LINS homepage: http://www.nonlinearscience.org.uk/} \]
Then \( p(x_1^*) = \frac{d}{c}. \) Hence \( E_2 \) exists if and only if \( d/c \) is in the range of \( p(x_1^*) \), \( y^* < (x_1^* g_1(0) + x_2^* g_2(0))/c/d \) and \( cm(0)x_1^* + y^*d < c(x_1^* g_1(0) + nx_2^*). \)

The equilibrium \( E_0(0, 0, 0) \) has variational matrix \( V(E_0) \) given by

\[
V(E_0) = \begin{pmatrix}
g_1(0) - m(0) & n & 0 
m(0) & g_2(0) - n & 0 
0 & 0 & -d
\end{pmatrix}
\]

which has one negative and two positive eigenvalues whenever \( g_1(0) - m(0) > 0 \) and \( g_2(0) - n > 0 \) giving a point with non-empty unstable manifolds and a stable manifold.

The equilibrium \( E_1(\pi_1, \pi_2, 0) \) has variational matrix \( V(E_1) \) given by

\[
V(E_1) = \begin{pmatrix}
g_1(\pi_1) + \pi_1 g_1'(\pi_1) - m(0) & n & m'(0)\pi_1 - p_1(\pi_1) 
m(0) & g_2(\pi_2) + \pi_2 g_2'(\pi_2) - n & m'(0)\pi_1 
0 & 0 & d + cp_1(\pi_1)
\end{pmatrix}
\]

which has two negative eigenvalues and one positive eigenvalue whenever \( -d + cp_1(\pi_1) > 0 \), giving a point with non-empty stable manifolds and unstable manifold. It can be easily shown that \( E_1 \) is globally asymptotically stable in the \( x_1 - x_2 \) plane whenever it exists. Indeed, let \( H(x_1, x_2) = 1/x_1 x_2 \). Obviously \( H(x_1, x_2) > 0 \) if \( i = 1, 2 \); we denote

\[
F_1(x_1, x_2) = x_1 g_1(x_1) - m(0)x_1 + nx_2, \\
F_2(x_1, x_2) = x_2 g_2(x_2) + m(0)x_1 - nx_2,
\]

\[
\Delta(x_1, x_2) = \frac{\partial}{\partial x_1}(F_1 H) + \frac{\partial}{\partial x_2}(F_2 H). 
\]

Then

\[
H(x_1, x_2) F_1(x_1, x_2) = \frac{g_1(x_1)}{x_2} - \frac{m(0)}{x_2} + \frac{n}{x_1}, \\
H(x_1, x_2) F_2(x_1, x_2) = \frac{g_2(x_2)}{x_1} + \frac{m(0)}{x_1} - \frac{n}{x_2}.
\]

Therefore

\[
\Delta(x_1, x_2) = \frac{g_1'(x_1)}{x_2} - \frac{n}{x_1} + \frac{g_2'(x_2)}{x_1} - \frac{m(0)}{x_2^2} < 0.
\]

Therefore, by the Bendixson-Dulac criterion, there will be no periodic orbit in the \( x_1 - x_2 \) plane. Since \( E_1 \) is locally asymptotically stable in the above plane so it is globally asymptotically stable.

The equilibrium \( E_2(x_1^*, x_2^*, y^*) \) has variational matrix \( V(E_2) \) given by

\[
V(E_2) = \begin{pmatrix}
g_1(x_1^*) + x_1^* g_1'(x_1^*) - m(y^*) - y^* p'(x_1^*) & n & -m'(y^*)x_1^* - p(x_1^*) 
m(y^*) & g_2(x_2^*) + x_2^* g_2'(x_2^*) - n & m'(y^*)x_1^* 
0 & 0 & 0
\end{pmatrix}
\]

The characteristic equation for the variational matrix \( V(E_2) \) is given by

\[
\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0
\]

where,

\[
a_1 = -(g_1(x_1^*) + x_1^* g_1'(x_1^*) - m(y^*) - y^* p'(x_1^*) + g_2(x_2^*) + x_2^* g_2'(x_2^*) - n), \\
a_2 = (g_1(x_1^*) + x_1^* g_1'(x_1^*) - m(y^*) - y^* p'(x_1^*)(g_2(x_2^*) + x_2^* g_2'(x_2^*) - n) - nm(y^*) \\
+ (m'(y^*)x_1^* + p(x_1^*)cy^* p'(x_1^*)), \\
a_3 = -(m'(y^*)x_1^*cy^* p'(x_1^*) + (m'(y^*)x_1^* + p(x_1^*)cy^* p'(x_1^*))(g_2(x_2^*) + x_2^* g_2'(x_2^*) - n))
\]

Clearly \( E_2 \) is locally asymptotically stable if \( a_i > 0, i = 1, 2, 3 \) and \( a_1 a_2 - a_3 > 0 \).

We now show that the solutions of system(1) are bounded.

*IUNS email for contribution: editor@nonlinearscience.org.uk*
Theorem 1 The solutions of system (1) are bounded.

Proof. We define a function \( W = x_1 + x_2 + \frac{1}{2} y \).

The time derivative along a solution of (1) is

\[
W = x_1 g_1(x_1) + x_2 g_2(x_2) - \frac{1}{2} y.
\]

For each \( \lambda > 0 \) the following inequality is fulfilled:

\[
\dot{W} + \lambda W = x_1 (g_1(x_1) + \lambda) + x_2 (g_2(x_2) + \lambda) + (\lambda - d) - \frac{1}{2} y.
\]

If we choose \( \lambda < d \), then right side is bounded for all \((x_1, x_2, y) \in \mathbb{R}^3_+\). Thus we find that

\[
\dot{W} + \lambda W \leq m_1 + m_2,
\]

where \( m_i = \max_{x_i \in [0, k_i]} x_i (g_1(x_i) + \lambda), i = 1, 2 \). Applying a theorem on differential inequalities [3], we obtain

\[
0 \leq W \leq \frac{m_1 + m_2}{\lambda} + W(x_1(0), x_2(0), y(0)) e^{\lambda t} \quad \text{and for } t \to \infty, 0 \leq W \leq \frac{m_1 + m_2}{\lambda}.
\]

Hence system (1) is bounded.

Now we state condition which guarantees the persistence of system (1).

Theorem 2 Suppose \( E_1 \) exist. Further suppose that \( g_1(0) > m(0) \) and \( g_2(0) > n \) and \( d < cp_1(\mathcal{T}_1) \) then system (1) is uniformly persistent.

Proof. Suppose that \( x \) is a point in the positive octant and \( o(x) \) is the orbit through \( x \) and \( \Omega \) is the omega limit set of the orbit through \( x \). Note that \( \Omega(x) \) is bounded. We claim that \( E_0 \notin \Omega(x) \). If \( E_0 \in \Omega(x) \), then by Butler-McGehee lemma [9], there exists a point \( p \) in \( \Omega(x) \cap W^s(E_0) \) where \( W^s(E_0) \) denotes the strong stable manifold of \( E_0 \). Since \( o(p) \) lies in \( \Omega(x) \) and \( W^s(E_0) \) is the y axis, we conclude that \( o(p) \) is unbounded, which is a contradiction.

Next \( E_1 \notin \Omega(x) \), for otherwise, since \( E_1 \) is a saddle point, by Butler-McGehee lemma, there exist a point \( p \) in \( \Omega(x) \cap W^s(E_0) \) where \( W^s(E_0) \) denotes the strong stable manifold of \( E_0 \). Since \( o(p) \) lies in \( \Omega(x) \) and \( W^s(E_1) \) is the \( x_1-x_2 \) plane implies that an unbounded orbit lies in \( \Omega(x) \) a contradiction.

Thus, \( \Omega(x) \) does not intersect any of the coordinate planes and hence system (1) is persistent. Since (1) is bounded, by main theorem in Butler et al. [4], this implies that the system is uniformly persistent (permanent). ■

We conclude this model with an example of our results.

Example 1 Consider the system

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 (a_1 - b_1 x_1) - (a_0 + \frac{a y}{k + y}) x_1 + n x_2 - \frac{y x_1}{e + x_1}, \\
\frac{dx_2}{dt} &= x_2 (a_2 - b_2 x_2) + (a_0 + \frac{a y}{k + y}) x_1 - n x_2, \\
\frac{dy}{dt} &= y (-d + \frac{c x_1}{e + x_1})
\end{align*}
\]

(7)

\( x_i(0) > 0, i = 1, 2, y(0) > 0 \).

Here \( g_1(x_1) = a_1 - b_1 x_1, i = 1, 2, m(y) = a_0 + \frac{a y}{k + y}, p(x) = \frac{x_1}{e + x_1} \).

Considering \( a_1 = 2, b_1 = 1.5, a_0 = 1.5, \alpha = 1, k = 1, n = 1, a_2 = 2, b_2 = 1, c = 1, e = 1, d = 0.5 \). We find that \( g_1(0) > m(0) > 0 \) and \( g_2(0) > n \) and \( d < cp_1(\mathcal{T}_1) \) where \( \mathcal{T}_1 = 1.3333 \) and all the equilibria for system (7) exist and are given by,

\( E_0(0, 0, 0), E_1(1.3333, 2, 0), E_2(1, 2, 1) \).

At this \( E_2 \), Eq. (7) reduces to

\( \lambda^3 + 6.25 \lambda^2 + 9.4375 \lambda + 0.5 = 0 \).

Roots of this equation are \(-0.6642, -4.51741, -1.66615 \). This implies that \( E_2 \) is locally asymptotically stable (See Fig. 1).

4 Discussion

The main focus of this paper was to analyze a generalized prey-predator system with a reserved area. The analysis, consisted of equilibrial stability and persistence criteria. The persistence criteria may be interpreted biologically. The condition \( g_1(0) > m(0) \) and \( g_2(0) > n \) indicate that the migration rate of both the prey in absence of predators remain below a certain threshold value. Again the condition \( d < cp_1(\mathcal{T}_1) \) means that predator can invade into the system. This paper generalizes the model studied in [7].
Acknowledgements

This work was supported by UGC, Govt. of India (Grant No.PSW-059/11-12(ERO),08.08.2011). The author is grateful to anonymous reviewers for their helpful comments.

References


