High Order Variable Mesh Approximation for the Solution of 1D Non-linear Hyperbolic Equation

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Abstract: In this paper, we propose a new high order Numerov type three-level implicit compact discretization on a non-uniform mesh for the solution of one-space dimensional non-linear hyperbolic partial differential equation of the form $u_{tt} = u_{xx} + g(x, t, u, u_x, u_t)$ subject to appropriate initial and Dirichlet boundary conditions. We use only three evaluations of the function $g$ and three grid points at each time level in a compact cell. We also discuss how our method is able to handle the wave equation in polar coordinates. Numerical results are provided to justify the usefulness of the proposed method.

Keywords: Variable mesh; Non-linear hyperbolic equation; High order method; Numerov type approximation; Wave equation in polar coordinates; Vander Pol equation; Dissipative equation; Maximum absolute errors

Mathematics Subject Classifications (2010): 65M06; 65M12

1 Introduction

We consider the following one-space dimensional nonlinear hyperbolic partial differential equation

$$u_{tt} = u_{xx} + g(x, t, u, u_x, u_t), \quad 0 < x < 1, t > 0$$ (1)

subject to the initial conditions

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq 1$$ (2)

and the boundary conditions

$$u(0, t) = g_0(t), \quad u(1, t) = g_1(t), \quad t \geq 0$$ (3)

We assume that $u(x, t) \in C^0$ and $\phi(x)$ and $\psi(x)$ are sufficiently differentiable function of as higher order as possible.

The numerical solution of one space dimensiona second order non-linear hyperbolic equation in cartesian, cylindrical and spherical coordinates are of great importance in many fields of engineering and sciences. Greenspan [1] has obtained approximate solution of wave equation using boundary value technique. Later, Ciment and Leventhal [2, 3] have used operator compact implicit method to solve the wave equation. Using five evaluations of function $g$ and 9-grid points Jain \textit{et al} [6] and Mohanty \textit{et al} [7] have developed high order approximations for the solution of non-linear hyperbolic equations. Later, using nine evaluations of the function $g$ and a single computation cell, Mohanty and Arora [8] have derived high order numerical approximation for the general second order non-linear hyperbolic equation. Mohebbi and Dehghan [5] have also studied high order compact method for the solution of linear hyperbolic equation. Most recently, using less algebra (minimum number of grid points and minimum evaluations of function $g$) Mohanty and Singh [9, 10] have developed high accuracy numerical methods for the one and two space dimensional quasi-linear hyperbolic equations. It is well known that any explicit method for the wave equation is conditionally stable. Twizell [4] has derived a new explicit difference method for the wave equation with extended stability range. Because of the stability restriction and instability

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due to non-uniform mesh, not much numerical methods for the solution of second order hyperbolic equations have been developed so far. Third order accurate variable mesh method for the solution of nonlinear two point boundary value problems have been discussed by several authors (see [11–14]). Recently, Mohanty [15] has discussed high accuracy two level implicit variable mesh method for the solution of 1D nonlinear parabolic equations. To the authors knowledge no high order methods on a variable mesh for the solution of 1D nonlinear hyperbolic equations have been discussed in the literature so far. In this paper, using nine-grid points, we discuss a new three-level implicit Numerov type discretization of order two in time and three in space on a variable mesh for the solution of non-linear hyperbolic equation (1). In this method we require only three evaluations of the function $g$. In next section, we give formulation of the method. In section 3, we give the complete derivation of the method. In section 4, we discuss the application of the proposed method to one dimensional wave equation in polar coordinates.

In this section, we modify our technique in such a way that the solution retains its order and accuracy everywhere in the solution region. In section 5, we examine our method over a set of linear and nonlinear second order hyperbolic equations whose exact solutions are known and compared the results with the results of other known methods. Concluding remarks are given in section 6.

2 Discretization procedure

Let $k > 0$ be the mesh spacing in the time direction so that $t_j = jk$, $0 < j < J$, $J$ being a positive integer. Further, we discretize the unit interval $[0, 1]$ such that $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. Let $h_l = x_l - x_{l-1}$, $l = 1(1)N$ and $\sigma_l = (h_{l+1}/h_l) > 0$ be the mesh ratio parameter in the space direction. We replace the region $\Omega$ by a set of grid points $(x_l, t_j)$ denoted by $(l, j)$. The values of the exact solution of $u(x, t)$ at the grid point $(l, j)$, are denoted by $U^j_l$. Let $u^j_l$ be the approximation solution at the same grid point.

We denote:

$P_l = \sigma_l^2 + \sigma_l - 1, \quad Q_l = (1 + \sigma_l)(1 + 3\sigma_l + \sigma_l^2), \quad R_l = \sigma_l(1 + \sigma_l - \sigma_l^2), \quad S_l = \sigma_l(1 + \sigma_l)$

At the grid point $(l, j)$, we consider the following approximations:

\begin{align}
U^j_{l+1} &= (U^j_l + U^j_{l-1})/(2k) \quad (4a) \\
U^j_{l+1} &= (U^j_{l+1} - U^j_{l+1})/(2k) \quad (4b) \\
U^j_{l-1} &= (U^j_{l-1} - U^j_{l-1})/(2k) \quad (4c) \\
U^j_{l+1} &= (U^j_{l+1} - 2U^j_l + U^j_{l-1})/(k^2) \quad (4d) \\
U^j_{l+1} &= (U^j_{l+1} - 2U^j_{l-1} + U^j_{l+1})/(k^2) \quad (4e) \\
U^j_{l+1} &= (U^j_{l+1} - 2U^j_{l-1} + U^j_{l+1})/(k^2) \quad (4f)
\end{align}

\begin{align}
U^j_{x1} &= (U^j_{l+1} - (1 - \sigma_l^2)U^j_l - \sigma_l^2U^j_{l-1})/(h_lS_l) \quad (5a) \\
U^j_{x1+1} &= ((1 + 2\sigma_l)U^j_{l+1} - (1 + \sigma_l)^2U^j_l + \sigma_l^2U^j_{l-1})/(h_lS_l) \quad (5b) \\
U^j_{x-1} &= (-U^j_{l+1} + (1 + \sigma_l)^2U^j_l - \sigma_l(2 + \sigma_l)U^j_{l-1})/(h_lS_l) \quad (5c) \\
U^j_{xxl} &= \frac{2(U^j_{l+1} - (1 + \sigma_l)U^j_l + \sigma_lU^j_{l-1})}{\sigma_l(1 + \sigma_l)(h_l^2)} \quad (5d)
\end{align}

\begin{align}
\overline{U}^j_{l+1} &= g(\sigma_{l+1}, t_j, U^j_{l+1}, U^j_{x1+1}, U^j_{x1+1}, U^j_{l+1}) \quad (6a) \\
\overline{U}^j_{l-1} &= g(\sigma_{l-1}, t_j, U^j_{l-1}, U^j_{x1-1}, U^j_{x1-1}, U^j_{l-1}) \quad (6b)
\end{align}
\[
\bar{\bar{u}}_{zlx}^j = \bar{u}_{zlx}^j + \frac{h_l \sigma_l (\sigma_l^2 + \sigma_l + 1)}{6Q_l} (\bar{G}_{l+1}^j - \bar{G}_{l-1}^j) - \frac{h_l \sigma_l (\sigma_l^2 + \sigma_l + 1)}{6Q_l} (\bar{u}_{ttl+1}^j - \bar{u}_{ttl-1}^j)
\]  
(7a)

\[
\bar{G}^j = g(x_l, t_j, U_l^j, \bar{u}_{zlx}^j, \bar{G}^j)
\]  
(7b)

Then at each grid point \((l, j), l = 1, 2, \ldots, N; j = 2, \ldots, J\), a Numerov type approximation with accuracy of \(O(k^2 + k^2 h_l + h^2_l)\) for the solution of differential equation (1) may be written as

\[
U_{l+1}^j - (1 + \sigma_l)U_l^j + \sigma_l U_{l-1}^j = \frac{h_l^2}{12} \left[ P_l \bar{u}_{ttl+1}^j + Q_l \bar{u}_{ttl}^j + R_l \bar{u}_{ttl-1}^j \right] - \frac{h_l^2}{12} \left[ P_l G_{l+1}^j + Q_l G_l^j + R_l G_{l-1}^j \right] + O(k^2 h_l^2 + k^2 h_l^3 + h_l^5)
\]  
(8)

where \(T_l^j = O(k^2 h_l^2 + k^2 h_l^3 + h_l^5)\)

### 3 Derivation of the approximation

At the grid point \((l, j)\), let us denote

\[
\alpha_l^j = \left( \frac{\partial g}{\partial u_x} \right)_l^j
\]  
(9)

Further at the grid point \((l, j)\), the exact solution \(U_l^j\) satisfies

\[
U_{xtl}^j - U_{xxl}^j = g(x_l, t_j, U_l^j, U_{xl}^j, U_{ttl}^j) \equiv G_l^j
\]  
(say)

Using Taylor expansion about the grid point \((l, j)\), first we obtain

\[
U_{l+1}^j - (1 + \sigma_l)U_l^j + \sigma_l U_{l-1}^j = \frac{h_l^2}{12} \left[ P_l \bar{u}_{ttl+1}^j + Q_l \bar{u}_{ttl}^j + R_l \bar{u}_{ttl-1}^j \right] - \frac{h_l^2}{12} \left[ P_l G_{l+1}^j + Q_l G_l^j + R_l G_{l-1}^j \right] + O(k^2 h_l^2 + k^2 h_l^3 + h_l^5)
\]  
(11)

With the help of the approximation (4a)-(5d), from (6a) and (6b), we obtain

\[
\bar{G}_{l+1}^j = G_{l+1}^j - \frac{h_l^2 \sigma_l (1 + \sigma_l)}{6} \frac{\partial^2 U_l^j}{\partial x^2} \alpha_l^j + O(k^2 + k^2 h_l + h_l^3)
\]  
(12a)

\[
\bar{G}_{l-1}^j = G_{l-1}^j - \frac{h_l^2 (1 + \sigma_l)}{6} \frac{\partial^2 U_l^j}{\partial x^2} \alpha_l^j + O(k^2 + k^2 h_l + h_l^3)
\]  
(12b)

Now we define the approximations:

\[
\bar{u}_{zlx}^j = U_{zlx}^j + a_l h_l (\bar{G}_{l+1}^j - \bar{G}_{l-1}^j) + b_l h_l (\bar{u}_{ttl+1}^j - \bar{u}_{ttl-1}^j)
\]  
(13)

where \(a_l\) and \(b_l\) are free parameters to be determined.

By using the approximation (5a) and (12a), (12b) from (13), we get

\[
\bar{u}_{zlx}^j = U_{zxl}^j + \frac{h_l^2}{6} T_l + O(k^2 h_l + h_l^3)
\]

where

\[
T_l = (\sigma_l - 6a_l (1 + \sigma_l))U_{xxl}^j + 6(a_l + b_l) (1 + \sigma_l) U_{txl}^j
\]  
(14)

Thus from 7, we obtain
Finally by the help of the approximations (12a), (12b) and (15) from (8) and (11), we get

$$ T_i^j = \frac{h_i^2}{72} \left\{ -\sigma_i (1 + \sigma_i) P_i + (\sigma_i - 6a_i (1 + \sigma_i)) Q_i - (1 + \sigma_i) R_i \right\} U_{j+1}^{i+1} \right) + 6(a_i + b_i)(1 + \sigma_i)Q_i U_{j+1}^{i+1} \right\} \alpha_i^j + O(k^2 h_i^2 + k^2 h_i^3 + h_i^5) \right) $$

In order to obtain the difference method of $O(k^2 + k^2 h_i + h_i^3)$ the coefficient of must be zero, which gives the values of parameter

$$ a_i = -b_i = \frac{\sigma_i (\gamma_i^2 + \sigma_i + 1)}{6Q_i} $$

Thus we obtain the difference method of $O(k^2 + k^2 h_i + h_i^3)$ for the differential equation (1) and the local truncation error reduces to $T_i^j = O(k^2 h_i^2 + k^2 h_i^3 + h_i^5)$.

Note that, the initial and Dirichlet boundary conditions are given by (2) and (3), respectively. Incorporating the initial and boundary conditions, we can write the method (8) in a tri-diagonal matrix form. If the differential equation (1) is linear, we can solve the linear system using Gauss-elimination (tri-diagonal solver) method; in the non-linear case, we can use Newton-Raphson iterative method to solve the non-linear system (see [15]-[19]).

### 4 Application to wave equation in polar coordinates

In this section, we aim to discuss the stable difference schemes for a class of one space linear hyperbolic equations with singular coefficients and ensure that the numerical methods developed here retain their order and accuracy.

Let us consider the equation of the form

$$ u_{tt} = u_{rr} + D(r)u_r + E(r)u + f(r,t), \quad 0 < r < 1, \quad t > 0 \quad (18) $$

subject to appropriate initial and Dirichlet boundary conditions given by (2) and (3), respectively, where $D(r) = \alpha/r$, $E(r) = -\alpha/r^2$. For $\alpha = 0$, the equation above represents time dependent wave equation and for $\alpha = 1$ and 2, the equation (18) represents wave equation in cylindrical and spherical polar coordinates, respectively.

Replacing the variable $x$ by $r$ and applying the approximation (8) to the differential equation (18), and neglecting the local truncation error, we obtain the scheme

$$ \frac{12}{h_i^2} \left[ u_{i+1}^{j+1} - (1 + \sigma_i)u_i^j + \sigma_i u_{i-1}^j \right] = \left[ P_i \pi_{i+1}^{j+1} + Q_i \pi_{i+1}^{j+1} + R_i \pi_{i+1}^{j+1} \right] \right) - (P_i + Q_i D_i \alpha_i h_i)(D_i+1 \pi_{i+1}^{j+1} + E_i \pi_{i+1}^{j+1} + f_i^{j+1}) \right) - Q_i(D_i \alpha_i h_i) \right( E_i u_i^j + f_i^j \right) \right) - (R_i - Q_i D_i \alpha_i h_i)(D_i-1 \pi_{i-1}^{j+1} + E_i-1 u_{i+1}^j + f_i^{j-1}) \right) - Q_i D_i b_i h_i \left( \pi_{i+1}^{j+1} - \pi_{i-1}^{j+1} \right) \right), \quad i = 1(1)N, \quad j = 1, 2, \ldots \quad (19) $$

where the values of $P_i$, $Q_i$, $R_i$, $a_i$ and $b_i$ are already defined in the previous section and $D_i = D(r_i)$, $D_i \pm 1 = D(r_i \pm 1)$, $E_i = E(r_i)$, $E_i \pm 1 = E(r_i \pm 1)$, $f_i^j = f(r_i, t_j)$, $f_i^{j \pm 1} = f(r_i^{\pm 1}, t_j)$ etc.

Note that the linear variable mesh scheme (19) is of $O(k^2 + k^2 h_i + h_i^3)$ accuracy for the solution of the hyperbolic differential equation (18), however, the scheme fails to compute when the solution is to be determined at $l = 1$ due to zero division. We overcome this difficulty by using the following approximations.

Let,

$$ LINS homepage: \text{http://www.nonlinearscience.org.uk/} \quad (15) $$
\[D_t \equiv D_0\]
\[D_{t+1} = D_t + \sigma th_tD_{rt} + \frac{\sigma^2 h^2}{2} D_{rrt} + O(h_t^3) \equiv D_t\]  \hspace{1cm} (20a)
\[D_{t-1} = D_t - h_tD_{rt} + \frac{h^2}{2} D_{rrt} - O(h_t^3) \equiv D_2\]  \hspace{1cm} (20b)
\[E_t \equiv E_0\]
\[E_{t+1} = E_t + \sigma th_tE_{rt} + \frac{\sigma^2 h^2}{2} E_{rrt} + O(h_t^3) \equiv E_1\]  \hspace{1cm} (20c)
\[E_{t-1} = E_t - h_tE_{rt} + \frac{h^2}{2} E_{rrt} - O(h_t^3) \equiv E_2\]  \hspace{1cm} (20d)
\[f^i_1 = f(r_l, t_j) \equiv F_0\]  \hspace{1cm} (20e)
\[f^i_{t+1} = f^i_1 + \sigma h_tf^i_{rt} + \frac{\sigma^2 h^2}{2} f^i_{rrt} + O(h_t^3) \equiv F_1\]  \hspace{1cm} (20f)
\[f^i_{t-1} = f^i_1 - h_tf^i_{rt} + \frac{h^2}{2} f^i_{rrt} - O(h_t^3) \equiv F_2\]  \hspace{1cm} (20g)

Where \(D_{rt} = \frac{dD(r_t)}{dr}, D_{rrt} = \frac{d^2D(r_t)}{dr^2}\), \(E_{rt} = \frac{dE(r_t)}{dr}, E_{rrt} = \frac{d^2E(r_t)}{dr^2}\), \(f^i_{rt} = \frac{\partial f(r_l, t_j)}{\partial r}, f^i_{rrt} = \frac{\partial^2 f(r_l, t_j)}{\partial r^2}\) etc.

Now substituting the approximations (20a)-(20h) in (19) and neglecting high order terms, we obtain

\[
12 \frac{h^2}{l^2} \left[ u^j_{l+1} - (1 + \sigma_l)u^j_{l} + \sigma_l u^j_{l-1} \right] = \left[ P_l\overline{\pi}_{ltt+1}^j + Q_l\overline{\pi}_{ltt+1}^j + R_l\overline{\pi}_{ltt+1}^j \right] \\
- (P_l + Q_lD_0a_lh_t)(D_1\overline{\pi}_{l+1}^{j} + E_1u^j_{l+1} + F_1) \\
- Q_l(D_0u^j_{l+1} + E_0u^j_{l} + F_0) \\
- (R_l - Q_lD_0a_lh_t)(D_2\overline{\pi}_{ltt-1}^j + E_2u^j_{ltt-1} + F_2) \\
- Q_lD_0h_t(\overline{\pi}_{ltt+1}^j - \overline{\pi}_{ltt-1}^j), \quad l = 1(1)N, \quad j = 1, 2, \ldots \]  \hspace{1cm} (21)

Note that the numerical method (21) is of \(O(k^2 + k^2h^2 + h^3)\) accuracy and free from the terms \(1/(r_{l+1})\), hence very easily solved for \(l = 1(1)N\) in the solution region \(0 < r < 1, t > 0\). This technique shows that the proposed numerical method is applicable to singular problems and we do not require the presence of any fictitious points outside the solution region to handle the numerical scheme near the boundary.

## 5 Numerical Illustrations

Substituting the approximations (4a), (4d), (5a) and (5b) directly into the differential equation (1) we obtain a lower order variable mesh method

\[
\overline{U}_{ltl}^j = \overline{U}_{xxt}^j + g(x_t, t_j, U^j_1, U^j_x, U^j_t) + O(k^2 + h_t), \quad l = 1(1)N, \quad j = 1, 2, \ldots \]  \hspace{1cm} (22)

In this section, we have solved some benchmark problems using the method described by equation (8) and compared our results with the results obtained by using the method (22) for the solution of 1-D non-linear wave equation. The exact solutions are provided in each case. The right hand side homogeneous functions, initial and boundary conditions may be obtained using the exact solution as a test procedure. The linear difference equations have been solved using a tri-diagonal solver, whereas non-linear difference equations have been solved using the Newton-Raphson method. While using the Newton-Raphson method, the iterations were stopped when absolute error tolerance \(\leq 10^{-12}\) was achieved. All computations were carried out using double precision arithmetic.

Note that, the proposed method (8) for second order hyperbolic equations is a three-level scheme. The value of \(u\) at \(t = 0\) is known from the initial condition. To start any computation, it is necessary to know the numerical value of \(u\) of required accuracy at \(t = k\). In this section, we discuss an explicit scheme of \(O(k^2)\) for \(u\) at first time level, i.e., at \(t = k\)
in order to solve the differential equation (1) using the method (8), which is applicable to problems in Cartesian and polar coordinates.

Since the values of \( u \) and \( u_t \) are known explicitly at \( t = 0 \), this implies that all their successive tangential derivatives are known at \( t = 0 \), i.e. the values of \( u, u_x, u_{xx}, \ldots, u_t, u_{tx}, \ldots, \) etc. are known at \( t = 0 \).

An approximation for \( u \) of order \( O(k^3) \) at \( t = k \) may be written as

\[
 u^1_t = u^0_t + ku^0_t + \frac{k^2}{2}(u^0_t) + O(k^3)
\]  

From equation (1), we have

\[
 (u^0_t)^0 = \left[u_{xx} + g(x,t,u,u_t)\right]^0
\]  

Thus using the initial values and their successive tangential derivative values, from (24) we can obtain the value of \((u^0_t)^0\), and then ultimately, from (23) we can compute the value of \(u\) at first time level, i.e. at \( t = k \). Replacing the variable \( x \) by \( r \) in (23), we can also obtain an approximation of \( O(k^2) \) for \( u \) at \( t = k \) for problems in \( rt \)-plane.

Since

\[
 1 = x_{N+1} - x_0 = (x_{N+1} - x_N) + (x_N - x_{N-1}) + \cdots + (x_1 - x_0)
\]

\[
 = h_{N+1} + h_N + \cdots + h_1 = h_1(1 + \sigma_1 + \sigma_1\sigma_2 + \sigma_1\sigma_2\sigma_3 + \cdots + \sigma_1\sigma_2\cdots\sigma_N)
\]

Thus

\[
 h_1 = \frac{1}{1 + \sigma_1 + \sigma_1\sigma_2 + \cdots + \sigma_1\sigma_2\cdots\sigma_N}
\]

This determines the starting value of the first step length in \( x \)-direction and the subsequent step lengths in \( x \)-direction are calculated by

\[
 h_2 = \sigma_1 h_1, h_3 = \sigma_2 h_2, \ldots \quad \text{etc.}
\]

For simplicity, we may consider \( \sigma_1 = \sigma \) (a constant), then \( h_1 \) reduces to

\[
 h_1 = \frac{1 - \sigma}{1 - \sigma^{N+1}}, \quad \sigma \neq 1
\]

Therefore, by prescribing the value of \( N \) and \( \sigma \), we can calculate \( h_1 \) from the above relation and the remaining mesh points in \( x \)-direction is determined by \( h_{t+1} = \sigma h_t, \) \( l = 1(1)N \). We have chosen the value \( \sigma = 1.02 \). Throughout our computation we used the time step \( k = 1.6/(N + 1)^2 \).

**Example 1** (Wave equation in polar coordinates)

\[
 u_{tt} = u_{rr} + \frac{\alpha}{r}u_r + f(r,t), \quad 0 < r < 1, \quad t > 0
\]

The exact solution is \( u = \cosh r \sin t \). The maximum absolute errors are tabulated in Table 1 at \( t = 1 \) for \( \alpha = 0.1 \) and 2.

**Example 2** (Van der Pol type nonlinear wave equation)

\[
 u_{tt} = u_{xx} + \gamma(u^2 - 1)u_t + f(x,t), \quad 0 < x < 1, \quad t > 0
\]

with exact solution \( u = e^{-\gamma t} \sin \pi x \). The maximum absolute errors are tabulated in Table 2 at \( t = 2 \) for \( \gamma = 1, 2 \) and 3.

**Example 3** (Dissipative nonlinear wave equation)

\[
 u_{tt} = u_{xx} - 2uu_t + f(x,t), \quad 0 < x < 1, \quad t > 0
\]

with exact solution \( u = \sin(\pi x) \sin t \). The maximum absolute errors are tabulated in Table 3 at \( t = 1 \) and 2.

**Example 4** (Non-linear wave equation)

\[
 u_{tt} = u_{xx} + \gamma(u(u_x + u_t) + f(x,t), \quad 0 < x < 1, \quad t > 0
\]

with exact solution \( u = e^\gamma \cosh x \). The maximum absolute errors are tabulated in Table 4 at \( \gamma = 2, 5 \) and 10 at \( t = 1 \).
### Table 1: The maximum absolute errors

<table>
<thead>
<tr>
<th>$N + 1$</th>
<th>Method (8)</th>
<th>Method (21) with $E(r) = 0$</th>
<th>$\alpha = 0, 1$ and 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>08</td>
<td>0.1045(-03)</td>
<td>0.6100(-02)</td>
<td>0.1200(-01)</td>
</tr>
<tr>
<td>16</td>
<td>0.6586(-05)</td>
<td>0.1600(-02)</td>
<td>0.3600(-02)</td>
</tr>
<tr>
<td>32</td>
<td>0.4124(-06)</td>
<td>0.6162(-03)</td>
<td>0.1200(-02)</td>
</tr>
<tr>
<td>64</td>
<td>0.2579(-07)</td>
<td>0.2643(-03)</td>
<td>0.5281(-03)</td>
</tr>
</tbody>
</table>

### Table 2: The maximum absolute errors

<table>
<thead>
<tr>
<th>$N + 1$</th>
<th>Method (8)</th>
<th>Method (22)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 1$</td>
<td>$\gamma = 2$</td>
<td>$\gamma = 3$</td>
</tr>
<tr>
<td>04</td>
<td>0.3700(-03)</td>
<td>0.2380(-03)</td>
</tr>
<tr>
<td>08</td>
<td>0.2676(-04)</td>
<td>0.1610(-04)</td>
</tr>
<tr>
<td>16</td>
<td>0.2138(-05)</td>
<td>0.1210(-05)</td>
</tr>
<tr>
<td>32</td>
<td>0.2363(-06)</td>
<td>0.1236(-06)</td>
</tr>
</tbody>
</table>

### Table 3: The maximum absolute errors

<table>
<thead>
<tr>
<th>$N + 1$</th>
<th>Method (8)</th>
<th>Method (22)</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 1$</td>
<td>$t = 2$</td>
<td>$t = 1$</td>
<td>$t = 2$</td>
<td></td>
</tr>
<tr>
<td>08</td>
<td>0.8583(-04)</td>
<td>0.1064(-04)</td>
<td>0.9300(-02)</td>
<td>0.1410(-01)</td>
</tr>
<tr>
<td>16</td>
<td>0.6570(-05)</td>
<td>0.8302(-05)</td>
<td>0.2500(-02)</td>
<td>0.3700(-02)</td>
</tr>
<tr>
<td>32</td>
<td>0.7055(-06)</td>
<td>0.9182(-06)</td>
<td>0.7250(-03)</td>
<td>0.1100(-02)</td>
</tr>
<tr>
<td>64</td>
<td>0.1108(-06)</td>
<td>0.1810(-06)</td>
<td>0.2781(-03)</td>
<td>0.3941(-03)</td>
</tr>
</tbody>
</table>

### Table 4: The maximum absolute errors

<table>
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<tr>
<th>$N + 1$</th>
<th>Method (8)</th>
<th>Method (22)</th>
<th>$\gamma = 2$</th>
<th>$\gamma = 5$</th>
<th>$\gamma = 10$</th>
<th>$\gamma = 2$</th>
<th>$\gamma = 5$</th>
<th>$\gamma = 10$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.4849(-03)</td>
<td>0.1500(-01)</td>
<td>0.1100(-02)</td>
<td>0.7500(-02)</td>
<td>0.2670(-00)</td>
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<tr>
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<td>0.4366(-05)</td>
<td>0.3496(-04)</td>
<td>0.1300(-02)</td>
<td>0.3195(-03)</td>
<td>0.2200(-02)</td>
<td>0.6850(-01)</td>
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<tr>
<td>32</td>
<td>0.3208(-06)</td>
<td>0.2922(-05)</td>
<td>0.1315(-03)</td>
<td>0.1077(-03)</td>
<td>0.6869(-03)</td>
<td>0.2120(-01)</td>
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</tr>
<tr>
<td>64</td>
<td>0.1874(-07)</td>
<td>0.3900(-06)</td>
<td>0.2077(-04)</td>
<td>0.4569(-04)</td>
<td>0.2748(-03)</td>
<td>0.8400(-02)</td>
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<td></td>
</tr>
</tbody>
</table>

*IJNS email for contribution: editor@nonlinearscience.org.uk*
6 Concluding remarks

Available numerical methods on a variable mesh for the numerical solution of second order non-linear wave equations are of $O(k^2 + h_l)$ accurate. In this article, using the same variable mesh and same number of grid points and three evaluations of the function $g$ (as compared to five and nine evaluations of the function $g$ discussed in [4-6] for constant mesh), we have derived a new stable Numerov type discretization of $O(k^2 + k^2 + h_l + h_l^3)$ accuracy for the solution of non-linear wave equation (1). The proposed method produces stable numerical solution for nonlinear equations, which is exhibited from the computed results. The proposed numerical method is also applicable to wave equation in polar coordinates which produces stable and oscillation free solutions even in the vicinity of the singularity, whereas the corresponding lower order method (22) is unstable.

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References