Thin Film Flow of a Third Grade Fluid Through Porous Medium Over an Inclined Plane

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Abstract: This communication aims to present the analytic solution for thin film flow of an incompressible third grade fluid through porous medium on an inclined plane. In the flow through porous medium the equations are nonlinear and are solved for the velocity field using both the traditional perturbation method and the homotopy perturbation method. The results obtained by two methods are in close agreement with each other and graphs are plotted to discuss the effect of different parameters.

Keywords: porous medium; inclined plane; homotopy perturbation method

1 Introduction

In recent years, especially with the emergence of polymer industry, petroleum industries and other types of pulp industries, the non-Newtonian fluids have become very much important. Due to complexity of non-Newtonian fluids, it becomes difficult to suggest a single model which exhibits all properties of non-Newtonian fluids, therefore various empirical and semi empirical models have been proposed. Non-Newtonian fluids can mainly be classified into two main classes such as differential type fluids and rate type fluids. Among these two classes, the differential type fluids have received great attention from scientists and engineers. A second grade fluid is one of the most acceptable fluid in this sub clam of non-Newtonian fluids. This is because of its mathematical simplicity in comparison to third grade and fourth grade fluids. However, there are studies available in literature in which the authors have successfully treated the challenging nonlinear equations governing the flow of a third grade fluid [1-4].

During last few decades, lot of studies have been devoted to the viscous flows through porous medium [5, 6]. This is because of its technological applications in diverse areas of science and technology as: in food processing, heat storage, oil recovery, in packing fruits and vegetables and in various processes in chemical industry and environmental sciences. Civil engineers, mining engineers, petroleum engineers and hydro geologists are interested in seepage problems in rock mass, sand beds and subterranean and aquifers. Being inspired from such practical application, we have considered the viscous flow of third grade fluid through porous medium past an inclined plane. The problem was first studied by Siddiqui et al [7] in which they studied third grade flow over an inclined plane through open half space. In this study, we have replaced the open half space of Siddiqui et al [7] by porous half space. The flow modelling is based on modified Darcy’s law for third grade fluid.

\[ \nabla p = -\frac{\mu \phi \nabla}{k}. \]

While dealing with the non-Newtonian fluids or of the great challenge in the solution of governing nonlinear differential equations. Number of the numerical and analytical techniques have been proposed by various researchers. However, an efficient analytic solution still finds great appreciations. Keeping this fact in mind, we have solved the governing nonlinear equations of present problem using the two powerful analytic techniques namely, the traditional perturbation method [8] and homotopy perturbation method [9,10]. It is important to mention here that the two solutions are in a complete agreement and the previous results of Siddiqui et al [7] can easily be recovered by substituting the porous

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medium parameter equal to zero. In this study, it is also observer that the homotopy perturbation method is a powerful analytical technique that is simple and straightforward and does not require the existence of any small or large parameter as does traditional perturbation method. Homotopy perturbation method has successfully been applied to a number of nonlinear problems arising in the science and engineering by various researchers [11-16]. This proves the validity and acceptability of HPM as a useful solution technique.

The paper is distributed in five sections. In section 2, governing equations of motion are given. In section 3, the solution is obtained by perturbation method and homotopy perturbation method. Discussion on results is given in section 4 and some concluding remarks are given in section 5.

2 Formulation of flow problem

For an incompressible viscous fluid, the governing equations are the mass and momentum conservation laws in the absence of heat transfer phenomenon such laws are described through the following equations named as continuity equation and momentum equation.

\[ \nabla \cdot V = 0, \]  \hspace{1cm} (2.1)

\[ \frac{\rho D}{Dt} V = -\nabla p + \rho f + div\tau - \frac{\mu \phi}{k} V, \]  \hspace{1cm} (2.2)

where \( \rho \) is the constant density, \( V \) the velocity vector, \( p \) the pressure, \( \tau \) the stress tensor \( f \) the body force and \( \frac{D}{Dt} \) denoting the material derivative. The stress tensor defining a third grade fluid is given by

\[ \tau = \sum_{i=1}^{3} S_i, \]  \hspace{1cm} (2.3)

Here, \( \mu \) is the coefficient of viscosity and \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and \( \beta_3 \) are material constants. The Rivlin-Ericksen tensors \( A_n \) are defined by \( A_0 = I \) the identity tensor and

\[ A_n = \frac{DA_{n-1}}{Dt} + A_{n-1} (\nabla V) + (\nabla V)^f A_{n-1}, \ n \geq 1, \]

where \( A_0 = I \) is the identity tensor.

We consider a semi-infinite flat plate inclined at an angle \( \alpha \) with the horizontal.

The ambient air is considered stationary so that the flow is driven due to gravity alone. The surface tension is assumed to be negligible and we have taken the film of uniform thickness \( \delta \). In this way, we have the velocity field of the form

\[ V = (u(y), 0, 0) \]  \hspace{1cm} (2.4)

Substituting the value of \( V \) and \( \nabla \cdot \tau \) in Eq. (2.2), we obtain

\[ \mu \frac{\partial^2 u}{\partial y^2} + 6 (\beta_2 + \beta_3) \left( \frac{\partial u}{\partial y} \right)^2 \left( \frac{\partial^2 u}{\partial y^2} \right) + \rho g \sin \alpha - \frac{\partial u}{k} = 0 \]  \hspace{1cm} (2.5)

subject to the boundary conditions

\[ \begin{cases} u(y) = 0 \text{ at } y = 0, \\ \frac{\partial u}{\partial y} = 0 \text{ at } y = \delta. \end{cases} \]  \hspace{1cm} (2.6)

The dimensionless variables are defined through

\[ \begin{align*}
  u &= \frac{\nu u^*}{\delta^*}, & y &= \delta y^*, \\
  \beta &= \frac{\nu^{\beta_2 + \beta_3}}{\delta^*}, & \phi \sin \alpha &= \frac{m^* v^2}{\delta^*}. \end{align*} \]  \hspace{1cm} (2.7)

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By using Eq. (2.7) in Eqs. (2.5) and (2.6), we have
\[ \frac{\partial^2 u^*}{\partial y^*} + 6\beta^* \left( \frac{\partial u^*}{\partial y^*} \right)^2 \left( \frac{\partial^2 u^*}{\partial y^*} \right) + m^* - \lambda = 0 \] (2.8)
\[ u^*(0) = 0 \text{ at } y^* = 0, \] (2.9)
\[ \frac{du^*}{dy^*} = 0 \text{ at } y^* = 1. \] (2.10)

where \( \lambda = \frac{\phi \delta^2}{k} \). Eq. (2.9) comes from no slip condition and Eq. (2.10) is obtained by using \( \tau_{yx} = 0 \) at \( y^* = 1 \).

3 Solution of problem

In order to solve the system Eqs. (2.8)-(2.10) analytically, we use two analytic techniques, namely, the perturbation method and the homotopy perturbation method.

3.1 Solution by perturbation method

The basic requirement for the application of solution by perturbation method is the existence of small or large parameter in the governing equation; we therefore regard \( \beta \) as the small parameter and expand \( u(y^*, \beta) \) in the Poincare-type series of the form
\[ u(y, \beta) = u_0^*(y^*) + \beta u_1^*(y^*) + \beta^2 u_2^*(y^*) + \cdots \] (3.1)

After the substitution of equation (3.1) into (2.8) we get zeroth, first and second order equations

**Zeroth order problem**

\[ O(\beta^0) : \frac{d^2 u_0^*}{dy^*} + m^* - \lambda u_0^* = 0, \] (3.2)

with boundary conditions
\[ \begin{cases} u_0^*(0) = 0 \text{ at } y^* = 0, \\ \frac{du_0^*}{dy^*} = 0 \text{ at } y^* = 1. \end{cases} \] (3.3)

**First order problem**

The first order problem is given by
\[ O(\beta^1) : \frac{d^2 u_1^*}{dy^*} + 6 \left( \frac{du_0^*}{dy^*} \right)^2 \left( \frac{d^2 u_0^*}{dy^*} \right) - \lambda u_1^* = 0, \] (3.4)

subject to boundary conditions
\[ \begin{cases} u_1^*(0) = 0 \text{ at } y^* = 0, \\ \frac{du_1^*}{dy^*} = 0 \text{ at } y^* = 1. \end{cases} \] (3.5)

**Second order problem**

The second order problem is given by
\[ O(\beta^2) : \frac{d^2 u_2^*}{dy^*} + 6 \left( \frac{du_0^*}{dy^*} \right)^2 \left( \frac{d^2 u_0^*}{dy^*} \right) + 12 \left( \frac{du_0^*}{dy^*} \right) \left( \frac{du_1^*}{dy^*} \right) \left( \frac{d^2 u_0^*}{dy^*} \right) - \lambda u_2^* = 0, \] (3.6)

subject to boundary conditions
\[ \begin{cases} u_2^*(y^*) = 0 \text{ at } y^* = 0, \\ \frac{du_2^*}{dy^*} = 0 \text{ at } y^* = 1. \end{cases} \] (3.7)
Zeroth order solution
The solution of system of Eqs.(3.2) & (3.3) is
\[ u_0 = \left( \frac{m}{\lambda} \right) + e^{y\sqrt{\lambda}} \left( -\frac{m}{1 + e^{2\sqrt{\lambda}}\lambda} \right) + e^{-y\sqrt{\lambda}} \left( -me^{2\sqrt{\lambda}} \lambda \right). \]

First order solution
The solution of system of Eqs.(3.4) & (3.5) is
\[ u_1 = \left( \frac{3e^{-3y\sqrt{\lambda}}m^3}{4 \left( 1 + e^{2\sqrt{\lambda}} \right)^3 \lambda^2} \right) \]
\[ + \left( \frac{3e^{-3y\sqrt{\lambda}}m^3 \left( 4e^{2(1+2y)\sqrt{\lambda}}y^{\sqrt{\lambda}} - 4e^{2(1+2y)\sqrt{\lambda}}y^{\sqrt{\lambda}} \right)}{4 \left( 1 + e^{2\sqrt{\lambda}} \right)^3 \lambda^2} \right) \]
\[ + e^{y\sqrt{\lambda}} \left( -3m^3 \left( 1 + 2e^{2\sqrt{\lambda}} + 2e^{4\sqrt{\lambda}} + e^{6\sqrt{\lambda}} - 8e^{4\sqrt{\lambda}} \lambda \right) \right) \]
\[ + e^{-y\sqrt{\lambda}} \left( -3m^3 \left( 1 + 2e^{2\sqrt{\lambda}} + 2e^{4\sqrt{\lambda}} + e^{6\sqrt{\lambda}} + 8e^{4\sqrt{\lambda}} \lambda \right) \right). \]

The second order solution is too long and in order to make the research paper compact, we have omitted long expressions. In this way the solution series is given by
\[ u(y) = u_0(y) + \beta u_1(y) + \beta^2 u_2(y). \]

3.2 Solution by homotopy perturbation method
In order to apply homotopy perturbation method, we select the linear operator
\[ L = \frac{d^2}{dy^2} - \lambda \]
and construct the following homotopy:
\[ L(V) - L(u_0) + qL(u_0) + q \left[ 6\beta^2 \left( \frac{dV}{dy} \right)^2 + \left( \frac{d^2V}{dy^2} \right) + m \right] = 0, \quad (3.8) \]
where \( q \in [0, 1] \) is the embedding parameters and \( u_0 \) is the initial guess approximation
\[ u_0 = \left( \frac{m}{\lambda} \right) + e^{y\sqrt{\lambda}} \left( -\frac{m}{1 + e^{2\sqrt{\lambda}}\lambda} \right) + e^{-y\sqrt{\lambda}} \left( -me^{2\sqrt{\lambda}} \lambda \right). \]
Substituting \( V(y) = v_0 + qv_1 + q^2v_2 + q^3v_3 + \cdots \) in Eq. (3.8), where \( v' \)s are independent of \( q' \)s.

Zeroth order problem
\[ L(v_0) - L(u_0) = 0 \]
with boundary conditions
\[ \begin{cases} v_0(0) = 0 & \text{at } y = 0, \\ \frac{dv_0}{dy} = 0 & \text{at } y = 1. \end{cases} \]
First order problem

\[ L(v_1) + L(u_0) + \left[ 6\beta^* \left( \frac{dv_0}{dy} \right)^2 \left( \frac{d^2v_0}{dy^2} \right) + m^* \right] = 0 \]

with boundary conditions

\[ \begin{aligned}
&v_1(0) = 0 \quad \text{at} \quad y = 0, \\
&\frac{dv_1}{dy} = 0 \quad \text{at} \quad y = 1.
\end{aligned} \]

Second order problem

\[ L(v_2) + \beta^* \left[ 6 \left( \frac{dv_0}{dy} \right)^2 \left( \frac{d^2v}{dy^2} \right) + 12 \left( \frac{dv_0}{dy} \right) \left( \frac{d^2v_0}{dy^2} \right) \right] = 0 \]

with boundary conditions

\[ \begin{aligned}
&v_2(0) = 0 \quad \text{at} \quad y = 0, \\
&\frac{dv_2}{dy} = 0 \quad \text{at} \quad y = 1.
\end{aligned} \]

Solving the above problems in conjunction with the corresponding boundary conditions.

Zeroth order solution

\[ v_0(y) = \left( \frac{m}{\lambda} \right) + e^{y\sqrt{\lambda}} \left( \frac{-m}{1 + e^{2\sqrt{\lambda}}} \lambda \right) + e^{-y\sqrt{\lambda}} \left( \frac{-me^{2\sqrt{\lambda}}}{1 + e^{2\sqrt{\lambda}}} \lambda \right). \quad (3.9) \]

First order solution

\[ v_1(y) = \left( 3e^{-3y\sqrt{\lambda}}m^3 \left( e^{6\sqrt{\lambda}} + e^{6y\sqrt{\lambda}} + 2e^{2(2+y)\sqrt{\lambda}} + 2e^{2(1+2y)\sqrt{\lambda}} \right) / 4 \left( 1 + e^{2\sqrt{\lambda}} \right)^3 \lambda^2 \\
+ \left( 3e^{-3y\sqrt{\lambda}}m^3 \left( 4e^{2(2+y)\sqrt{\lambda}}y\sqrt{\lambda} - 4e^{2(1+2y)\sqrt{\lambda}}y\sqrt{\lambda} \right) / 4 \left( 1 + e^{2\sqrt{\lambda}} \right)^3 \lambda^2 \\
+ e^{y\sqrt{\lambda}} \left( -3m^3 \left( 1 + 2e^{2\sqrt{\lambda}} + 2e^{2\sqrt{\lambda}} + 2e^{2\sqrt{\lambda}} + 8e^{2\sqrt{\lambda}} \right) / 4(1 + e^{2\sqrt{\lambda}})4\lambda^2, \right. \right. \]

\[ \]
In Figure 1, velocity function $u(y)$ is plotted against $y$ for different values of non-Newtonian parameter $\beta$. Clearly, increasing values of $\beta$ cause to decrease the velocity in this region and as a consequence the velocity gradient also decreases at the plate due to which the skin friction also decreases at the plate and at the other edge of the thin film, the velocity gradient vanishes as given in the boundary data. Figure 2 depicts that by increasing porous medium parameter $\lambda$, velocity decreases in this region. This is because of the fact that increasing values of $\lambda$ to respond to large opening the porous space. In Figure 3, velocity is plotted for the different values of parameter $m$, by increasing the value of $m$, the velocity increases. This is because of the reason that increasing value of $m$ correspond to the increasing angle of inclination, which shows that by increasing the angle of inclination of inclined plane, the velocity increases significantly.

5 Concluding Remarks

A thin film flow of a non-Newtonian third grade fluid through porous medium over an inclined plane has been studied analytically. The effect of non-Newtonian parameter is to decrease the flow velocity and hence the skin friction at the plate. An increase in porous medium parameter also causes the velocity to decrease.

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Figure 3: Dimensionless velocity profiles with fixed values of parameters $\beta = 0.75$, $\lambda = 0.01$ and different values of the parameter $m = 0.4, 0.6, 0.8$.

References


