An Efficient Iterative Technique for Solving Some Nonlinear Problems

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(Received 5 July 2011, accepted 7 November 2011)

Abstract: In this article, we have provided approximate analytical solutions of some well known nonlinear problems by using He’s variational iteration method (VIM). This method is based on the use of Lagrange multiplier for identification of optimal value of a parameter in a functional. This procedure is a powerful tool for solving several mathematical models arises from problems in science and engineering. It provides a sequence of functions which converges to the exact solution of the model equation. The advantage of this method over other iterative methods lies in its simplicity and ability to solve nonlinear problems accurately and efficiently.

Keywords: variational iteration method; regularized long wave equation; Burger’s equation; modified Korteweg-de Vries equation

1 Introduction

Solution of nonlinear problems was supposed to be the most difficult field of research. However, with the development of high speed computers and availability of numerous numerical methods, many problems in science and engineering can be solved numerically. In recent years, researchers look forward for some new methods which are very cost effective and simpler in implementation to solve nonlinear problems. They succeeded to develop techniques known as iterative techniques like, Homotopy Perturbation Method (HPM)[1], Homotopy Analysis Method (HAM)[2], Adomian Decomposition Method (ADM)[3], Tanh Method[4] and Variational Iteration Method (VIM)[5, 6]. The variational iteration method is extensively applied to solve large class of linear and nonlinear differential equations in recent years.

VIM has many advantages over the contemporary iterative techniques. As in the past many years ADM is extensively used to solve a wide range of linear and nonlinear ordinary differential equations, partial differential equations and integral equations, but usually the solutions provided by ADM do not satisfy the boundary conditions. However, VIM is of very simple implementation and provides us an approximate solution that satisfy initial and boundary conditions. Another advantage of VIM over ADM is, VIM provides the solution of the problem without calculating Adomian’s Polynomial. Additionally, VIM solves the problem without any need to discretization of the variables. Therefore, the results obtained by VIM are not affected by computation round off errors. Large computer memory and much time are also not required by the method. Likewise VIM does not require restrictive assumptions and specific transformations for nonlinear terms as required by other techniques. In this work, we implement VIM in solving three important equations, i.e., Regularized long wave equation, Burger’s equation and Modified Korteweg-de Vries equation.

2 Variational iteration method (VIM)

In order to describe the working rule for VIM, we suppose the nonlinear differential equation of the form

\[ L (u (x, t)) - N (u (x, t)) = \varphi (x, t) \]  

(1)
where $L$ is a linear operator, $N$ is a nonlinear operator and $\varphi(x,t)$ is nonhomogeneous analytic function. By using VIM, we construct a correction functional given as below

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\eta) [Lu_n(x,\eta) + N\tilde{u}_n(x,\eta) - \varphi_n(x,\eta)] \, d\eta$$  \hspace{1cm} (2)

where $\lambda$ is Lagrange multiplier, which can be find by using optimality and variational theory, the term $u_n(x,t)$ shows the nth approximation and $\tilde{u}_n(x,\eta)$ represents a restricted variation, which gives $\delta\tilde{u}_n(x,\eta) = 0$. The successive iterations provide an approximate solution $u_{n+1}(x,t)$, $n \geq 0$. Finally the series solution is given as

$$u(x,t) = \lim_{n \to \infty} u_n(x,t).$$  \hspace{1cm} (3)

## 3 Applications

### 3.1 RLW equation

The RLW equation is a nonlinear partial differential equation modeled for nonlinear dispersive waves. This equation has applications in different areas like, magnetohydrodynamics waves in plasma, ion-acoustic waves in plasma, pressure waves in liquid gas bubble mixtures, longitudinal dispersive waves in elastic rods and rotating flow down in tube. The theory of development of this equation is given in [7, 8].

Let us consider the equation

$$u_t + uu_x + u_{xx} - u_{xxt} = 0, \quad x \in \mathbb{R}$$  \hspace{1cm} (4)

along with initial condition

$$u(x,0) = u_0(x) = 3\beta \text{sech} (\gamma x),$$  \hspace{1cm} (5)

where $\beta > 0$ is a constant and $\gamma = \frac{1}{2} \left( \frac{\beta}{\beta + 1} \right)^{1/2}$. By using VIM we have the correction functional for Eq.(4), given as under

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\eta) \left[ \frac{\partial u_n(x,\eta)}{\partial \eta} + \frac{\partial \tilde{u}_n(x,\eta)}{\partial \eta} + \tilde{u}(x,\eta) \frac{\partial u_n(x,\eta)}{\partial x} - \frac{\partial^3 \tilde{u}(x,\eta)}{\partial x^2 \partial \eta} \right] \, d\eta.$$  \hspace{1cm} (6)

Using restricted variation, stationary conditions and integration by parts, the Lagrange multiplier for Eq.(6) is found to be $\lambda = -1$, so the Eq.(6) becomes

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[ \frac{\partial u_n(x,\eta)}{\partial \eta} + \frac{\partial \tilde{u}_n(x,\eta)}{\partial \eta} + \tilde{u}(x,\eta) \frac{\partial u_n(x,\eta)}{\partial x} - \frac{\partial^3 \tilde{u}(x,\eta)}{\partial x^2 \partial \eta} \right] \, d\eta.$$  \hspace{1cm} (7)

For $n=0,1,2$ in Eq.(7), we obtained the following approximants

$$u_1(x,t) = \frac{3\beta}{\cosh^4(\gamma x)} \left[ \cosh^2(\gamma x) + \gamma t \sinh(\gamma x) \cosh(\gamma x) + 3\beta \gamma t \sinh(\gamma x) \right]$$  \hspace{1cm} (8)

$$u_2(x,t) = \frac{3\beta}{2 \cosh^4(\gamma x)} \left[ \begin{array}{c} -12\gamma^2 t \sinh(\gamma x) \cosh^3(\gamma x) - 72\beta\gamma^3 t \sinh(\gamma x) \cosh^2(\gamma x) \\
+ 24\beta \gamma^3 t \sinh(\gamma x) \cosh^4(\gamma x) + 2\beta \gamma^3 t \sinh(\gamma x) \cosh^4(\gamma x) \\
+ 18 \beta^2 \gamma^4 t \sinh(\gamma x) \cosh^3(\gamma x) + 36 \beta^2 \gamma^4 t \sinh(\gamma x) \cosh^2(\gamma x) \\
+ 6 \beta^2 \gamma^4 t \sinh(\gamma x) \cosh^4(\gamma x) - 2\gamma^2 t^2 \cosh^5(\gamma x) - 18 \beta \gamma^2 t^2 \cosh^4(\gamma x) \\
- 4 \beta \gamma^3 t^2 \sinh(\gamma x) \cosh^5(\gamma x) - 30 \beta \gamma^3 t^2 \sinh(\gamma x) \cosh(\gamma x) \\
- 54 \beta \gamma^3 t^2 \sinh(\gamma x) \cosh^5(\gamma x) + 2 \cosh^6(\gamma x) \\
+ 2 \gamma t \sinh(\gamma x) \cosh^3(\gamma x) + 2 \gamma t \sinh(\gamma x) \cosh^5(\gamma x) \\
+ 27 \beta^2 \gamma^2 t^2 \cosh^4(\gamma x) + 12 \beta \gamma^2 t^2 \cosh^4(\gamma x) + 2 \gamma^2 t^2 \cosh^6(\gamma x) \end{array} \right]$$  \hspace{1cm} (9)

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Using restricted variation, stationary conditions and integration by parts, the Lagrange multiplier for Eq.(14) is found to

\[ u_\lambda (x,t) = \frac{\beta}{280 \cosh^{15}(\gamma x)} \left[ 840 \cosh^{14}(\gamma x) + 840 \gamma t \sinh(\gamma x) \cosh^{13}(\gamma x) + 10080 \gamma^5 t \sinh(\gamma x) \cosh^9(\gamma x) \\
-60480 \gamma^2 t \sinh(\gamma x) \cosh^{10}(\gamma x) - 30240 \gamma^4 t \sinh(\gamma x) \cosh^{10}(\gamma x) \\
+907200 \gamma^6 t \sinh(\gamma x) \cosh^8(\gamma x) + ... \right]. \]

The exact solution of Eq.(4) is given as

\[ u(x,t) = 3 \beta \sech(\gamma(x - (1 + \beta)t)). \]

Error Analysis:

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3.2 Burger’s equation

The Burger’s equation is a fundamental partial differential equation from fluid mechanics. The equation was first known to Forsyth (1906) and had been discussed by Bateman [13](1915). The first steady-state solution of Burger’s equation was given by Bateman. However it is named for Johannes Matinus Burgers [14] (1895-1981), due to his extensive work. Burger’s equation models many problems of a fluid flow, which involves either shocks or viscous dissipation and wave propagation problems [9], acoustics, gas dynamics, heat conduction, longitudinal elastic waves in an isotropic solid and turbulence. Let us consider the Burger’s equation together with initial condition given as

\[ u_t + uu_x - u_{xx} = 0, \quad u(x,0) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{x}{4} \right). \]

The correction functional for Eq.(12) is given as

\[ u_{\kappa+1}(x,t) = u_{\kappa} + \int_0^t \lambda(\zeta) \left[ \frac{\partial u_{\kappa}(x,\zeta)}{\partial \zeta} + u_{\kappa} \frac{\partial u_{\kappa}(x,\zeta)}{\partial x} - \frac{\partial^2 u_{\kappa}(x,\zeta)}{\partial x^2} \right] d\zeta. \]

Using restricted variation, stationary conditions and integration by parts, the Lagrange multiplier for Eq.(14) is found to be \( \lambda = -1 \). Substituting the value of \( \lambda \) in Eq.(14), we get the following iterative formula

\[ u_{\kappa+1} = u_{\kappa}(x,t) - \int_0^t \left[ \frac{\partial u_{\kappa}(x,\zeta)}{\partial \zeta} + u_{\kappa} \frac{\partial u_{\kappa}(x,\zeta)}{\partial x} - \frac{\partial^2 u_{\kappa}(x,\zeta)}{\partial x^2} \right] d\zeta. \]

Now we use the initial condition \( u(x,0) \) as an initial guess

\[ u_0(x,t) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{x}{4} \right). \]

For \( \kappa = 0,1,2 \) in Eq.(15), we get following results

\[ u_1(x,t) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{1}{4} x \right) - \frac{t}{16} \left[ -1 + \tanh^2 \left( \frac{1}{4} x \right) \right]. \]
\[ u_2(x,t) = \frac{1}{2} \frac{1}{2} \tanh \left( \frac{1}{4} x \right) - \frac{t}{16} \left[ -1 + \tanh^2 \left( \frac{1}{4} x \right) \right] \\
+ \frac{t^2}{128} \left[ \tanh \left( \frac{1}{4} x \right) - \tanh^3 \left( \frac{1}{4} x \right) \right] \\
+ \frac{t^3}{1536} \left[ \tanh \left( \frac{1}{4} x \right) - 2 \tanh^3 \left( \frac{1}{4} x \right) + \tanh^5 \left( \frac{1}{4} x \right) \right]. \tag{18} \]

\[ u_3(x,t) = \frac{1}{2} \left\{ 1 - \tanh^3 \left( \frac{1}{4} x \right) \right\} - \frac{t}{16} \left[ -1 + \tanh^2 \left( \frac{1}{4} x \right) \right] \\
+ \frac{t^2}{128} \left[ \tanh \left( \frac{1}{4} x \right) - \tanh^3 \left( \frac{1}{4} x \right) \right] \\
- \frac{t^3}{3072} \left[ 1 - 4 \tanh^2 \left( \frac{1}{4} x \right) + 3 \tanh^4 \left( \frac{1}{4} x \right) \right] \\
+ \frac{t^4}{98304} \left[ -5 - 10 \tanh \left( \frac{1}{4} x \right) + 35 \tanh^2 \left( \frac{1}{4} x \right) + 38 \tanh^3 \left( \frac{1}{4} x \right) \right] \\
+ \frac{t^5}{491520} \left[ -10 \tanh^2 \left( \frac{1}{4} x \right) + \frac{4}{3} \tanh \left( \frac{1}{4} x \right) - 22 \tanh^2 \left( \frac{1}{4} x \right) + \frac{2}{3} \tanh \left( \frac{1}{4} x \right) + 7 \tanh^3 \left( \frac{1}{4} x \right) \right] \\
+ \frac{t^6}{2359296} \left[ -10 \tanh \left( \frac{1}{4} x \right) + 7 \tanh^3 \left( \frac{1}{4} x \right) - 15 \tanh \left( \frac{1}{4} x \right) \right] \\
+ \frac{t^7}{66060288} \left[ -34 \tanh^3 \left( \frac{1}{4} x \right) + 21 \tanh \left( \frac{1}{4} x \right) - 5 \tanh \left( \frac{1}{4} x \right) \right]. \tag{19} \]

The exact solution of Eq. (12) is given as

\[ u(x,t) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{1}{4} \left( x - \frac{1}{2} t \right) \right). \tag{20} \]

**Error analysis**

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### 3.3 Modified KdV equation

The mKdV equation has number of applications in many fields of nonlinear science. It has been used to describe acoust[10], large amplitude internal waves in ocean, schottky barriers transmission lines and models of traffic over-crowding [11, 12] Let us consider the modified KdV equation with initial condition

\[ u_t + 6u^2u_x + u_{xxx} = 0, \quad x \in R, \quad u(x, 0) = 2 \text{sech} (2x). \tag{21} \]

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The correction functional for Eq.(21) is given as

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\zeta) \left[ \frac{\partial u_n(x, \zeta)}{\partial \zeta} + 6 \tilde{u}_n^2(x, \zeta) \frac{\partial \tilde{u}_n(x, \zeta)}{\partial x} + \frac{\partial^3 \bar{u}_n(x, \zeta)}{\partial x^3} \right] d\zeta. \]  

(23)

Using restricted variation, stationary conditions and integration by parts, the Lagrange multiplier for Eq.(23) is found to be \( \lambda = -1 \). Substituting the value of \( \lambda \) in Eq.(23), we get the following iterative formula

\[ u_{n+1} = u_n(x, t) - \int_0^t \left[ \frac{\partial u_n(x, \zeta)}{\partial \zeta} + 6 \tilde{u}_n^2(x, \zeta) \frac{\partial \tilde{u}_n(x, \zeta)}{\partial x} + \frac{\partial^3 \bar{u}_n(x, \zeta)}{\partial x^3} \right] d\zeta. \]  

(24)

Now we use the initial condition \( u(x, 0) \) as an initial guess

\[ u_0(x, t) = 2 \text{sech} (2x). \]  

(25)

For \( \kappa = 0, 1, 2 \) in Eq.(15), we get following results

\[ u_1(x, t) = \frac{2}{\cosh^2(2x)} [\cosh(2x) + 8t \sinh(2x)] \]  

(26)

\[ u_2(x, t) = \frac{2}{\cosh^2(2x)} \left[ \begin{array}{c} \cosh^6(2x) + 8t \sinh(2x) \cosh^5(2x) - 64t^2 \cosh^4(2x) + 32t^2 \cosh^6(2x) \\ -512t^4 \sinh(2x) \cosh(2x) + 3072t^3 \sinh(2x) \cosh^4(2x) \\ -18432t^4 \cosh^2(2x) + 6144t^4 \cosh^2(2x) + 12288t^4 \end{array} \right] \]  

(27)

\[ u_3(x, t) = \frac{2}{15015 \cosh^{22}(2x,t)} \left[ \begin{array}{c} 15015 \cosh^{21}(2x) + 120120t \sinh(2x) \cosh^{20}(2x) + 480480t^2 \cosh^{21}(2x) \\ -96096t^2 \cosh^{19}(2x) + 1281280t^3 \sinh(2x) \cosh^{20}(2x) \\ -7687680t^5 \sinh(2x) \cosh^{19}(2x) + \ldots \end{array} \right]. \]  

(28)

Exact solution of Eq.(21) is given as

\[ u(x, t) = 2 \text{sech} (2x - 8t). \]  

(29)

**Error analysis**

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### 4 Conclusion

In this work, we have successfully applied variational iteration method for solving three nonlinear equations namely (i) the nonlinear regularized long wave equation, (ii) the Burger’s equation and (iii) the modified KdV equation. We obtain all the solutions in form of a convergent series. Results based on VIM procedure are compared with the exact solutions. The agreement in both results is very good which shows the efficiency of VIM in solving nonlinear problems. An interesting state about VIM is that it works very effectively and with the small number of iterations, it can converge to the accurate solution of the problem. VIM is a computational method which introduces the solution in the form of a convergent series with elegantly computable terms. An additional advantage of the method is that VIM can also be used by those who do not have a deep knowledge of calculus of variations in pure mathematics, only some basic knowledge of advanced calculus can make one eligible to apply the method. The flexibility, convenience, accuracy and adaptation provided by the method have made it a strong candidate for approximate analytical solution.

LINS homepage: http://www.nonlinearscience.org.uk/
References


