Exact Traveling Wave Solution of Degasperis-Procesi Equation

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Abstract: In this paper, exact traveling wave solution of Degasperis–Procesi equation can be investigated via the first-integral method. By using the first-integral method which is based on the ring theory of commutative algebra, We construct exact traveling wave solution for Degasperis–Procesi equation, and the obtained solution agrees well with the previously known result.

Keywords: Degasperis–Procesi equation; first integral method; exact traveling wave solution

1 Introduction

Degasperis and Procesi [1] showed, by the use of the method of asymptotic integrability, that the PDE

\[ u_t - u_{txxt} + (b + 1) uu_x = bu_x u_{xx} + uu_{xxx}, \]  

(1)

can not be completely integrable unless \( b = 2 \) or \( b = 3 \). The case \( b = 2 \) is the following Camassa-Holm (CH) shallow water equation (see [2])

\[ u_t - u_{xxxt} + 3 uu_x = 2 u_x u_{xx} + uu_{xx}, \]  

(2)

which is well known to be integrable and to possess multi-peakon solutions. The case \( b = 3 \) is the following Degasperis-Procesi (DP) shallow water equation

\[ u_t - u_{xxxt} + 4 uu_x = 3 u_x u_{xx} + uu_{xxx}. \]  

(3)

Although, the DP equation (3) has a similar form to the CH equation (2), two equations are pretty different. For two equations, the different isospectral problem and the fact that there is no simple transformation of equation (3) into equation (2) imply that equation (3) is different from equation (2) in the integrable structures and the form of the conservation laws. The DP equation (3) is very interesting as it is an integrable shallow water equation and presents a quite rich structure. Degasperis, Holm and Hone [3-5] proved that equation (3) is integrable by constructing its lax pair, and admits multi-peakon solutions, and explained connection with a negative flow in the Kaup-kupershmidt hierarchy via a reciprocal transformation. Landmark and Szmigielski [6] presented an inverse scattering approach for computing \( n \)-peakon solutions of the equation (3). The blow-up phenomenon of equation (3) was discussed and the global existence of the solution was proved in [7]. In [8, 9] the Cauchy problem for equation (3) was demonstrated. Much work on the DP equation (3) has been done [10-17].

The investigation of traveling wave solutions plays an important role in understanding the physical phenomena of nonlinear systems. Various powerful methods, such as inverse scattering method, Hirota method, symmetry method, homogeneous balance method and Painleve method, etc, have been presented in finding the explicit exact solution, in particular, solitary wave solutions, of nonlinear evolution equations. In this paper, the first-integral method was used to investigate exact traveling wave solution of nonlinear equations. The first integral method, which is based on the ring theory of commutative algebra, was first proposed by Feng [18]. This method was further developed by the same author and some other mathematicians [19–20].

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in this paper, our main purpose is to focus our attention on traveling wave solutions of Eq. (3). The rest of this paper is organized as follows: in section 2, by using the first integral method, the exact peaked wave solution for Eq. (3) was established, which is full agreement with the previously known result. Finally, some conclusions are given in section 3.

2 The Degasperis-Procesi (DP) equation

Next we consider exact traveling wave solutions of the DP equation (3). Substituting the transformation \( u = \psi(\xi) \) with \( \xi = x - ct \) into Eq. (3), we obtain the following ordinary differential equations

\[
-c\psi' + c\psi''' + 4\psi\psi' = 3\psi^2 + \psi\psi''
\]  

(4)

Integrating (4) once, (4) becomes

\[
-c\psi' + c\psi'' + 2\psi^2 = \frac{3}{2}\psi^2 + \psi\psi'' - \frac{1}{2}\psi'^2
\]  

(5)

Let \( \varphi = x \) and \( \frac{dx}{dt} = y \), then Eq. (5) is equivalent to the following two-dimensional autonomous system

\[
\begin{aligned}
\frac{dx}{dt} & = y \\
\frac{dy}{dt} & = -cx + 2x^2 - y^2
\end{aligned}
\]  

(6)

Making the following transformation \( d\xi = (x - c)dt \), then system (6) becomes

\[
\begin{aligned}
\frac{dy}{dt} & = (x - c)y \\
\frac{dx}{dy} & = -cx + 2x^2 - y^2
\end{aligned}
\]  

(7)

We are applying the Division Theorem [21] to seeking the first integral to Eq. (3). Suppose that \( x = x(\tau) \) and \( y = y(\tau) \) are the nontrivial solutions to (7), and \( q(x, y) = \sum_{i=0}^{m} a_i(x) y^i \) is an irreducible polynomial in \( C[x, y] \) such that

\[ q(x, y) = \sum_{i=0}^{m} a_i(x) y^i = 0, \]  

(8)

where \( a_i(x) (i = 0, 1, 2, \cdots, m) \) are polynomials of \( x \) and \( a_m(x) \neq 0 \). (8) is also called the first integral to (7). We start our study by assuming \( m = 2 \) in (8). Note that \( \frac{dy}{dt} \) is a polynomial in \( x \) and \( y \), and \( q[x(\tau), y(\tau)] = 0 \) implies \( \frac{dy}{dt} \bigg|_{(7)} = 0 \). By the using of the Division Theorem, there exists a polynomial \( H(x, y) = g(x) + h(x) y \) in \( C[x, y] \) such that

\[
\frac{dy}{dt} \bigg|_{(7)} = \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \right) + \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \right) \]  

(9)

On comparing the coefficients of \( y^i (i = 3, 2, 1, 0) \) on both sides of (9), we have

\[ y^3 : (x - c)\hat{a}2(x)2 - 2a2(x) = a2(x)h(x) \]  

(10)

\[ y^2 : (x - c)\hat{a}1(x) - a1(x) = a2(x)g(x) + a1(x)h(x) \]  

(11)

\[ y^1 : (x - c)\hat{a}0(x) + 2(-cx + 2x^2)2a2(x) = a1(x)g(x) + a0(x)h(x) \]  

(12)

\[ y^0 : (-cx + 2x^2)2a1(x) = a0(x)g(x) \]  

(13)

Since \( a_i(x) (i = 0, 1, 2) \) are polynomials, from (10) we deduce that \( h(x) \) is a constant. For simplification, taking \( h(x) = A \) and solving (10), we have \( a2(x) = R1(x - c)^{A+2} \), where \( R1 \) is an integration constant. From (11), (12) and (13) we have \( a1(x) = (x - c)^{A+1} \left[ \int R1g(x)dx + R2 \right] \), \( deg a1(x) = deg g(x) + A + 2 \) and \( deg a0(x) = A + 4 \). Balancing the degrees of \( deg g(x) \) from (13), we conclude that \( deg g(x) = 1 \). We will deduce a contradiction with (13) if \( deg g(x) \neq 1 \).
Assuming \( g(x) = A_1 x + A_0 \) with \( A_1 \neq 0 \), from (10), (11) and (12) we have \( a_0(x) = R_1(x - c)^4 \left[ \frac{1}{3} A_1^2 - 1 \right] x^4 + \left( \frac{1}{2} A_1 A_0 + 2c \right) x^3 + \left( \frac{1}{2} A_1 R_1 + A_0^2 - 2c^2 \right) x^2 + A_0 R_2 x + R_3 \]. By substituting \( a_2(x), a_1(x), a_0(x) \) and \( g(x) \) into Eq. (13) and equating the coefficients of \( x^i \) (\( i = 5, 4, 3, 2, 1, 0 \)) to zero, we obtain the corresponding system of nonlinear algebraic equations. Solving the system simultaneously, we get the solution set

\[
A_1 = \pm 4, \quad A_0 = \mp 2c, \quad R_2 = 0, \quad R_3 = 0. \quad (14)
\]

Now, taking the solution set (14) into account, Eq. (8) becomes

\[
y^2 \pm 2xy + x^2 = 0 \quad (15)
\]

which is a first integral of Eq. (7). Solving Eq. (15), we get

\[
y = \mp x, \quad (16)
\]

Finally, combining Eq. (7) with Eq. (16) and changing to the original variables, we obtain traveling wave solutions to Eq. (3) as

\[
u = \varphi(\xi) = ce^{-|\xi|} = ce^{-|x-ct|}. \quad (17)
\]

The result is identical with one another in [3], the peakon expressed by (17) are showed in Fig. 1 (under some parameter conditions \( c = 1 \)).

![Figure 1: The profile of the peakon](image)

If assuming \( m = 1 \) in (8), the argument is identical, so we omit it. If assuming \( m = 3, 4 \) in (8), respectively, using the similar arguments as earlier we obtain that (7) does not have any first integral in the form (8). We have no need of discussion for the cases \( m \geq 5 \) due to the fact that the polynomial equation with the degree greater than or equal to 5 is generally not solvable.

### 3 Conclusions

In this paper, we study exact traveling wave solution of Degasperis–Procesi equation by using the first-integral method, and the obtained solution agrees well with the previously known one in the literature. We can see that the first-integral method used in this paper is very effective and can be widely applied to nonlinear problems.

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References


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