Stability Analysis and Optimal Vaccination Strategies for an SIR Epidemic Model with a Nonlinear Incidence Rate

Hassan Laarabi1 *, Abdelhadi Abta2, Mostafa Rachik1, Jamal Bouyaghroumi1, El Houssine Labriji1
1 Université Hassan II Mohammadia Faculté des Sciences Ben M’Sik Département de Mathématiques et Informatique
Casablanca, Morocco
2 Université Chouaib Doukkali Faculté des Sciences Département de Mathématiques et Informatique El Jadida, Morocco
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Abstract: An SIR epidemic model with saturated incidence rate and a vaccination program is formulated, where the susceptibles are assumed to satisfy the logistic equation. The incidence term is of saturated form with the infected individuals. First, we have discussed the existence and the stability of both the disease free and endemic equilibrium. Second, the impact of vaccination in reducing $R(u)$ is tackled. Then, to achieve control of the disease, a control problem is formulated and it is shown that an optimal control exists for our model. The optimality system is derived and solved. Finally, numerical simulations are performed to illustrate and verify the analytical results.

Keywords: SIR model; Optimal Control; Vaccination; Stability.

1 Introduction

Mathematical modeling is of considerable importance in the study of epidemiology because it may provide understanding of the underlying mechanisms which influence the spread of diseases and may suggest control strategies. Mathematical models take into account main factors that govern the development of a disease, such as transmission and recovery rates, and predict how the disease will spread over a period of time. In recent years, many attempts have been made to develop realistic mathematical models for investigating the transmission dynamics of infectious diseases, and the asymptotic behaviors of these epidemic models are studied [39]. A key role is played by the incidence rate, namely a function describing the mechanism of transmission of the disease. Generally, such a function depends on both the susceptible and infected classes. In many epidemic models, the bilinear incidence is frequently used [18,34]. This incidence rate is based on the law of mass action. More specifically, if $S$ and $I$ are the numbers of susceptible and infected individuals in a population, and if $\beta$ is the per capita contact rate, then it is assumed that the infection spreads with the rate $\beta SI$. This contact law is more appropriate for communicable diseases such as influenza, etc., but not for sexually transmitted diseases. Furthermore, the mass action incidence may not yield appropriate results for a variety of reasons. For example, the underlying assumption of homogeneous mixing may not always hold. In this case, the necessary population structure and the heterogeneous mixing may be incorporated into a model with a specific form of non-linear transmission. The bilinear incidence rate can also be affected by the control policies adopted by public health authorities. For instance, during the outbreak of SARS in 2003, protection measures such as closing schools, closing restaurants, postponing conferences, isolating the infected, etc., were taken by the Chinese government [37] and greatly reduced the contact rate per unit time. Therefore, a number of non-linear incidence rates was suggested by researchers. Thus, after studying the cholera epidemic spread in Bari in 1973, Capasso and Serio[9] introduced the saturated incidence $Sg(I)$ into epidemic models. This is important because the number of effective contacts between infected individuals and susceptible individuals may be saturated at high infection levels due to the crowding of infected individuals or due to the protection measures by the susceptible individuals. To incorporate the effect of the behavioral changes of the susceptible individuals, Liu et al.[33] proposed the general incidence rate $\frac{\beta SI}{1+qI}$, where $p$ and $q$ are positive constants and $k$ is nonnegative. The special cases
2.1 Stability analysis of SIR model

where \( \delta \) is the death rate of recovered populations.

The Epidemic Model with Vaccine Dynamics

The model to be studied takes the following form

\[
\begin{align*}
\dot{S}(t) &= rS(t)(1 - \frac{S(t)}{K}) - \frac{\alpha S(t)I(t)}{1 + aI(t)} - uS(t), \quad S(0) = S_0 \geq 0 \\
\dot{I}(t) &= \frac{\alpha S(t)I(t)}{1 + aI(t)} - (\beta + \gamma)I(t), \quad I(0) = I_0 \geq 0 \\
\dot{R}(t) &= \gamma I(t) - \delta R(t) + uS(t), \quad R(0) = R_0 \geq 0
\end{align*}
\]

The model has a susceptible group designated by \( S \), an infected group \( I \), and a recovered group \( R \) with permanent immunity. \( r \) is the intrinsic growth rate of susceptible, \( K \) is the carrying capacity of the susceptible in the absence of infective, \( \alpha \) is the maximum values of per capita reduction rate of \( S \) due to \( I \), \( a \) is half saturation constants, \( u \) is the vaccination coverage of susceptible, \( \beta \) is the death rate of infected populations, \( \gamma \) is the natural recover rate from infection, \( \delta \) is the death rate of recovered populations.

2.1 Stability analysis of SIR model

In this section, we discuss the local stability of the disease-free equilibria and an endemic equilibrium of system (1). System (1) always has two disease-free equilibria

\[
E_0 = (0, 0, 0)
\]

and

\[
E_1 = \left( \frac{K(r - u)}{r}, 0, \frac{uK(r - u)}{r\delta} \right)
\]

exists if \( r > u \). Further, if

\[
\mathcal{R}(u) := \frac{K\alpha(r - u)}{r(\beta + \gamma)} > 1,
\]

system (1) admits a unique endemic equilibrium \( E^* = (S^*, I^*, R^*) \), with

\[
S^* = \frac{(\beta + \gamma)(1 + aI^*)}{\alpha},
\]

\[
I^* = \frac{[aK\alpha(r - u) - 2a(\beta + \gamma) - \alpha^2 K] + \sqrt{\Delta}}{2ra(\beta + \gamma)},
\]

\[
R^* = \frac{\gamma}{\delta} I^* + \frac{u}{\delta} S^*,
\]

\[
\Delta = [aK\alpha(r - u) - 2a(\beta + \gamma) - \alpha^2 K]^2 + 4a^2(\beta + \gamma)(K\alpha(r - u) - r(\beta + \gamma)).
\]
Now let us start to discuss the local behavior of the equilibria points $E_0, E_1 = \left( \frac{K(r-u)}{r}, 0, \frac{uK(r-u)}{r^2} \right)$ and $E^* = \left( S^*, I^*, R^* \right)$ of the system.

Since the first and the second equation of system (1) are independent of $R$, we consider only these two equations for stability analysis.

The general variational matrix $M$ for the system (1) is given as follows:

$$ M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} $$

where $a_{11} = r(1 - S) - \frac{\alpha I}{1 + aI} - u - \frac{K}{r} S, a_{12} = -\frac{2S}{1 + aI^*}, a_{21} = \frac{\alpha S}{1 + aI^*}, a_{22} = \frac{\alpha S}{1 + aI^*} - (\beta + \gamma)$.

The variational matrix $M_0$ corresponding to $E_0$ is given by

$$ M_0 = \begin{bmatrix} -(r - u) & 0 \\ 0 & -(\beta + \gamma) \end{bmatrix} $$

So, when $r < u$, becomes stable and in that case disease free equilibrium does not exist.

The variational matrix $M_1$ corresponding to $E_1$ is given by

$$ M_1 = \begin{bmatrix} -(r - u) & -\frac{\alpha K(r-u)}{r} \\ 0 & -(\beta + \gamma) \end{bmatrix} $$

Here the disease free equilibrium $E_1$ is locally asymptotically stable as long as the reproduction number $R(u) < 1$ and unstable when $R(u) > 1$. So, the disease-free equilibrium is asymptotically stable in the absence of the endemic equilibrium.

The variational matrix $M_2$ corresponding to $E^*$ is given by

$$ M_2 = \begin{bmatrix} -\frac{rS^*}{K} & \frac{\alpha S^*}{1 + aI^*} \\ \frac{\alpha I^*}{1 + aI^*} & -\frac{\alpha S^*}{(1 + aI^*)^2} \end{bmatrix} $$

Here, using the fact that $\beta + \gamma = \frac{\alpha S^*}{1 + aI^*}$, we have

$$ \beta + \gamma = \frac{\alpha S^*}{1 + aI^*} = \frac{\alpha aS^*I^*}{(1 + aI^*)^2}. $$

Therefore, it’s follow that

$$ \text{det}(M_2) = \frac{rS^*}{K} \left[ (\beta + \gamma) - \frac{\alpha S^*}{(1 + aI^*)^2} \right] + \frac{\alpha^2 S^* I^*}{(1 + aI^*)^2} > 0 $$

and

$$ \text{Trace}(M_2) = -\frac{rS^*}{K} - \left[ (\beta + \gamma) - \frac{\alpha S^*}{(1 + aI^*)^2} \right] < 0. $$

Hence the endemic equilibrium $E^*$ is locally asymptotically stable for $R(u) > 1$.

In Fig.1 and Fig.2, we give some examples of numerical simulation where $E^*$ is locally asymptotically stable.

### 2.2 Vaccine induced reproduction number

Already we have defined the vaccine induced reproduction number

$$ R(u) = \frac{K\alpha(r-u)}{r(\beta + \gamma)}. \quad (2) $$
Now, \( R_0'(u) = \frac{-K\alpha}{(\beta + \gamma)} < 0 \), and hence \( R(u) \) is a decreasing function of \( u \). This shows the impact of vaccination in reducing the vaccine induced reproduction number. Also in the absence of vaccination i.e when, \( u = 0 \),

\[
R(0) = R_0 = \frac{K\alpha}{(\beta + \gamma)}
\]  

From (2) it is clear that the introduction of vaccination implies \( R(u) \leq R_0 \) for \( 0 < u < r \) and consequently, if \( R_0 < 1 \) then \( R(u) < 1 \). Therefore, \( E_1 \) i.e. the disease-free equilibrium is asymptotically stable if \( R(u) < 1 \). Also for \( R_0 = 1, R(u) \leq 1 \) and in that case the disease-free equilibrium is asymptotically stable.

### 3 The Optimal Vaccination

Optimal control techniques are of great use in developing optimal strategies to control various kinds of diseases. To solve the challenges of obtaining an optimal vaccination strategy, we use optimal control theory.

We consider the control variable \( u(t) \in U_{ad} \) to be the percentage of susceptible individuals being vaccinated per unit of time. Here

\[
U_{ad} = \{u | u(t) \text{ is measurable}, 0 \leq u(t) \leq u_{max} < \infty, t \in [0, t_{end}] \}
\]

indicates an admissible control set. Now, we consider an optimal control problem to minimize the objective functional

\[
J(u) = \int_0^{t_{end}} [A_1S(t) + A_2I(t) + \frac{1}{2}\tau u^2(t)]dt
\]  

subject to system (1). Here \( A_1 \) and \( A_2 \) are positive constants to keep a balance in the size of \( S(t) \) and \( I(t) \), respectively. The square of the control variable reflects the severity of the side effects of the vaccination. In the objective functional, \( \tau \) is a positive weight parameter which is associated with the control \( u(t) \). The objective of our work is to minimize the infected and susceptible individuals and to maximize the total number of recovered individual by using possible minimal control variables \( u(t) \).

#### 3.1 Existence of an Optimal Control

For existence, we consider a control system (1) with initial conditions. Then, we rewrite our system (1) in the following form:

\[
\phi_t = B\phi + F(\phi)
\]  

where

\[
\phi = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}
\]
We set

\[
B = \begin{bmatrix}
 r - u(t) & 0 & 0 \\
0 & -(\beta + \gamma) & 0 \\
u(t) & \gamma & -\delta \\
\end{bmatrix}
\]

\[
F(\phi) = \begin{bmatrix}
 -rS^2(t)K & \frac{aS(t)I(t)}{1+aI(t)} \\
\frac{aS(t)I(t)}{1+aI(t)} & -\frac{\alpha S(t)I(t)}{1+aI(t)} \\
0 & 0 \\
\end{bmatrix}
\]

and \( \phi_t \) denote derivative of \( \phi \) with respect to time \( t \). Equation (5) is a non-linear system with a bounded coefficient.

We set

\[
D(\phi) = B\phi + F(\phi)
\]

(6)

Now,

\[
F(\phi_1) - F(\phi_2) = \begin{bmatrix}
\frac{r}{\gamma} \left( S_2^2(t) - S_1^2(t) \right) + \frac{aS_2(t)I_2(t)}{1+aI_2(t)} - \frac{aS_1(t)I_1(t)}{1+aI_1(t)} + \frac{aS_1(t)I_1(t)}{1+aI_1(t)} - \frac{aS_1(t)I_2(t)}{1+aI_2(t)} \\
0 \\
\end{bmatrix}
\]

Therefore

\[
\begin{align*}
&|F(\phi_1) - F(\phi_2)| = \frac{r}{\gamma} \left| S_2^2(t) - S_1^2(t) \right| + 2\alpha \left| S_2(t)I_2(t) - S_1(t)I_1(t) + aI_1(t)I_2(t) \right| + \frac{aS_1(t)I_1(t)}{1+aI_1(t)} - \frac{aS_1(t)I_2(t)}{1+aI_2(t)} \\
\leq & \frac{r}{\gamma} \left| S_2^2(t) - S_1^2(t) \right| + 2\alpha \left| S_2(t)I_2(t) - S_1(t)I_1(t) + aI_1(t)I_2(t) \right| + \frac{aS_1(t)I_1(t)}{1+aI_1(t)} - \frac{aS_1(t)I_2(t)}{1+aI_2(t)} \\
\leq & \frac{r}{\gamma} \left| S_2^2(t) - S_1^2(t) \right| + 2\alpha \left| S_2(t)I_2(t) - S_1(t)I_1(t) + aI_1(t)I_2(t) \right| + \frac{aS_1(t)I_1(t)}{1+aI_1(t)} - \frac{aS_1(t)I_2(t)}{1+aI_2(t)} \\
\leq & \frac{r}{\gamma} \left| S_2(t) - S_1(t) \right| \left| S_2(t) + S_1(t) \right| + 2\alpha \left| I_2(t) + aI_1(t)I_2(t) \right| + 2\alpha \left| S_1(t) \left| I_2(t) - I_1(t) \right| \\
\leq & \frac{r}{\gamma} \left| S_2(t) - S_1(t) \right| \left| S_2(t) + S_1(t) \right| + 2\alpha \left| I_2(t) + aI_1(t)I_2(t) \right| + 2\alpha \left| S_1(t) \left| I_2(t) - I_1(t) \right| \\
\leq & \frac{2 \mu}{m} \left( \frac{r}{\gamma} + 2\alpha \left( \frac{a}{m^2} + \frac{a^2}{m^3} \right) \right) \left| S_1(t) - S_2(t) \right| + 2\alpha \left| I_2(t) \right| \left| I_2(t) - I_1(t) \right| \\
\leq & M \left( \left| S_1(t) - S_2(t) \right| + \left| I_1(t) - I_2(t) \right| \right),
\end{align*}
\]

where

\[
M = 2 \mu \left( r \left( \frac{1}{\gamma} + \alpha \left( \frac{a}{m^2} \right) \right) \right)
\]

Also, we get

\[
|D(\phi_1) - D(\phi_2)| \leq V |\phi_1 - \phi_2|, \text{ where } V = \max \left( M, \|B\| \right) < \infty.
\]

Thus, it follows that the function \( D \) is uniformly Lipschitz continuous. From the definition of the control \( u(t) \) and the restriction on \( S(t), I(t) \) and \( R(t) \geq 0 \), we see that a solution of the system (5) exists (Birkhoff and Rota, 1989).
Let us go back to the optimal control problem, (1) and (4). In order to find an optimal solution, first we find the Lagrangian and Hamiltonian for the optimal control problem (1) and (4). In fact, the Lagrangian of the optimal problem is given by \( L(S, I, u) = A_1S(t) + A_2I(t) + \frac{1}{2} \tau u^2(t) \).

We seek the minimal value of the Lagrangian. To accomplish this, we define the Hamiltonian \( H \) for the control problem:

\[
H(S, I, R, u, \lambda_1, \lambda_2, \lambda_3, t) = L(S, I, u) + \lambda_1(t) \frac{dS(t)}{dt} + \lambda_2(t) \frac{dI(t)}{dt} + \lambda_3(t) \frac{dR(t)}{dt},
\]

where \( \lambda_1(t), \lambda_2(t) \) and \( \lambda_3(t) \) are the adjoint functions to be determined suitably.

**Theorem 1** There exists an optimal control \( u^*(t) \) such that

\[
J(u^*(t)) = \min_{u \in U_{ad}} J(u(t))
\]

subject to the control system (1) with initial conditions.

**Proof.** To prove the existence of an optimal control we use the result in (Lukes, 1982). Note that the control and the state variables are nonnegative values. In this minimizing problem, the necessary convexity of the objective functional in \( u(t) \) is satisfied.

The control space

\[
U_{ad} = \{ u | u(t) \text{ is measurable, } 0 \leq u(t) \leq u_{max} < \infty, t \in [0, t_{end}] \}
\]

is also convex and closed by definition. The optimal system is bounded which determines the compactness needed for the existence of the optimal control. In addition, the integrand in the functional (4), \( A_1S(t) + A_2I(t) + \frac{1}{2} \tau u^2(t) \) is convex on the control \( u(t) \). Also, we can easily see that, there exist a constant \( \rho > 1 \), positive numbers \( \omega_1 \) and \( \omega_2 \) such that

\[
J(u(t)) \geq \omega_2 + \omega_1 (|u|^2) \hat{u}_t.
\]

We conclude that there exists an optimal control. \( \blacksquare \)

### 3.2 Characterization of the optimal control

In the previous section we showed the existence of an optimal control which maximize the functional (4) subject to system (1). In order to derive the necessary conditions for this optimal control, we apply Pontryagin’s maximum principle to the Hamiltonian

\[
H(t, x(t), u(t), \lambda(t)) = f(t, x(t), u(t)) + \lambda(t)g(t, x(t), u(t)).
\]

If \((x^*(t), u^*(t))\) is an optimal solution of an optimal control problem, then there exists a non-trivial vector function \( \lambda(t) = (\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)) \) satisfying the following equalities:

\[
x'(t) = \frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x}, \quad 0 = \frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial u}, \quad \lambda'(t) = -\frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial \lambda},
\]

where \( \cdot' \) denotes the derivative with respect to time \( t \). Which gives after derivation

\[
\begin{align*}
    u^*(t) &= 0 & & \text{if } \frac{\partial H}{\partial u} < 0 \\
    u^*(t) \in [0, u_{max}] & & \text{if } \frac{\partial H}{\partial u} = 0 \\
    u^*(t) &= u_{max} & & \text{if } \frac{\partial H}{\partial u} > 0
\end{align*}
\]

Now, we apply the necessary conditions to the Hamiltonian \( H(7) \).

**Theorem 2** Let \( S^*(t) \), \( I^*(t) \) and \( R^*(t) \) be optimal state solutions with associated optimal control variable \( u^*(t) \) for the optimal control problem (1) and (4). Then, there exist adjoint variables \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) that satisfy

\[
\begin{align*}
    \frac{d\lambda_1(t)}{dt} &= -A_1 - \lambda_1(t) \left( r - \frac{2S^*}{k} - \frac{\alpha S^*}{1 + aT_f} - u^*(t) \right) - \lambda_2(t) \left( \frac{\alpha S^*}{1 + aT_f} \right) - \lambda_3(t) u^*(t) \\
    \frac{d\lambda_2(t)}{dt} &= -A_2 + \lambda_1(t) \left( \frac{\alpha S^*}{1 + aT_f} \right) - \lambda_2(t) \left( \frac{\alpha S^*}{1 + aT_f} - (\beta + \gamma) \right) - \lambda_3(t) \gamma \\
    \frac{d\lambda_3(t)}{dt} &= \lambda_3(t) \delta
\end{align*}
\]
with transversality conditions

\[ \lambda_i(t_{\text{end}}) = 0, \quad i = 1, 2, 3. \]  

(10)

**Furthermore, the optimal control** \( u^*(t) \) **is given by**

\[ u^*(t) = \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_3(t))S^*(t)}{\tau}, u_{\text{max}} \right), 0 \right). \]

(11)

**Proof.** We use the Hamiltonian (7) in order to determine the adjoint equations and the transversality conditions. From setting \( S(t) = S^*(t), I(t) = I^*(t) \) and \( R(t) = R^*(t) \), and differentiating the Hamiltonian with respect to \( S, I \) and \( R \), respectively, we obtain (2). And by using the optimality conditions we find

\[
\frac{\partial H}{\partial u} = \tau u^*(t) - \lambda_1(t)S^* + \lambda_3(t)S^* = 0, \quad \text{at } u = u^*(t)
\]

which gives

\[ u^*(t) = \frac{\left( \lambda_1(t) - \lambda_3(t) \right)S^*(t)}{\tau}. \]

Using the property of the control space, we obtain

\[
\begin{cases}
  u^*(t) = 0, & \text{if } \frac{\lambda_1(t) - \lambda_3(t)S^*(t)}{\tau} \leq 0 \\
  u^*(t) = \frac{\lambda_1(t) - \lambda_3(t)S^*(t)}{\tau}, & \text{if } 0 < \frac{\lambda_1(t) - \lambda_3(t)S^*(t)}{\tau} < u_{\text{max}} \\
  u^*(t) = u_{\text{max}}, & \text{if } \frac{\lambda_1(t) - \lambda_3(t)S^*(t)}{\tau} \geq u_{\text{max}}
\end{cases}
\]

So the optimal control is characterized as

\[ u^*(t) = \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_3(t))S^*(t)}{\tau}, u_{\text{max}} \right), 0 \right). \]

The optimal control and the state are found by solving the optimality system, which consists of the state system (1), the adjoint system (2), initial conditions at \( t = 0 \), boundary conditions (10), and the characterization of the optimal control (11).

So the optimality system is given by

\[
\begin{align*}
  \dot{S}^*(t) &= rS^*(t)(1 - \frac{S^*(t)}{k}) - \frac{aI^*(t)\tau}{1 + aI^*(t)} - \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_3(t))S^*(t)}{\tau}, u_{\text{max}} \right), 0 \right)S^*(t), \\
  \dot{I}^*(t) &= \frac{\alpha S^*(t)I^*(t)}{1 + aI^*(t)} - (\beta + \gamma)I^*(t), \\
  \dot{R}^*(t) &= \gamma I^*(t) - \delta R^*(t) + \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_3(t))S^*(t)}{\tau}, u_{\text{max}} \right), 0 \right)S^*(t), \\
  \lambda_1(t) &= -A_1 - \lambda_1(t) \left( r(1 - \frac{2S^*(t)}{k}) - \frac{aI^*(t)}{1 + aI^*(t)} - \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_3(t))S^*(t)}{\tau}, u_{\text{max}} \right), 0 \right) \right) \\
  &\quad - \lambda_2(t) \left( \frac{aI^*(t)}{1 + aI^*(t)} \right) - \lambda_3(t) \max \left( \min \left( \frac{(\lambda_1(t) - \lambda_3(t))S^*(t)}{\tau}, u_{\text{max}} \right), 0 \right), \\
  \lambda_2(t) &= -A_2 + \lambda_1(t) \left( \frac{\alpha S^*(t)}{1 + aI^*(t)} \right) - \lambda_2(t) \left( \frac{\alpha S^*(t)}{1 + aI^*(t)} \right) - (\beta + \gamma) - \lambda_3(t) \gamma, \\
  \lambda_3(t) &= -\lambda_3(t) \delta
\end{align*}
\]

with \( \lambda_1(t_{\text{end}}) = 0, \lambda_2(t_{\text{end}}) = 0, \lambda_3(t_{\text{end}}) = 0, S(0) = S_0, I(0) = I_0, R(0) = R_0 \)
4 Numerical illustration

4.1 The improved GSS1 method

The resolution of the optimality system is created improving the Gauss Seidel – like implicit finite-difference method developed by Guebel et al. (2001) and denoted GSS1 method. It consists on discretizing the interval $[t_0, t_{end}]$ by the points $t_k = kl + t_0 (k=0,1, ..., n)$, where $l$ is the time step. Next, we define the state and adjoint variables $S(t)$, $I(t)$, $R(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ and the control $u(t)$ in terms of nodal points $S^k$, $I^k$, $R^k$, $\lambda_1^k$, $\lambda_2^k$, $\lambda_3^k$ and $u^k$ with $S^0$, $I^0$, $R^0$, $\lambda_1^0$, $\lambda_2^0$, $\lambda_3^0$ and $u^0$ as the state and adjoint variables and the controls at initial time $t_0$. $S^n$, $I^n$, $R^n$, $\lambda_1^n$, $\lambda_2^n$, $\lambda_3^n$ and $u^n$ as the state and adjoint variables and the controls at final time $t_{end}$.

As it is well known, the approximation of the time derivative by its first-order forward-difference is given, for the first state variable $S(t)$, by $\frac{dS(t)}{dt} = \lim_{t \to 0} \frac{S(t+t)-S(t)}{t}$.

We use the scheme developed by Guebel et al. to adapt it to our case as following:

$$\frac{S^{k+1} - S^k}{l} = rS^{k+1}\left(1 - \frac{S^{k+1}}{k}\right) - \frac{\alpha S^{k+1} I^k}{1 + a I^k} - u^k S^{k+1}$$  \hspace{0.5cm} (13)
$$\frac{I^{k+1} - I^k}{l} = \frac{\alpha S^{k+1} I^{k+1}}{1 + a I^{k+1}} - (\beta + \gamma) I^{k+1}$$  \hspace{0.5cm} (14)
$$\frac{R^{k+1} - R^k}{l} = \gamma I^{k+1} - \delta R^{k+1} + u^k S^{k+1}$$ \hspace{0.5cm} (15)

By applying an analogous technology, we approximate the time derivative of the adjoint variables by their first-order backward-difference and we use the appropriated scheme as follows:

$$\frac{\lambda_1^{n-k} - \lambda_1^{n-k-1}}{l} = -A_1 - \lambda_1^{n-k-1}\left(r\left(1 - \frac{2 S^{k+1}}{k}\right) - \frac{\alpha I^{k+1}}{1 + a I^{k+1}} - u^k\right) - \lambda_2^{n-k}\left(\frac{\alpha I^{k+1}}{1 + a I^{k+1}}\right) - \lambda_3^{n-k} u^k$$  \hspace{0.5cm} (16)
$$\frac{\lambda_2^{n-k} - \lambda_2^{n-k-1}}{l} = -A_2 + \lambda_1^{n-k-1}\left(\frac{\alpha S^{k+1}}{(1 + a I^{k+1})^2}\right) - \lambda_2^{n-k-1}\left(\frac{\alpha S^{k+1}}{(1 + a I^{k+1})^2} - (\beta + \gamma)\right) - \lambda_3^{n-k} \gamma$$ \hspace{0.5cm} (17)
$$\frac{\lambda_3^{n-k} - \lambda_3^{n-k-1}}{l} = \lambda_3^{n-k-1} \delta$$ \hspace{0.5cm} (18)

4.2 Numerical results

We use the following data:

$$S_0 = 50; I_0 = 35; R_0 = 15; a = 2.3; \alpha = 1.49; \tau = 50; r = 2.5$$
$$A_1 = 0.1; A_2 = 0.5; \gamma = 0.611; \beta = 0.031; \delta = 0.01; k = 100$$

We obtain the following figures

**Figure 3:** The plot represents the susceptible populations both with control and without control.

**Figure 4:** The plot represents the infected populations both with control and without control.
When viewing the graphs, remember that each of the individuals without control is marked by dashed black lines. The control individuals are marked by blue lines. The recovered individuals, with control, increase rapidly while the susceptible and infected individuals, with control, rapidly decrease. This is due to the vaccination strategy.

5 Conclusion

In this paper we do not consider any special disease but our aim is to set up an optimal control problem relative to epidemic model with saturated incidence rate and saturated treatment function, so it is to minimize the infected and susceptible populations and to maximize recovered populations.

A comparison between optimal control and no control is presented. It is easy to see that the optimal vaccination is much more effective for reducing the number of infected and susceptible individuals and increasing the number of recovered individuals. In order to illustrate the overall picture of the epidemic, the numbers of infected, susceptible and recovered individuals under the optimal control and no control are shown in figures.

Appendix

Let us consider the function \( N(t) = S(t) + I(t) + R(t) \). Now the time derivative of \( N(t) \) along solution of (1) is

\[
\frac{dN(t)}{dt} = \frac{dS(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{k}\right) - \delta R(t) - \beta I(t)
\]

Therefore,

\[
\frac{dN(t)}{dt} + cN(t) = \left[r\left(1 - \frac{S(t)}{k}\right) + c\right]S(t) + (c - \delta)R(t) + (c - \beta)I(t).
\]

Let \( m = \min(\beta, \delta) \) and \( c = m \)

therefore

\[
\frac{dN(t)}{dt} + mN(t) \leq \left[r\left(1 - \frac{S(t)}{k}\right) + m\right]S(t).
\]

or,

\[
\frac{dN(t)}{dt} + mN(t) \leq \mu, \text{ where } \mu = \frac{k(r + m)^2}{4r}.
\]

Applying the theory of differential inequality (Birkhoff and Rota, 1982) we obtain

\[
0 < N(t) < (1 - e^{-mt})\frac{\mu}{m} + N(0)e^{-mt}
\]

Which, upon letting \( t \to +\infty \), yield \( 0 < N(t) < \frac{\mu}{m} \)

So, we have that all the solutions of the systems (1) that starts in \( \mathbb{R}^3_+ \) are confined to the region

\[
\Omega = \{(S(t), I(t), R(t)) \in \mathbb{R}^3_+/S(t) + I(t) + R(t) \leq \frac{\mu}{m}\}
\]
References


