Assessment of Homotopy Analysis Method and Modified Homotopy Perturbation Method for Strongly Nonlinear Oscillator

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(Received 4 May 2013, accepted 10 December 2013)

Abstract: Some recently developed analytical methods namely; homotopy analysis method, homotopy perturbation method and modified homotopy perturbation method are applied successfully for solving strongly nonlinear oscillators. The analytical results obtained by using HAM are compared with those of HPM, mHPM as well as with numerical results. In this study, different examples with strong nonlinearity are assessed in detail to illustrate the effectiveness and convenience of the methods. Comparing the three methods, the attention is focused on the accuracy of the results and applicability of the methods. It has been observed that the solution using HAM which is valid in a small region is more accurate than the solutions obtained using other two methods. Whereas mHPM works very well for a wide range and it is computationally more efficient.

Keywords: Nonlinear equation; Homotopy analysis method; Homotopy perturbation method; Analytic solution; Nonlinear oscillator

1 Introduction

Modeling of natural phenomena in science and engineering mostly leads to nonlinear problems. Solutions of these nonlinear problems are more difficult compared to linear ones, especially through the analytical approach. Therefore, endless efforts are devoted either to find ways to solve them or to decrease the error in the solutions.

At present, there exist few well known analytical approaches for solving nonlinear problems, viz. perturbation technique; non-perturbation techniques (e.g. Lyapunov’s small parameter method, δ-expansion method, Adomian’s decomposition method), etc. Perturbation technique is widely applied in science and engineering because in this technique the original nonlinear equations are replaced by an infinite number of linear sub-problems which are easy to solve. Unfortunately, not every nonlinear even if there exists a small parameter, the sub-problem might have no solutions or might be rather complicated so that only a few of the sub-problems can be solved. Thus, it is not guaranteed that one can always apply perturbation approximations efficiently for a given nonlinear problem. Many new techniques have been proposed recently to eliminate the “small parameter” assumption, such as the artificial parameter method [1], homotopy analysis method [2, 3], variational iteration method [4], homotopy perturbation method [5, 6] etc. Among these methods homotopy perturbation method (HPM) and homotopy analysis method (HAM) are two powerful methods which give us acceptable analytical results with convenient convergence and stability.

Homotopy perturbation method [5, 6] first proposed by He for solving nonlinear differential equation as well as nonlinear integral equations. This method is a coupling of the traditional perturbation method and homotopy in topology. Unlike the traditional perturbation technique, this method does not require any small parameter. It includes a significant advantage to this method. HPM applied successfully to a wide class of nonlinear differential equations which includes nonlinear heat transfer equations [7, 8], nonlinear dispersive equations, nonlinear wave equation [9], nonlinear oscillator [10], nonlinear Schrodinger equations [11] and to other fields. Recently, some modification of this method so called modified homotopy perturbation method (mHPM) [12], has been done to facilitate the accurate calculation and accelerate the rapid convergence of the series solution.

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IUNS.2013.12.30/771
Homotopy analysis method, another powerful analytic approach, is a recently introduced analytical method for solving nonlinear equations, which has undergone rapid development since its inception in 1995 by Liao [3]. This method which logically contains Lyapunov’s small parameter method [13], the $\delta$-expansion method [14], and Adomian’s decomposition method [15] has been successfully applied to many nonlinear problems. Unlike the previous methods, HAM does not depend on any small parameter. Further, as compared to perturbation methods, HAM includes greater flexibility in the selection of a proper set of basis functions for the solution and it is also a much simpler method in the control of the convergence rate and region of approximation series as described by Liao.

In this article, at first a brief description of the analytical methods HPM, mHPM and HAM are provided in Section 2; and then in Section 3, these three methods are applied successfully to evaluate the analytical solution of strong nonlinear differential equations. In Section 4, two different examples are considered and worked out in detail by using describe methods to illustrate the efficiency and convenience of the methods. Section 4 also contains a comparison among the presented methods.

2 Analysis of the three methods

In this section, a brief introduction has been provided on the basic concept of the methods homotopy perturbation method (HPM), modified homotopy perturbation method (mHPM) and homotopy analysis method (HAM).

2.1 Basic concept of homotopy perturbation method

The combination of the classical perturbation technique and homotopy method is called homotopy perturbation method, which has eliminated the limitations of the traditional perturbation technique. To illustrate the basic concept of this method for solving nonlinear differential equation, we consider the following general differential equation

$$ A(u) + f(r) = 0, \quad r \in \Omega, $$

(1)

with boundary condition

$$ B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, $$

(2)

where $A$ is a general differential operator, $B$ is a boundary operator, $f(r)$ is a known forcing term and $\Gamma$ is the boundary of the domain $\Omega$.

The operator $A$ can be divided into two parts of $L$ and $N$, where $L$ is the linear part and $N$ is the nonlinear part. Therefore the relation (1) can be written as

$$ L(u) + N(u) + f(r) = 0. $$

(3)

By the homotopy technique proposed by Liao [2], He construct a homotopy of Eq. (1) as follows

$$ (1 - q)L[\phi(t; q) - u_0(t)] + q[A(\phi) + f(r)] = 0, \quad q \in [0, 1], $$

(4)

where $q$ is the embedding parameter and $u_0$ is the initial approximation of $u$.

From relation (4), it is clear that as the embedding parameter $q$ varies from 0 to 1 the solution $\phi(t; q)$ varies from the initial approximation $u_0$ to the solution $u$. According to [6], the embedding parameter $q$ can be considered as a small parameter and so it is natural to assume that the solution of Eq. (4) can be expressed in terms of power series in $q$:

$$ \phi = \phi_0 + q\phi_1 + q^2\phi_2 + q^3\phi_3 + \ldots $$

(5)

Therefore by setting $q = 1$ the solution of Eq. (1) can be obtained as

$$ u = \lim_{q \to 1} \phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \ldots $$

(6)

More detailed information about HPM and the convergence of the series of (6) can be found in [4, 5].
2.2 Basic idea of modified homotopy perturbation method

To explain the modified homotopy perturbation method (mHPM), we consider the same general differential equation (1). The modified homotopy perturbation method is similar to the standard HPM. In this method we add and subtract a linear term such as \( C u(t) \) (where \( C \) is a constant) to the main governing differential equation as suggested by Ganji et al. [12]. In this way the governing differential equation will remain unchanged and the term which is added to the linear operator, prevent from divergent solution. It may be noted that by using this method much accurate solution compare to the standard HPM can be obtained and the rate of convergence is also high compared to standard HPM. The only difference between the HPM and mHPM is the linear operator which depends on the governing equation as well as the boundary conditions. So the additional linear term is changed from problem to problem. The mHPM can be very helpful when the standard HPM is failure to solve the governing equation.

2.3 Basic concept of homotopy analysis method

Homotopy analysis method (HAM) was proposed by means of homotopy, a fundamental concept of Topology. Liao [3] has developed a new kind of homotopy so called the zero-order deformation equation

\[
(1 - q)\mathcal{L}[\phi(t; q) - u_0(t)] = qh\mathcal{H}(t)\mathcal{N}[\phi(t; q)],
\]

for a nonlinear equation

\[
\mathcal{N}[u(t)] = 0,
\]

where \( h \) is the nonzero convergence parameter, \( \mathcal{H}(t) \) is the auxiliary function and \( q \in [0, 1] \) is the embedding parameter.

It is clear that when \( q = 0 \) then \( \phi(t; 0) = u_0(t) \) and when \( q = 1 \) then \( \phi(t; 1) = u(t) \), which indicates that as the embedding parameter \( q \) increases from 0 to 1, the solution \( \phi(t; q) \) varies from the initial approximation \( u_0(t) \) to the actual solution \( u(t) \) of Eq. (8). Using Taylor’s theorem \( \phi(t; q) \) can be expanded with respect to the embedding parameter \( q \) as follows

\[
\phi(t; q) = \phi(t; 0) + \sum_{m=1}^{\infty} \frac{1}{m!} \left. \frac{\partial^m \phi(t; q)}{\partial q^m} \right|_{q=0} q^m.
\]

Differentiating the zero-order deformation equation (7) \( m \)-times with respect to \( q \) and then setting \( q = 0 \) and finally dividing them by \( m! \), we have the so-called \( m \)-th order deformation equation

\[
\mathcal{L}[u_m(t) - \chi_m u_{m-1}(t)] = h\mathcal{H}(t)R_m(\vec{u}_{m-1}),
\]

where

\[
u_m(t) = \frac{1}{m!} \left. \frac{\partial^m \phi(t; q)}{\partial q^m} \right|_{q=0},\]

\[
\chi_m = \begin{cases} 
0 & \text{when } m \leq 1, \\
1 & \text{otherwise},
\end{cases}
\]

and

\[
R_m(\vec{u}_{m-1}) := \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\phi(t; q)]}{\partial q^{m-1}} \right|_{q=0}
\]

with \( \vec{u}_{m-1} := \{u_0(t), u_1(t), \ldots, u_{m-1}(t)\} \). More detailed information about HAM can be found in [3].

The auxiliary parameter, \( h \), the auxiliary function, \( \mathcal{H} \), the initial approximation, \( u_0 \) and the auxiliary linear operator, \( \mathcal{L} \), are chosen in such a way that Eq. (9) converges at \( q = 1 \). Then, at \( q = 1 \), Eq. (9) becomes

\[
u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t).
\]

Convergence of this method is available in [3].
3 Application of the methods

In this section, HAM, HPM and mHPM are applied successfully for solving strongly nonlinear governing equation of nonlinear oscillators to demonstrate the effectiveness of the methods; and to compare the obtained results.

Let us consider the governing equation of motion of undamped oscillator with nonlinearity of high order

\[
\frac{d^2u}{dt^2} + \alpha u + Mu^3 = 0
\]  

(15)

with initial conditions

\[
u(0) = A \quad \text{and} \quad \left. \frac{du}{dt} \right|_{t=0} = 0.
\]

(16)

There are several literatures on this nonlinear equation with different values of \(\alpha\) and \(M\). For example, this kind of nonlinear equation is studied by means of Adomian decomposition method [16], artificial parameter decomposition [17], variational iteration method [18] and He’s parameter expanding [19]. It is noted that when \(\alpha = 1\) and \(M = 1\) then the equation (15) represents the equation of motion of free undamped Duffing’s oscillator [19]. Equation (15) represents a hard spring for \(\alpha = 1\), \(M > 0\) and it represents a soft spring when \(\alpha = 1\), \(M < 0\).

The analytical solution of the nonlinear differential equation (15) subjected to the initial conditions (16) is obtained by using described three methods in the next sub-sections.

3.1 Solution of the nonlinear differential equation (15) using HPM

This section is devoted to evaluate the solution of the nonlinear differential equation (15) subjected to the conditions (16) using HPM. In view of the relation (15), it is very straightforward to choose the linear and nonlinear operator as follows

\[
\mathcal{L} = u'' + \alpha u \quad \text{and} \quad \mathcal{N} = Mu^3.
\]

(17)

Then the homotopy can be constructed as

\[
\mathcal{L}(v) - \mathcal{L}(u_0) + q\mathcal{L}(u_0) + qMu^3 = 0,
\]

(18)

where the initial approximation can be chosen of the form

\[
u_0(t) = A\cos(\alpha t).
\]

(19)

Homotopy perturbation method assumes that the solution of equation (18) can be expressed as a series of the power of \(q\), i.e.

\[
v = v_0 + qv_1 + q^2v_2 + q^3v_3 + \ldots
\]

(20)

The approximate solution of Eq. (15), therefore, can be easily obtained:

\[
u = \lim_{q \to 1} v = v_0 + v_1 + v_2 + \ldots
\]

(21)

Now substituting relation (20) into relation (18), and equating the coefficients of \(q\) with identical power of \(q\) we have the following set of linear differential equations

\[
q^0 : \mathcal{L}(v_0) = \mathcal{L}(u_0) = 0 \quad \text{with} \quad v_0(0) = A, v_0'(0) = 0
\]

(22)

\[
q^1 : \mathcal{L}(v_1) + \mathcal{L}(u_1) + Mu_0^3 = 0 \quad \text{with} \quad v_1(0) = 0, v_1'(0) = 0
\]

(23)

\[
q^2 : \mathcal{L}(v_2) + 3Mu_0^2v_1 = 0 \quad \text{with} \quad v_2(0) = 0, v_2'(0) = 0
\]

(24)

\[
\vdots
\]

From the relations (22) and (23), we obtain

\[
v_0 = u_0 = A\cos(\alpha t)
\]

(25)

\[
\frac{d^2v_1}{dt^2} + v_1 + A(-\alpha^2 + 1)\cos(\alpha t) + MA^3\cos^3(\alpha t) = 0
\]

(26)

\[
\vdots
\]
The solution of equation (26) can be obtained by using variational iteration method [4] as

\[ v_1 = \frac{MA^3}{4(9\alpha^2 - 1)} \cos(3\alpha t) + (-\alpha^2 + 1 + \frac{3}{4}MA^2)\frac{A}{\alpha^2 - 1} \cos(\alpha t). \]  

(27)

In a similar manner \( v_2, v_3 \) and so on can be find out. Then the solution of equation (15) can be readily obtained as

\[ u(t) \approx \sum_{j=0}^{n} v_j(t) \]  

(28)

where \( n \) is the order of approximation.

### 3.2 Solution of the nonlinear differential equation (15) using mHPM

In the present subsection, we apply mHPM for solving the nonlinear differential equation (15). To use mHPM, linear operator of the nonlinear differential equation (15) has to be modified. The linear operator is modified as

\[ L = u'' + Cu + \alpha u. \]  

(29)

It can be shown that for this example, \( C \) will be \( MA^2 \). So \( L \) becomes (from Eq.29)

\[ L = u'' + (MA^2 + \alpha)u. \]

Then the homotopy can be expressed as follows

\[ (1 - q)[L(v) - L(u_0)] + q[u'' + \alpha u + Mu^3] = 0. \]  

(30)

Substituting relations (20) and (16) in (30), and equating the identical power of \( q \) we obtain

\[ q^0 : v_0'' + (\alpha + MA^2)v_0 = 0 \text{ with } v_0(0) = A, v_0'(0) = 0 \]
\[ q^1 : v_1'' + \alpha v_1 - MA^2(v_0 - v_1) + Mv_0^3 = 0 \text{ with } v_1(0) = 0, v_1'(0) = 0 \]
\[ q^2 : v_2'' + \alpha v_2 - MA^2(v_1 - v_2) + 3Mv_0^2v_1 = 0 \text{ with } v_2(0) = 0, v_2'(0) = 0 \]
\[ \vdots \]

Next the solution of the system (31) can be find out by using variational iteration method and the first two terms of the solution are as follows

\[ v_0 = A \cos \left( \sqrt{(MA^2 + \alpha)}t \right) \]  

(32)

\[ v_1 = A^3 \frac{32(MA^2 + \alpha)^2}{2(MA^2 + \alpha)^2 + 16(MA^2 + \alpha)^2} \left[ 4t(MA^2 + \alpha) \sin \left( \sqrt{(MA^2 + \alpha)}t \right) - \sqrt{(MA^2 + \alpha)} \cos \left( \sqrt{(MA^2 + \alpha)}t \right) + \sqrt{(MA^2 + \alpha)^3} \cos \left( 3\sqrt{(MA^2 + \alpha)}t \right) \right]. \]  

(33)

Finally we have the solution of Eq. (15) subjected to the given initial conditions (16) as

\[ u(t) \approx \sum_{j=0}^{n} v_j(t). \]  

(34)

### 3.3 Solution of the nonlinear differential equation (15) using HAM

In this section the nonlinear second order differential equation (15) has been solved by using HAM. Under the rule of solution expression [3] and from equation (15), it is straightforward to choose a non-linear as well as a linear operator as

\[ N[\phi(t; q)] = \frac{\partial^2 \phi(t; q)}{\partial t^2} + \alpha \phi(t; q) + M \phi^3(t; q), \]  

(35)

\[ L[\phi(t; q)] = \frac{\partial^2 \phi(t; q)}{\partial t^2}, \]  

(36)
with the property
\[ \mathcal{L}[c_0 + c_1 t] = 0, \]
where \( c_0, c_1 \) are two integral constants and the value of these constants can be evaluated from the specific condition given in Eq. (16).

The solution of the m-th order deformation equation (10) after using relation (36), can be written as
\[ u_m = \chi_m u_{m-1} + h \int_0^t \int_0^s \mathcal{H}(w) R_m \left( \varphi_{m-1}(w) \right) dw + c_1 t + c_0 \] (37)
subjected to the conditions
\[ u_m(0) = 0, \quad u_m'(0) = 0, \] (38)
where
\[ R_m \left( \varphi_{m-1}(w) \right) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N} [\phi(t; q)]}{\partial q^{m-1}} \right|_{q=0} \]
\[ = \frac{d^2 u_{m-1}}{dt^2} + \alpha u_{m-1} + M \sum_{i=0}^{m-1} u_{m-1-i} \sum_{j=0}^{i} i_{i-j} u_j \] (39)

Using rule of solution expression [3] and relation (16), we choose the initial approximation as
\[ u_0(t) = A, \] (40)
which makes certain the conditions (16).

Now form the relations (37), (39) and (40) for \( m \geq 1 \), we can successively obtain
\[ u_1(t) = \frac{h}{2} A(MA^2 + \alpha) t^2, \] (41)
\[ u_2(t) = \frac{h}{2} A(MA^2 + \alpha) t^2 + \frac{k^2}{24} At^2(3MA^2 t^2 + \alpha^2 + 12), \]
\[ u_3(t) = \frac{h}{2} A(MA^2 + \alpha) t^2 + \frac{h^2}{24} At^2(3MA^2 t^2 + \alpha^2 + 12) \]
\[ + \frac{Ah^2}{240} \left[ t^4 \left\{ \frac{h}{3} (27A^4 M^2 + 24A^2 Ma + \alpha^2) t^2 + 10(1 + 2h)(\alpha + 3A^2 M) \right\} \right] \]
\[ + 120(1 + h) \]
\[ \vdots \]

Then we have the analytic solution of (15) as
\[ u(t) \approx \sum_{m=0}^{n} u_m(t), \] (43)
where \( n \) is the order of approximation and in the present work \( n = 3 \) is considered.

Here it can be noted that the solution i.e. Eq. (43) of Eq. (15) obtained using HAM depends on the auxiliary parameter \( h \). The auxiliary parameter \( h \) determines the convergence region and the rate of approximation in the approximate solution. The obtained results are discussed in the succeeding section.

4 Numerical examples with results discussion

This section contained some different examples of nonlinear oscillator to illustrate the effectiveness of the described three methods.

Example 1 We first consider a nonlinear differential equation which represent the motion of free undamped Duffing’s oscillator [12, 16, 17]
\[ u'' + u + u^3 = 0, \quad u(0) = A, \quad u'(0) = 0. \] (44)
Ghosh et. al. [16] solved such kind of problem by using an explicit numeric-analytic technique based on Adomian Decomposition Method (ADM). They have shown that the series solution obtained by using ADM converges rapidly in a very small region whereas it has very slow convergent rate in the wider region and the truncated series solution is also inaccurate in that wide region. Ganji et. al. [12] also considered the undamped Duffing’s oscillator equation and obtained the solution using homotopy perturbation method. They showed that the HPM solution is unable to give the accurate solution of the problem even for a short region.

In the present study, the analytical solution of Eq. (44) using HAM is achieved through the relations (41)-(43), considering \( \alpha = 1, M = 1 \). The solution of Eq. (44) using HPM is also obtained from (28), considering five terms of the series solution with \( \alpha = 1, M = 1 \). The solution of Eq. (44) using mHPM is attained from (34) considering just three terms of the series solution and substituting \( \alpha = 1, M = 1 \). Fig. 2 depicts the comparison of the results obtained from three analytical methods with that of the numerical method. In this work MATLAB function [20] is used to obtain the numerical solution of Eq. (44). The presented results in Fig.2 is for different values of \( t \) and \( A = 1.8 \). Since the HAM solution depends on \( \alpha \), the curves for different orders of approximations are presented in Fig. 1. It is clear from the Fig. 1 that HAM is convergent within a certain range of \( \alpha \).

Fig.3 compares the solution using HAM, HPM and mHPM with the numerical solution for a wider region of \( t \) by considering the same value of \( A \) i.e. \( A = 1 \). From Fig.3, we observed that HAM and HPM are just valid for a short region of \( t \) whereas the mHPM is still valid for a large range of \( t \) which indicates that the mHPM can solve the strong nonlinear Duffing equation with higher degree of accuracy by considering the first two terms of the series solution only. In this figure, it is also observed that the solution using mHPM is nearly same with the numerical solution.

The analytical solution using three described methods and the numerical solution are also shown in Fig.4 for a different value of \( A = 4 \). From Figs.2 and Fig.4, it can be observed that when the value of \( A \) increases, the validity region of the methods decreases and it confined to a smaller region. We also observed that (see figure 4) when \( A = 4 \), the validity of the present solution is restricted to \( t < 1 \). But one can increase the validity region of the method by considering the
Example 2 Let us consider the nonlinear governing equation of motion of oscillator as

\[ u'' + 3u + u^3 = 0 \quad u(0) = A, \quad u'(0) = 0. \]  

(45)

The comparison between the analytical solution using HAM, HPM, mHPM and the numerical solution for a small region of \( t \) is shown in Fig. 5 for \( A = 1.8 \). Whereas Figure 6 depicts the comparison of the same for a large region of \( t \). \( h \) curve for this example is shown in Fig. 7. Fi.8 depicts the analytical solution obtained using HAM, HPM, mHPM as well as the numerical solution for \( A = 4 \). For this case also, we observed from the figures that HAM which provides us much accurate solution than HPM as well as mHPM is valid only for a short range of \( t \) whereas the mHPM is still valid for a larger range of \( t \). It can be also observed that when the value of \( A \) increases then the validity region decreases.

In both examples, we noticed that mHPM needs only first three terms of the series solution to give an accurate solution which indicates that mHPM is computationally more efficient than others two methods.

5 Conclusions

In this study, three different analytical methods i.e. homotopy analysis method (HAM), homotopy perturbation method (HPM) and modified homotopy perturbation method (mHPM) are employed separately to evaluate the solutions of strongly nonlinear differential equations namely, nonlinear undamped oscillators. Obtained analytical solutions using described methods are compared with the numerical solution to check the accuracy of the results. The results demonstrate that for
a short region, HAM can give much better approximations for nonlinear differential equations than the previous solutions using HPM and mHPM. However, for a wide range, mHPM is only valid and provide the better solution than others. In HAM, the auxiliary parameter provides a convenient way to adjust and control the convergence region and rate of the solutions series. Moreover, HAM is faster than HPM. Compare to HPM, mHPM is computationally more efficient.

Acknowledgments

Srikumar Panda is grateful to the Council of Scientific and Industrial Research (CSIR), Government of India, for providing the Research Fellowship for pursuing Ph.D. at the Indian Institute of Technology Ropar, India. Author also thanks IIT Ropar for providing all the necessary facilities.

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