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On the Convergence of Compact Arithmetic Averaging Scheme for Semi-linear 2d-elliptic Equations and Estimates of Partial Derivatives

Navnit Jha*, Neelesh Kumar
Department of Mathematics, South Asian University, New Delhi-110021, India
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Abstract: A new third order accurate geometric mesh compact finite difference technique is proposed for the numerical solution of two dimensional semi-linear elliptic partial differential equations. The method is also used to obtain third and second order of accuracy for the first and second order partial derivatives respectively on geometric meshes. The essence of the method lies in the fact that it is applicable to the singular problems as well. The consideration of geometricity of the meshes in both the spatial directions enables the grid concentration adaptively, which leads to significant improvement in the numerical accuracy compared to the high order method on the uniform meshes. The convergence analysis of the method has been described with monotonic and irreducible property of the iteration matrix. Numerical illustrations with stationary Schrödinger equation, Helmholtz equation and singular problems show the reliability and accuracy of the new technique in terms of maximum absolute errors and computational order of convergence of the solution values and along with their partial order derivatives.

Keywords: Geometric mesh; Arithmetic average method; Compact scheme; Singularity; Helmholtz equation; Schrödinger equation

1 Introduction

The semi-linear elliptic partial differential equations to be numerically solved, is of the form

$$\nabla^2 U(x,y) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) U(x,y) = G(x,y,U(x,y)), \quad (x,y) \in \Omega \tag{1}$$

defined in a domain $\Omega = \{(x,y)|a < x < b, c < y < d\}$. The Dirichlet's boundary conditions are given by $U(x,y) = f(x,y), (x,y) \in \partial\Omega$. We assume that $U \in C^{(2)}(\Omega)$ and G along with its first order partial derivatives with respect to U are continuous and bounded.

Among many mathematical models, Helmholtz equation, stationary Schrödinger equation, Laplace and Poisson equations are elliptic partial differential equations of type (1). In general, the analytic solution for the arbitrary choice of $G(x,y,U)$ is difficult and thus we resort to the approximate solution by using finite difference method [1, 2], finite element method [3, 4], cubic spline [5], collocation spline [6–8] and radial basis functions [9]. There exists an ample solution technique devoted to the construction of compact scheme for solving elliptic partial differential equations in two space variables. In particular, we mention here the various improvements of classical finite difference scheme [10–12]. The discussion on Numerov's method [13] suggests us to incorporate linear combination of functional evaluations and solution values to improve the accuracy of the scheme. With this motive, an arithmetic average method for the solution of two point boundary value problems has been discussed by Chawla and Shivakumar [14]. The advantage of arithmetic average scheme has been seen with the presence of singular terms. This situation often arises when the Laplacian operator is considered in cylindrical and/or spherical polar coordinates. In the numerical solution, the mesh selection is an another important factor. In the past Shishkin mesh [15], Bakhvalov mesh [16], exponential expanding sequence of meshes [17]

*Corresponding author. *E-mail address*: navnitjha@sau.ac.in

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and Sundqvist's meshes [18] are considered in some or other physical situation, a one of the prominent situation is boundary or interior layers [19]. The application of various mesh strategy, significantly affect the accuracy of solution and order of the method.

In this paper, we have chosen a geometric mesh network for the discretization of two dimensional semi-linear elliptic boundary value problems. In some particular application areas, it is also known as exponential expanding sequence of meshes [20]. Such a mesh technique has been previously considered for one dimensional two point boundary value problems [21–27]. To the author's knowledge no such method has been developed for higher dimensional problems. Here, a new arithmetic averaging of meshes, finite difference approximations of third order accuracy is presented for solving semi-linear elliptic boundary value problems using seven off-step nodes and one central node. The special compact character and lower order method makes the scheme easier to implement. For a particular value of geometric mesh parameters, an accuracy of fourth order can be easily achieved. Therefore, the proposed technique can be regarded as a generalization of fourth order compact finite difference scheme.

The paper is organised with the description of geometric meshes and arithmetic average approximations of the solution values. Some discrete operators are developed to achieve compact difference scheme in the next section. Subsequently, the method is analysed for the convergence and obtain the bounds of discretization errors using matrix analysis. Some computational illustrations showing advantage of using proposed scheme have been shown. Finally the paper is concluded with remarks and further scope.

2 Geometric mesh discretization

The solution domain Ω is covered by the grid network $\{(x_l, y_m)|l = 0(1)L+1, m = 0(1)M+1\}$, where $L, M \in Z_+$ are given numbers. The geometric meshes are defined by the relations

$$x_{l+1} = x_l + \alpha h_l, \quad l = 0(1)L, \qquad h_{l+1} = \alpha h_l, \quad l = 1(1)L,$$
$$y_{m+1} = y_m + \beta k_m, m = 0(1)M, \quad k_{m+1} = \beta k_m, m = 1(1)M,$$

and thus, it is easy to compute the first mesh step size in x- and y- directions as given below

$$h_1 = \begin{cases} (b-a)(1-\alpha)/(1-\alpha^{L+1}) & \text{if } \alpha < 1 \\ (b-a)/(L+1) & \text{if } \alpha = 1 \\ (b-a)(\alpha-1)/(\alpha^{L+1}-1) & \text{if } \alpha > 1 \end{cases}; \quad k_1 = \begin{cases} (d-c)(1-\beta)/(1-\beta^{M+1}) & \text{if } \beta < 1 \\ (d-c)/(M+1) & \text{if } \beta = 1 \\ (d-c)(\beta-1)/(\beta^{M+1}-1) & \text{if } \beta > 1 \end{cases}$$

where $\alpha, \beta > 0$ are the geometric mesh parameters and $\lambda_{l,m} = k_m/h_l > 0$ is the mesh ratio parameter. The exact and approximate solution values at the node (x_l, y_m) are represented by $U_{l,m}$ and $u_{l,m}$ respectively. The higher order terms are denoted by

$$O_n = O_n(h_l, k_m) = \sum_{p+q=n} b_{p,q} h_l^p k_m^q$$

where $b_{p,q}$ are some terms occuring with the application of finite Taylor's expansion and p, q, n are integers, not necessarily positive. We shall use only seven off-step nodes and one central node to obtain functional approximations defined on the set

$$Q = \{(x_i, y_j) \colon (i, j) \in \{l, l \pm 1/2\} \times \{m, m \pm 1/2\} \sim \{l + 1/2, m + 1/2\}\}$$

Consider the following arithmetic average approximations on the off-step nodes obtained from the linear combination of $U(x, y)$ at nine grid points $(x_{l+i}, y_{m+j}), i, j = 0, \pm 1$.

$$\tilde{U}_{l+\frac{1}{2},m} = \frac{(\alpha+2)U_{l+1,m} + (\alpha+1)(\alpha+2)U_{l,m} - \alpha^2 U_{l-1,m}}{4(\alpha+1)}, \tag{2}$$

$$\tilde{U}_{l-\frac{1}{2},m} = \frac{-U_{l+1,m} + (\alpha+1)(2\alpha+1)U_{l,m} + \alpha(2\alpha+1)U_{l-1,m}}{4\alpha(\alpha+1)}, \tag{3}$$

$$\tilde{U}_{l,m+\frac{1}{2}} = \frac{(\beta+2)U_{l,m+1} + (\beta+1)(\beta+2)U_{l,m} - \beta^2 U_{l,m-1}}{4(\beta+1)}, \tag{4}$$

$$\tilde{U}_{l,m-\frac{1}{2}} = \frac{-U_{l,m+1} + (2\beta+1)(\beta+1)U_{l,m} + \beta(2\beta+1)U_{l,m-1}}{4\beta(\beta+1)}, \tag{5}$$

$$\begin{aligned}\tilde{U}_{l+\frac{1}{2},m-\frac{1}{2}} = &-\frac{\beta(\alpha-\beta)U_{l+1,m-1} + (\alpha+1)U_{l+1,m+1} + \alpha^2\beta(\beta+1)U_{l-1,m-1}}{4\beta(\alpha+1)(\beta+1)} \\ &+ \frac{(\beta+1)U_{l+1,m} + \beta U_{l,m} + \beta(\alpha+1)U_{l,m-1}}{4\beta},\end{aligned} \tag{6}$$

$$\begin{aligned}\tilde{U}_{l-\frac{1}{2},m+\frac{1}{2}} = &\frac{\alpha(\alpha-\beta)U_{l-1,m+1} - (\beta+1)U_{l+1,m+1} - \alpha(\alpha+1)\beta^2U_{l-1,m-1}}{4\alpha(\alpha+1)(\beta+1)} \\ &+ \frac{(\alpha+1)U_{l,m+1} + \alpha U_{l,m} + \alpha(\beta+1)U_{l-1,m}}{4\alpha},\end{aligned} \tag{7}$$

$$\begin{aligned}\tilde{U}_{l-\frac{1}{2},m-\frac{1}{2}} = &-\frac{\beta(\alpha-\beta)U_{l+1,m-1} + (\alpha+1)U_{l+1,m+1} + \beta(\beta+1)(2\alpha+1)U_{l-1,m-1}}{4\beta(\alpha+1)(\beta+1)} \\ &+ \frac{(\alpha-\beta)U_{l+1,m} + (2\alpha+1)\beta U_{l,m}}{4\alpha\beta}.\end{aligned} \tag{8}$$

With the help of Taylor's series expansion, it is easy to see that

$$\tilde{U}_{i,j} = U_{i,j} + O_3(h_i, k_j), (i,j) \in Q. \tag{9}$$

Next, we define some of the discrete operators using geometric meshes, which will be used to obtain our compact scheme. Let us define

$$\delta^{2,0}U_{l,m} = \frac{2}{\alpha(\alpha+1)}[U_{l+1,m} - (\alpha+1)U_{l,m} + \alpha U_{l-1,m}], \tag{10}$$

$$\delta^{0,2}U_{l,m} = \frac{2}{\beta(\beta+1)}[U_{l,m+1} - (\beta+1)U_{l,m} + \beta U_{l,m-1}], \tag{11}$$

$$\delta^{2,1}U_{l,m} = \frac{2}{\alpha(\alpha+1)}\left[U_{l+1,m} - U_{l+1,m-1} + \alpha(U_{l-1,m} - U_{l-1,m-1}) + (\alpha+1)(U_{l,m-1} - U_{l,m})\right], \tag{12}$$

$$\delta^{1,2}U_{l,m} = \frac{2}{\beta(\beta+1)}\left[U_{l,m+1} - U_{l-1,m+1} + \beta(U_{l,m-1} - U_{l-1,m-1}) + (\beta+1)(U_{l-1,m} - U_{l,m})\right], \tag{13}$$

$$\delta^{2,2}U_{l,m} = \frac{1}{\alpha\beta}\begin{bmatrix} (U_{l+1,m+1} + \alpha\beta U_{l-1,m-1} + \beta U_{l+1,m-1} + \alpha U_{l-1,m+1}) \\ +(\beta+1)((\alpha+1)U_{l,m} - U_{l+1,m} - \alpha U_{l-1,m}) - (\alpha+1)(U_{l,m+1} + \beta U_{l,m-1}) \end{bmatrix}. \tag{14}$$

Similar to the central difference operator, these operators are used to obtain partial derivatives and more precisely, we can express it as follow

$$\delta^{2,0}U_{l,m} = h_l^2 U_{xx_{l,m}} + O_3(h_l, k_m), \tag{15}$$

$$\delta^{0,2}U_{l,m} = k_m^2 U_{yy_{l,m}} + O_3(h_l, k_m), \tag{16}$$

$$\delta^{2,1}U_{l,m} = h_l^2 k_m U_{xxy_{l,m}} + O_4(h_l, k_m), \tag{17}$$

$$\delta^{1,2}U_{l,m} = h_l k_m^2 U_{xyy_{l,m}} + O_4(h_l, k_m), \tag{18}$$

$$\delta^{2,2}U_{l,m} = \frac{1}{4}(\alpha+1)(\beta+1)h_l^2 k_m^2 U_{xxyy_{l,m}} + O_5(h_l, k_m). \tag{19}$$

Following the technique of Stephenson [28], and using the above arithmetic average approximations along with the discrete compact operators defined above, the geometric mesh arithmetic average finite difference method for the semi-linear elliptic boundary value problems (1), is obtained for $l = 1(1)L, m = 1(1)M$ and expressed by the formula

$$h_l^{-2}\nabla^2_{l,m}U_{l,m} = \frac{1}{9}\lambda^2_{l,m}h_l^2 \sum_{(i,j)\in Q} d_{i,j}\tilde{G}_{i,j} + O_5(h_l, k_m), \tag{20}$$

where

$$\nabla^2_{l,m} = h_l^2\left[\delta^{0,2} + \frac{1}{3}(\alpha-1)\delta^{1,2} + \frac{\alpha^2+\alpha-1}{3(\alpha+1)(\beta+1)}\delta^{2,2}\right] + k_m^2\left[\delta^{2,0} + \frac{1}{3}(\beta-1)\delta^{2,1} + \frac{\beta^2+\beta-1}{3(\alpha+1)(\beta+1)}\delta^{2,2}\right],$$

is the geometric mesh discretization of Laplacian operator $\nabla^2 = \partial_{xx} + \partial_{yy}$,

$$\tilde{G}_{i,j} = G(x_i, y_j, \tilde{U}_{i,j}), (i,j) \in Q \tag{21}$$

and

$$\begin{aligned}
d_{l,m} &= \frac{1}{\alpha\beta}[4\beta^2(\alpha^2-2\alpha+1) - 4\alpha^2(2\beta-1) - 4(\alpha+\beta) + 9\alpha\beta],\\
d_{l+1/2,m} &= \frac{2}{\alpha(\alpha+1)}[\alpha(3\alpha+2\beta-2) - 2(\beta-1)],\\
d_{l-1/2,m} &= -\frac{2}{(\alpha+1)\beta}[2\alpha^2(\beta^2-2\beta+1) - 2\alpha\beta(\beta-1) - \beta - 2],\\
d_{l,m-1/2} &= -\frac{2}{\alpha(\beta+1)}[2\beta^2(\alpha^2-2\alpha+1) - 2\alpha\beta(\alpha-1) - \alpha - 2],\\
d_{l-1/2,m-1/2} &= \frac{4(\alpha-1)(\beta-1)(\alpha\beta-1)}{(\alpha+1)(\beta+1)},\\
d_{l+1/2,m-1/2} &= -\frac{4(\alpha-1)(\beta-1)}{\alpha(\alpha+1)},\\
d_{l,m+1/2} &= \frac{2}{\beta(\beta+1)}[\beta(3\beta-2) + 2\alpha(\beta-1) + 2],\\
d_{l-1/2,m+1/2} &= -\frac{4(\alpha-1)(\beta-1)}{\beta(\beta+1)}.
\end{aligned}$$

Note that, the detailed analysis shows that the local truncation error in the equation (20), is $O_8(h_l, k_m)$ for $\alpha = \beta = 1$, and thus we succeed to achieve fourth order accurate scheme in a uniform mesh network.

Next, we obtain numerical scheme for the first and second order partial derivatives of solution values. If we use only available nine grid points, then at most second and first order of accuracy can be achieved for the first and second order partial derivatives of solution variables respectively, in case of geometric meshes network. However, the use of functional approximations along with available nine grid points gives rise to third and second order of accuracy to the first and second order partial derivatives of solution variables respectively. Following the technique developed by Mohanty [29], we obtain the following approximations

$$\begin{aligned}
\tilde{U}_{x_{l,m}} = &-\frac{\alpha h_l}{3(\alpha+1)}(\tilde{G}_{l+1,m} - \tilde{G}_{l-1,m}) + \frac{U_{l+1,m} + (\alpha^2-1)U_{l,m} - \alpha^2 U_{l-1,m}}{\alpha(\alpha+1)h_l}\\
&- \frac{\alpha}{3\beta(\beta+1)\lambda^2_{l,m}h_l}\begin{bmatrix}(\beta+1)(U_{l,m} - U_{l-1,m})\\ +\beta(U_{l-1,m-1} - U_{l,m-1})\\ +U_{l-1,m+1} - U_{l,m+1}\end{bmatrix},
\end{aligned} \tag{22}$$

$$\begin{aligned}
\tilde{U}_{y_{l,m}} = &-\frac{\beta k_m}{3(\beta+1)}(\tilde{G}_{l,m+1} - \tilde{G}_{l,m-1}) + \frac{U_{l,m+1} + (\beta^2-1)U_{l,m} - \beta^2 U_{l,m-1}}{\beta(\beta+1)k_m}\\
&- \frac{\beta\lambda_{l,m}}{3\alpha(\alpha+1)h_l}\begin{bmatrix}(\alpha+1)(U_{l,m} - U_{l,m-1})\\ +\alpha(U_{l-1,m-1} - U_{l-1,m})\\ +U_{l+1,m-1} - U_{l+1,m}\end{bmatrix},
\end{aligned} \tag{23}$$

$$\begin{aligned}\tilde{U}_{xx_{l,m}} =& -\frac{2(\alpha-1)}{3(\alpha+1)}(\tilde{G}_{l+1,m}-\tilde{G}_{l-1,m})+\frac{2(U_{l+1,m}-(\alpha+1)U_{l,m}+\alpha U_{l-1,m})}{\alpha(\alpha+1)h_l^2} \\ &-\frac{2(\alpha-1)}{3\beta(\beta+1)k_m^2}\begin{bmatrix}(\beta+1)(U_{l,m}-U_{l-1,m}) \\ +\beta(U_{l-1,m-1}-U_{l,m-1}) \\ +U_{l-1,m+1}-U_{l,m+1}\end{bmatrix},\end{aligned} \tag{24}$$

$$\begin{aligned}\tilde{U}_{yy_{l,m}} =& -\frac{2(\beta-1)}{3(\beta+1)}(\tilde{G}_{l,m+1}-\tilde{G}_{l,m-1})-\frac{2(U_{l,m+1}-(\beta+1)U_{l,m}+\beta U_{l,m-1})}{\beta(\beta+1)k_m^2} \\ &-\frac{2(\beta-1)}{3\alpha(\alpha+1)h_l^2}\begin{bmatrix}(\alpha+1)(U_{l,m}-U_{l,m-1}) \\ +\alpha(U_{l-1,m-1}-U_{l-1,m}) \\ +U_{l+1,m-1}-U_{l+1,m}\end{bmatrix}.\end{aligned} \tag{25}$$

With the help of equation (21) and series expansion, we can estimate the truncation errors as follows

$$\tilde{U}_{x_{l,m}} = U_{x_{l,m}} + O_3(h_l, k_m), \tag{26}$$

$$\tilde{U}_{y_{l,m}} = U_{y_{l,m}} + O_3(h_l, k_m), \tag{27}$$

$$\tilde{U}_{xx_{l,m}} = U_{xx_{l,m}} + O_2(h_l, k_m), \tag{28}$$

$$\tilde{U}_{yy_{l,m}} = U_{yy_{l,m}} + O_2(h_l, k_m). \tag{29}$$

We can straightaway obtain second order accuracy for the mixed derivative U_{xy} from the linear combination of solution values on nine grid points and thus omitted from the discussion. The system of equation (20) gives rise to block tri-diagonal matrix and can be solved using standard Gauss-Seidel method. If $\widetilde{U}_{i,j}$ appears as a non-linear term, then Newton-Raphson iterative processes is considered. The solution is obtained in association with specified Dirichlet's boundary values.

3 Convergence analysis

This section concerned with the convergence property and error analysis of the geometric mesh arithmetic average method (20) discussed for the numerical solution of semi-linear elliptic problems of type (1). The proposed method (20), can be expressed as

$$\phi_{l,m} \equiv \phi_{l,m}\begin{pmatrix}U_{l-1,m-1}, U_{l,m-1}, U_{l+1,m-1}, \\ U_{l-1,m}, U_{l,m}, U_{l+1,m}, \\ U_{l-1,m+1}, U_{l,m+1}, U_{l+1,m+1}\end{pmatrix} = O_5(h_l, k_m), \tag{30}$$

where $l = 1(1)L, m = 1(1)M$ and

$$\phi_{l,m} = -h_l^{-2}\nabla^2_{l,m}U_{l,m} + \frac{1}{9}\lambda^2_{l,m}h_l^2\sum_{(i,j)\in Q} d_{i,j}\tilde{G}_{i,j}, \tag{31}$$

The scheme (30) in the matrix-vector notation is written as

$$\boldsymbol{\phi}(\boldsymbol{U}) + \boldsymbol{O_5} = \boldsymbol{0}, \tag{32}$$

where

$$\boldsymbol{U} = [U_{11}, U_{21}, \cdots, U_{L1}, \cdots\cdots, U_{1M}, U_{2M}, \cdots, U_{LM}]^T,$$

be the solution vector,

$$\boldsymbol{O_5} = [O_5^{11}, O_5^{21}, \cdots, O_5^{L1}, \cdots\cdots, O_5^{1M}, O_5^{2M}, \cdots, O_5^{LM}]^T,$$

be the truncation error vector and

$$\boldsymbol{\phi} = [\phi_{11}, \phi_{21}, \cdots, \phi_{L1}, \cdots\cdots, \phi_{1M}, \phi_{2M}, \cdots, \phi_{LM}]^T.$$

Our aim is to find the approximation $\boldsymbol{u}$ for the exact solution values $\boldsymbol{U}$, which are determined by solving

$$\boldsymbol{\phi}(\boldsymbol{u}) = \boldsymbol{0}. \tag{33}$$

From the equations (32) and (33), we obtain

$$\boldsymbol{\phi}(\boldsymbol{u}) - \boldsymbol{\phi}(\boldsymbol{U}) = \boldsymbol{O_5}. \tag{34}$$

Let $\epsilon_{l,m} = u_{l,m} - U_{l,m}, l = 1(1)L, m = 1(1)M$ be the finite precision errors occuring in the solution due to discretization and

$$\boldsymbol{\epsilon} = [\epsilon_{11}, \epsilon_{21}, \cdots\cdots, \epsilon_{L1}, \cdots, \epsilon_{1M}, \epsilon_{2M}, \cdots, \epsilon_{LM}]^T$$

be the error vector. We denote

$$\tilde{g}_{i,j} = g(x_i, y_j, u_{i,j}) \simeq \tilde{G}_{i,j}, (i,j) \in Q \tag{35}$$

and

$$\tilde{\epsilon}_{l+\frac{1}{2},m} = \frac{(\alpha+2)\epsilon_{l+1,m} + (\alpha+1)(\alpha+2)\epsilon_{l,m} - \alpha^2\epsilon_{l-1,m}}{4(\alpha+1)}, \tag{36}$$

$$\tilde{\epsilon}_{l-\frac{1}{2},m} = \frac{-\epsilon_{l+1,m} + (\alpha+1)(2\alpha+1)\epsilon_{l,m} + \alpha(2\alpha+1)\epsilon_{l-1,m}}{4\alpha(\alpha+1)}, \tag{37}$$

$$\tilde{\epsilon}_{l,m+\frac{1}{2}} = \frac{(\beta+2)\epsilon_{l,m+1} + (\beta+1)(\beta+2)\epsilon_{l,m} - \beta^2\epsilon_{l,m-1}}{4(\beta+1)}, \tag{38}$$

$$\tilde{\epsilon}_{l,m-\frac{1}{2}} = \frac{-\epsilon_{l,m+1} + (2\beta+1)(\beta+1)\epsilon_{l,m} + \beta(2\beta+1)\epsilon_{l,m-1}}{4\beta(\beta+1)}, \tag{39}$$

$$\begin{aligned}\tilde{\epsilon}_{l+\frac{1}{2},m-\frac{1}{2}} = &-\frac{\beta(\alpha-\beta)\epsilon_{l+1,m-1} + (\alpha+1)\epsilon_{l+1,m+1} + \alpha^2\beta(\beta+1)\epsilon_{l-1,m-1}}{4\beta(\alpha+1)(\beta+1)} \\ &+ \frac{(\beta+1)\epsilon_{l+1,m} + \beta\epsilon_{l,m} + \beta(\alpha+1)\epsilon_{l,m-1}}{4\beta},\end{aligned} \tag{40}$$

$$\begin{aligned}\tilde{\epsilon}_{l-\frac{1}{2},m+\frac{1}{2}} = &\frac{\alpha(\alpha-\beta)\epsilon_{l-1,m+1} - (\beta+1)\epsilon_{l+1,m+1} - \alpha(\alpha+1)\beta^2\epsilon_{l-1,m-1}}{4\alpha(\alpha+1)(\beta+1)} \\ &+ \frac{(\alpha+1)\epsilon_{l,m+1} + \alpha\epsilon_{l,m} + \alpha(\beta+1)\epsilon_{l-1,m}}{4\alpha},\end{aligned} \tag{41}$$

$$\begin{aligned}\tilde{\epsilon}_{l-\frac{1}{2},m-\frac{1}{2}} = &-\frac{\beta(\alpha-\beta)\epsilon_{l+1,m-1} + (\alpha+1)\epsilon_{l+1,m+1} + \beta(\beta+1)(2\alpha+1)\epsilon_{l-1,m-1}}{4\beta(\alpha+1)(\beta+1)} \\ &+ \frac{(\alpha-\beta)\epsilon_{l+1,m} + (2\alpha+1)\beta\epsilon_{l,m}}{4\alpha\beta},\end{aligned} \tag{42}$$

$$\tilde{E}_{l,m} = \eta_{l,m}\epsilon_{l,m}, \quad \eta_{l,m} = \left(\frac{\partial G}{\partial u}\right)_{(x_i,y_j)}, (i,j) \in Q. \tag{43}$$

With the help of Mean value theorem, one obtains

$$\boldsymbol{R} \equiv \phi(\boldsymbol{u}) - \phi(\boldsymbol{U}) = -h_l^{-2}\nabla^2_{l,m}\epsilon_{l,m} + \frac{1}{9}\lambda^2_{l,m}h_l^2 \sum_{(i,j)\in Q} d_{i,j}\tilde{E}_{i,j} \tag{44}$$

In the matrix vector notation, the equation (44) can be written as

$$\boldsymbol{\phi}(\boldsymbol{u}) - \boldsymbol{\phi}(\boldsymbol{U}) = \boldsymbol{P\epsilon}, \tag{45}$$

where $\boldsymbol{P} = [P_{i,j}], i, j = 1(1)LM$ is the block-tridiagonal matrix with the following non-zero elements for $\alpha, \beta \neq (\sqrt{5} \pm 1)/2$,

for $m = 2(1)M$:

$$P_{(m-1)L+l,(m-2)L+l-1} = \frac{\alpha^2 - \alpha - 1 + \lambda_{l,m}^2(\beta^2 - \beta - 1)}{3(\alpha+1)(\beta+1)} + O(h_l^2), l = 2(1)L,$$

$$P_{(m-1)L+l,(m-2)L+l} = -\frac{\alpha^2 + 3\alpha + 1 + \lambda_{l,m}^2(\beta^2 - \beta - 1)}{3\alpha(\beta+1)} + O(h_l^2), l = 1(1)L,$$

$$P_{(m-1)L+l,(m-2)L+l+1} = -\frac{\alpha^2 + \alpha - 1 - \lambda_{l,m}^2(\beta^2 - \beta - 1)}{3\alpha(\alpha+1)(\beta+1)} + O(h_l^2), l = 1(1)L-1,$$

for $m = 1(1)M$:

$$P_{(m-1)L+l,(m-2)L+l-1} = -\frac{\alpha^2 - \alpha - 1 + \lambda_{l,m}^2(\beta^2 + 3\beta + 1)}{3(\alpha+1)\beta} + O(h_l^2), l = 2(1)L,$$

$$P_{(m-1)L+l,(m-2)L+l} = \frac{\alpha^2 + 3\alpha + 1 + \lambda_{l,m}^2(\beta^2 + 3\beta + 1)}{3\alpha\beta} + O(h_l^2), l = 1(1)L,$$

$$P_{(m-1)L+l,(m-2)L+l+1} = \frac{\alpha^2 + \alpha - 1 - \lambda_{l,m}^2(\beta^2 + 3\beta + 1)}{3\alpha\beta(\alpha+1)} + O(h_l^2), l = 1(1)L-1,$$

for $m = 1(1)M - 1$:

$$P_{(m-1)L+l,(m-2)L+l-1} = \frac{\alpha^2 - \alpha - 1 - \lambda_{l,m}^2(\beta^2 + \beta - 1)}{3\beta(\alpha+1)(\beta+1)} + O(h_l^2), l = 2(1)L,$$

$$P_{(m-1)L+l,(m-2)L+l} = -\frac{\alpha^2 + 3\alpha + 1 - \lambda_{l,m}^2(\beta^2 + \beta - 1)}{3\alpha\beta(\beta+1)} + O(h_l^2), l = 1(1)L,$$

$$P_{(m-1)L+l,(m-2)L+l+1} = -\frac{\alpha^2 + \alpha - 1 + \lambda_{l,m}^2(\beta^2 + \beta - 1)}{3\alpha\beta(\alpha+1)(\beta+1)} + O(h_l^2), l = 1(1)L-1,$$

From the equations (34) and (45), we find

$$\boldsymbol{P\epsilon} = \boldsymbol{O_5} \tag{46}$$

Thus, for sufficiently small values of h_l, the lower, upper and main tridiagonal blocks have non-zero entries at sub-diagonal, diagonal and sup-diagonal, if $(\sqrt{5}-1)/2 < \alpha, \beta < (\sqrt{5}+1)/2$. Hence, the graph $\boldsymbol{\mathcal{G}(P)}$ of the matrix $\boldsymbol{P}$ is strongly connected [30]. The visual presentation of the graph $\boldsymbol{\mathcal{G}(P)}$ with $L = M = 3$ are shown in Figure 1. In a general case, the vertices in Figure 1, may be viewed with labels from the grid set $\{x_{l-1}, x_l, x_{l+1}\} \bigotimes \{y_{m-1}, y_m, y_{m+1}\}, l = 1(1)L, m = 1(1)M$. Consequently, the matrix $\boldsymbol{P}$ is irreducible [31].

Let $\eta = \min\limits_{l,m} \eta_{l,m}$ and S_q be the q^{th}row elements sum of the matrix $\boldsymbol{P}$, Thus, for sufficiently small values of h_l, the

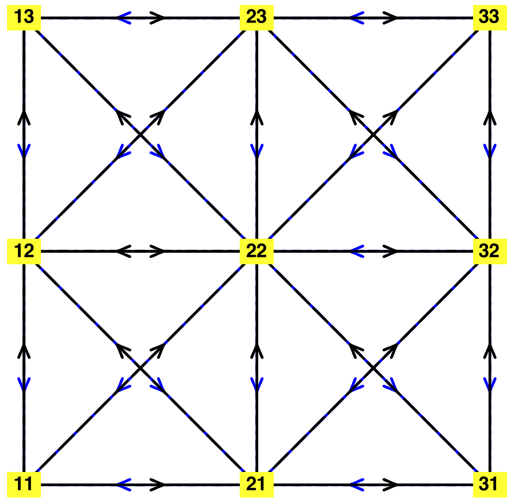

Figure 1: Connected graph of matrix $\boldsymbol{P}$

following inequalities can be established.

$$\begin{aligned}
S_1 &\geq \frac{5\alpha^2+5\alpha+1+\lambda_{l,m}^2(\beta^2+5\beta+5)}{3\alpha(\alpha+1)(\beta+1)}>0,\\
S_q &\geq \frac{2}{\beta+1}>0, q=2(1)L-1,\\
S_L &\geq \frac{\alpha^2+5\alpha+5+\lambda_{l,m}^2(\beta^2+5\beta+5)}{3(\alpha+1)(\beta+1)}>0,\\
S_{(r-1)L+1} &\geq \frac{2\lambda_{l,m}^2}{\alpha(\alpha+1)}>0, r=2(1)M-1,\\
S_{(r-1)L+q} &\geq \eta\lambda_{l,m}^2h_l^2\geq 0, r=2(1)M-1, q=2(1)L-1, \eta\geq 0,\\
S_{(r-1)L+L} &\geq \frac{2\lambda_{l,m}^2}{\alpha+1}>0, r=2(1)M-1,\\
S_{(M-1)L+1} &\geq \frac{5\alpha^2+5\alpha+1+\lambda_{l,m}^2(5\beta^2+5\beta+1)}{3\alpha\beta(\alpha+1)(\beta+1)}>0,\\
S_{(M-1)L+q} &\geq \frac{2}{\beta(\beta+1)}>0, q=2(1)L-1,\\
S_{(M-1)L+L} &\geq \frac{\alpha^2+5\alpha+5+\lambda_{l,m}^2(5\beta^2+5\beta+1)}{3\beta(\alpha+1)(\beta+1)}>0,
\end{aligned}$$

Hence, the matrix $\boldsymbol{P}$ is monotone [32]. Consequently, $\boldsymbol{P}^{-1}$ exists and is non-negative. If $P_{l,m}^{-1}$ be the (l,m)th element of $\boldsymbol{P}^{-1}$ and we define the matrix-vector norms as

$$\|\boldsymbol{P}^{-1}\|_\infty=\max_l\sum_{m=1(1)M}P_{l,m}^{-1}, \quad \|\boldsymbol{O}\|_\infty=\max_l\sum_{m=1(1)M}O_{l,m}=O_5 \tag{47}$$

From the obvious matrix identity $\boldsymbol{P^{-1}(PJ)=J}$, where $\boldsymbol{J}=[1,1,\ldots,1]^T$. We therefore, find that

$$\sum_{q=1(1)LM}P_{p,q}^{-1}S_q=1, p=1(1)LM, \tag{48}$$

Thus, the following bounds can be estimated with the help of equation (44) and series expansions

$$P_{p,1}^{-1}\leq\frac{1}{S_1}\leq\frac{3\alpha(\alpha+1)(\beta+1)}{\lambda_{l,m}^2(\beta^2+5\beta+5)+5\alpha^2+5\alpha+1}+O(h_l^2), \tag{49}$$

$$\sum_{q=2}^{L-1} P_{p,q}^{-1} \le \frac{1}{\underset{q=2(1)L-1}{min} S_q} \le \frac{\beta+1}{2} + O(h_l^2), \tag{50}$$

$$P_{p,L}^{-1} \le \frac{1}{S_L} \le \frac{3(\alpha+1)(\beta+1)}{\lambda_{l,m}^2(\beta^2+5\beta+5)+\alpha^2+5\alpha+5} + O(h_l^2), \tag{51}$$

$$\sum_{r=2}^{M-1} P_{p,(r-1)L+1}^{-1} \le \frac{1}{\underset{r=2(1)M-1}{min} S_{(r-1)L+1}} \le \frac{\alpha(\alpha+1)}{2\lambda_{l,m}^2} + O(h_l^2), \tag{52}$$

$$\sum_{r=2}^{M-1}\sum_{q=2}^{L-1} P_{p,(r-1)L+q}^{-1} \le \begin{cases} \sum_{q=1}^{LM} P_{p,q}^{-1} S_q = 1, \eta = 0, \\ \frac{1}{\underset{\substack{q=2(1)L-1\\ r=2(1)M-1}}{min} S_{(r-1)L+q}} \le \frac{1}{\eta\lambda_{l,m}^2} + O(h_l^\zeta), \zeta \ge 0, \eta > 0, \end{cases} \tag{53}$$

$$\sum_{r=2}^{M-1} P_{p,rL}^{-1} \le \frac{1}{\underset{r=2(1)M-1}{min} S_{rL}} \le \frac{\alpha+1}{2\lambda_{l,m}^2} + O(h_l^2), \tag{54}$$

$$P_{p,(M-1)L+1}^{-1} \le \frac{1}{S_{(M-1)L+1}} \le \frac{3\alpha\beta(\alpha+1)(\beta+1)}{\lambda_{l,m}^2(5\beta^2+5\beta+1)+5\alpha^2+5\alpha+1} + O(h_l^2), \tag{55}$$

$$\sum_{q=2}^{L-1} P_{p,(M-1)L+q}^{-1} \le \frac{1}{\underset{q=2(1)L-1}{min} S_{(M-1)L+q}} \le \frac{\beta(\beta+1)}{2} + O(h_l^2), \tag{56}$$

$$P_{p,LM}^{-1} \le \frac{1}{S_{LM}} \le \frac{3\beta(\alpha+1)(\beta+1)}{\lambda_{l,m}^2(5\beta^2+5\beta+1)+\alpha^2+5\alpha+5} + O(h_l^2) \tag{57}$$

As a result, from the above inequalities, we finally obtain

$$\| \boldsymbol{\epsilon} \|_\infty \le \| \boldsymbol{P^{-1}} \|_\infty \| \boldsymbol{O} \|_\infty \le O_3(h_l, k_m), \quad \text{provided} \quad \eta \ge 0. \tag{58}$$

This proves the third order accuracy of our scheme in the geometric mesh network. We conclude this section with the formal deduction that the proposed geometric mesh arithmetic average method is third order accurate for the numerical solution of the semi-linear elliptic partial differential equations, and it is convergent for sufficiently small values of h_l and $\frac{\partial G}{\partial u} > 0$ (in order to achieve $\eta \neq 0$, precisely $\eta > 0$) and $(\sqrt{5}-1)/2 < \alpha, \beta < (\sqrt{5}+1)/2$. In particular, it gives us fourth order of accuracy $O_4(h_l, k_m)$ for $\alpha = \beta = 1$ as a uniform mesh discretizations.

4 Experimental results

Numerical computations are performed for linear and semi-linear elliptic problems with singular and non singular cases. For solving linear and semi-linear difference equations, we have employed Gauss-Seidel and Newton-Raphson iterative methods. To test the accuracy, we have compared the approximate solutions with theoretical solutions in terms of maximum absolute errors ($\mathcal{E}^\infty$) of solution values and computational order of convergence (Θ), using the formula

$$\mathcal{E}^\infty = \max_{l,m} | u_{l,m} - U_{l,m} |, \quad \Theta = \log_2 \left(\frac{\mathcal{E}^\infty(L,M)}{\mathcal{E}^\infty(2L+1,2M+1)} \right).$$

The maximum absolute errors and convergence order for partial order derivatives of the solution variables are defined analogously. Results are obtained on geometric meshes $\alpha \neq 1, \beta \neq 1$ and uniform meshes $\alpha = \beta = 1$ as well. An error tolerance of 10^{-10} has been taken with zero vector as an initial guess. For simplicity, we have chosen L and M both equals. A similar observations are seen with L, M being distinct. All the computer programs are written in C and computed under Mac OS. The symbolic computations were carried out using $taylor()$ command in Maple software.

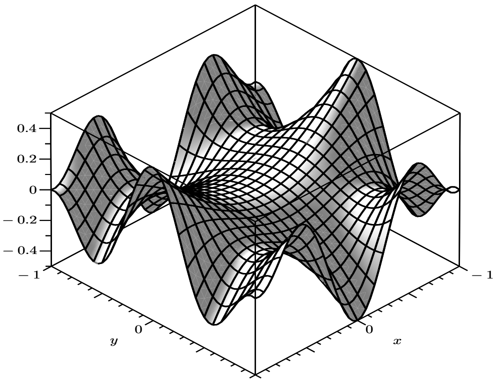

Figure 2: Graphical solutions at $\alpha = \beta = 1.01$

Example 1 *Consider the Helmholtz equation [33]*

$$\nabla^2 U = \theta^2 U, \quad \Omega = [0, 1]^2$$

with the analytic solution $U(x, y) = \frac{\cos(\pi x)\sinh(\sqrt{\theta^2+\pi^2}(1-y))}{\sinh(\sqrt{\theta^2+\pi^2})}$.

The maximum absolute errors and computational order of convergence with uniform meshes are obtained for $\theta = 2$ and various values of L in Table 1 for U, U_x, U_y, U_{xx} and U_{yy}. Table 2 presents the corresponding values in case of optimum geometric mesh ratio parameters. A significant improvement in the numerical accuracy of solution values has been observed with third order geometric mesh method compared with fourth order method of uniform meshes. Apart from the accuracy of solutions, computing time is also found to be encouraging with the geometric mesh approach over uniform mesh.

Table 1: Error estimates with uniform meshes for Example 1

L	$\mathcal{E}^\infty$	$\mathcal{E}_x^\infty$	$\mathcal{E}_y^\infty$	$\mathcal{E}_{xx}^\infty$	$\mathcal{E}_{yy}^\infty$	Θ	Θ_x	Θ_{xx}
7	1.46e-05	1.32e-02	1.42e-02	7.39e-02	1.47e-01	—	—	—
15	9.78e-07	2.18e-03	2.27e-03	2.47e-02	4.88e-02	3.9	2.6	1.6
31	6.00e-08	3.09e-04	3.16e-04	7.02e-03	1.39e-02	4.0	2.8	1.8

Table 2: Error estimates with geometric meshes for Example 1

L	α	β	$\mathcal{E}^\infty$	$\mathcal{E}_x^\infty$	$\mathcal{E}_y^\infty$	$\mathcal{E}_{xx}^\infty$	$\mathcal{E}_{yy}^\infty$
7	1.000	1.030	4.02e-07	1.26e-02	1.11e-02	7.06e-02	1.33e-01
15	1.000	1.014	6.19e-08	2.02e-03	1.71e-03	2.12e-02	4.17e-02
31	1.001	1.007	3.55e-09	3.06e-04	2.31e-04	6.91e-03	1.15e-02

Example 2 *Consider the stationary Schrödinger equation [34]*

$$\nabla^2 U + \theta^2(x^2 + y^2)U = 0, \quad \Omega = [0, 1]^2$$

with the analytic solution $U(x, y) = \frac{1}{2}(x^2 - y^2)\cos(\theta xy)$.

The maximum absolute errors are obtained for $\theta = 5$ and various values of L in Table 3 for U. The graphical illustrations of approximate and exact solution values on the domain $[-1, 1]^2$ has been shown in Figure 2 with $h_l = 1/8$ and $\theta = 5$.

Example 3 *Consider the linear singular equation*

$$\nabla^2 U - \left(\frac{1}{x(x-1)} + \frac{1}{y(y-1)}\right) U = F(x, y), \quad \Omega = [0, 1]^2$$

Table 3: Error estimates of solution values for Example 2

L	α	β	$\mathcal{E}^\infty$	Θ	α	β	$\mathcal{E}^\infty$	Θ
7	1.00	1.00	2.18e-04	—	1.01	1.01	2.05e-04	—
15	1.00	1.00	1.36e-05	4.0	1.01	1.01	1.09e-05	3.9
31	1.00	1.00	8.78e-07	4.0	1.01	1.01	8.58e-07	3.9

with the analytic solution $U(x, y) = xy(x - 1)(y - 1)$.

The right hand side function $F(x, y)$ *and boundary conditions may be obtained from the analytic solution as a test procedure. The solutions deteriorate with the application of uniform meshes, since the given PDE has singular behaviour near the boundary. Such a situation can easily be handled in geometric mesh network as shown in the Table 4, with the numerical values of maximum absolute errors and convergence order, evaluated for various values of L.*

Table 4: Error estimates of solution values for Example 3

L	α	β	$\mathcal{E}^\infty$	α	β	$\mathcal{E}^\infty$	Θ
7	1.00	1.00	overflow	0.92	0.92	5.94e-07	—
15	1.00	1.00	overflow	0.92	0.92	5.52e-08	3.4
31	1.00	1.00	overflow	0.92	0.92	5.94e-09	3.2

Example 4 *Consider the semi-linear equation [35]*

$$\nabla^2 U + e^{-U} = F(x, y), \quad \Omega = [0, \pi]^2$$

with the analytic solutions $U(x, y) = x(\pi - x)(\pi - y) \exp(x \cos y)$.

The unknown function $F(x, y)$ *and boundary conditions can be obtained from the analytic solution. The maximum absolute errors are obtained for various values of L in Table 5.*

Table 5: Error estimates of solution values for Example 4

L	α	β	$\mathcal{E}^\infty$	Θ	α	β	$\mathcal{E}^\infty$	Θ
7	1.00	1.00	6.46e-02	—	1.01	1.16	1.42e-02	—
15	1.00	1.00	4.72e-03	3.8	1.00	1.07	1.26e-03	3.5
31	1.00	1.00	2.95e-04	4.0	1.00	1.03	8.44e-05	3.9

5 Conclusions

A new lower order accurate geometric mesh compact finite difference method using off-step grids has been presented for the approximate solution of semi-linear elliptic partial differential equations with Dirichlet'S boundary conditions. The method has flexibility to achieve an accuracy of third or fourth order depending on the values of geometric mesh parameters within the specified range. The method has been used to obtain third order of accuracy for the first order partial derivative and second order of accuracy for the second order partial derivatives of solution values in the case of geometric meshes. Despite the lower truncation order, the proposed geometric mesh strategy shows superiority over uniform mesh scheme in terms of accuracy. Moreover, the method is directly applicable to the problems with singular coefficients. A simple modification in geometric mesh parameters α as α_l and β as β_m gives us a fully non uniform mesh discretization with the same order of accuracy. Applications of the proposed method to the more general cases viz quasi-linear elliptic problems with significant first order partial derivatives would be an interesting finding.

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