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Symmetry Reduction of (2+1)-Dimensional Dissipative Zabolotskaya-Khokhlov Equation

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Abstract: Symmetry reductions of the dissipative Zabolotskaya-Khokhlov equation are studied by the means of the Clarkson-Kruskal direct method and the corresponding reduction equations are solved directly with arbitrary functions and constants.

Keywords: symmetry reduction; Clarkson-Kruskal direct method; dissipative Zabolotskaya-Khokhlov equation

1 Introduction

The study on the exact solutions of the nonlinear evolution equations plays an important role in nonlinear science. Most of the phenomena in the world are nonlinear and complex. In order to learn much more about the nonlinear phenomena in the nature world, mathematician and physicists have proposed many effective methods to obtain more exact solutions of the nonlinear evolution equations, such as the inverse scattering transformation, the Hirota bilinear form, symmetry reduction, Darboux transformation and Bäcklund transformation, etc. It is well known that symmetry analysis is also one of the most effective methods for obtaining exact and analytical solutions of nonlinear systems. The classical method for finding symmetry reductions of nonlinear systems is the Lie group method and its generalized forms [1]. The direct and algorithmic method to find symmetry reductions is called Clarkson-Kruskal direct method (CK direct method), which can be used to obtain previously unknown reductions of nonlinear systems [2]. Many new symmetry reductions and exact solutions for a large number of physically significant nonlinear systems have been obtained by the means of the CK direct method.

The (3+1)-dimensional dissipative Zabolotskaya-Khokhlov equation was firstly derived by Kuznetsov [3]

$$u_{xt} + au_{xxx} + uu_{xx} + u_x^2 + bu_{yy} + bu_{zz} = 0, \tag{1}$$

with a, b being constants. When $a = 0$, Eq.(1) reduces to the Zabolotskaya-Khokhlov equation in the form as originally derived by Zabolotskaya and Khokhlov in [4] and Eq.(1) is also called the dispersionless Kadomtsev-Petviashvili equation if $u_{zz} = 0$. The dissipative Zabolotskaya-Khokhlov equation (DZK) has many applications in physical fields, in particular in the study of acoustic signals propagating through stratified media such as the ocean, and in the study of acoustic beams in various geometries. In Eq.(1), u is proportional to the perturbation pressure; the nonlinear term uu_{xx} models advective nonlinearity and au_{xxx} models dissipation. When $a \neq 0$ and $u_{zz} = 0$, Eq.(1) is just the (2+1)-dimensional dissipative Zabolotskaya-Khokhlov equation

$$u_{xt} + au_{xxx} + uu_{xx} + u_x^2 + bu_{yy} = 0, \tag{2}$$

which is also called (2+1)-dimensional Burgers equation. Eq.(2) describes the propagation of bi-dimensional waves in a viscous fluid and of nonlinear waves in radiative magneto-gasdynamics [5]. The classical and nonclassical Lie symmetry groups for Eq.(1) are studied and explained in [6, 7] and a general symmetry with undetermined functions is given in [8].

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In this paper, we will study the symmetry reduction for the (2+1)-dimensional DZK equation by the means of the CK direct method. The paper is organized as follows. In Section 2, several different types of symmetry reductions are studied by the CK direct method. Last section is summary and discussions.

2 Symmetry reduction of the (2+1)-dimensional DZK equation

In the soliton theory, there is much interest in obtaining exact analytical solutions of nonlinear systems. Symmetry reduction is either by seeking a similarity solution in a special form or, more generally, by exploiting symmetries of the nonlinear equations. On the other hand, many high dimension nonlinear systems can be reduced to low dimension differential equations or ordinary differential equations. In order to find the similarity solutions of a nonlinear system, one can use the standard classical Lie approach, nonclassical Lie approach, the Clarkson-Kruskal direct method and modified Clarkson-Kruskal method. The Clarkson-Kruskal direct method is the most convenient one to obtain the similarity reduction of a nonlinear system.

In this section, we will study the similarity reduction of the DZK equation (2) and five arbitrary constants and three arbitrary functions are included in the similarity reduction. With the help of Clarkson-Kruskal direct method, it is sufficient to seek a similarity reduction of the (2+1)-dimensional DZK equation in the special form

$$u(x,y,t)=\alpha(x,y,t)+\beta(x,y,t)P(\xi(x,y,t),\eta(x,y,t)), \tag{3}$$

where $\alpha=\alpha(x,y,t)$, $\beta=\beta(x,y,t)$, $\xi=\xi(x,y,t)$ and $\eta=\eta(x,y,t)$ are functions to be determined. Substituting Eq.(3) into Eq.(2) yields

$$\begin{aligned}
&a\beta\xi_x^3P_{\xi\xi\xi}+3a\beta\eta_x\xi_x^2P_{\xi\xi\eta}+3a\beta\xi_x\eta_x^2P_{\xi\eta\eta}+a\beta\eta_x^3P_{\eta\eta\eta}+\beta^2\xi_x^2PP_{\xi\xi}+\beta^2\eta_x^2PP_{\eta\eta}\\
&+2\beta^2\xi_x\eta_xPP_{\xi\eta}+(3a\beta\eta_x\xi_{xx}+2\alpha\beta\xi_x\eta_x+6a\beta_x\xi_x\eta_x)P_{\xi\eta}+(4\beta\beta_x\eta_x+\beta^2\eta_{xx})PP_\eta\\
&+(\beta\alpha_{xx}+2\alpha_x\beta_x+b\beta_{yy}+a\beta_{xxx}+\alpha\beta_{xx}+\beta_{xt})P+(4\beta\beta_x\xi_x+\beta^2\xi_{xx})PP_\xi\\
&+(\beta_x^2+\beta\beta_{xx})P^2+(3a\beta\eta_x\eta_{xx}+\alpha\beta\eta_x^2+b\beta\eta_y^2+\beta\eta_x\eta_t+3a\beta_x\eta_x^2)P_{\eta\eta}\\
&+(\alpha\beta\xi_x^2+b\beta\xi_y^2+3a\beta_x\xi_x^2+\beta\xi_x\xi_t+3a\beta\xi_x\xi_{xx})P_{\xi\xi}+2\beta^2\xi_x\eta_xP_\xi P_\eta+\beta^2\xi_x^2P_\xi^2\\
&+(2b\beta\xi_y\eta_y+3a\beta\xi_x\eta_{xx}+3a\beta\xi_{xx}\eta_x+2\alpha\beta\xi_x\eta_x+6a\xi_x\eta_x\beta_x+\beta\xi_x\eta_t+\beta\eta_x\xi_t)P_{\xi\eta}\\
&+(\beta\xi_{xt}+3a\beta_x\xi_{xx}+3a\beta_{xx}\xi_x+a\beta\xi_{xxx}+b\beta\xi_{yy}+\alpha\beta\xi_{xx}+2b\beta_y\xi_y+\beta_x\xi_t+\beta_t\xi_x+2\alpha\beta_x\xi_x\\
&+2\beta\alpha_x\xi_x)P_\xi+(2\beta\alpha_x\eta_x+2\alpha\beta_x\eta_x+\beta_t\eta_x+\beta_x\eta_t+2b\beta_y\eta_y+\alpha\beta\eta_{xx}+b\beta\eta_{yy}+a\beta\eta_{xxx}\\
&+3a\beta_{xx}\eta_x+3a\beta_x\eta_{xx}+\beta\eta_{xt})P_\eta+\beta^2\eta_x^2P_\eta^2+\alpha_x^2+a\alpha_{xxx}+\alpha\alpha_{xx}+b\alpha_{yy}+\alpha_{xt}=0.
\end{aligned} \tag{4}$$

In order to require this equation be a partial differential equation for the function $P(\xi,\eta)$, the ratios of the coefficients of different derivatives and powers of $P(\xi,\eta)$ have to be functions of ξ,η only. We use the coefficient of $P_{\xi\xi\xi}$ as the normalizing coefficient and require that other coefficients be of the form $a\beta\xi_x^3\Gamma(\xi,\eta)$, where $\Gamma(\xi,\eta)$ is a function of ξ,η to be determined. There are five freedoms in the determination of functions α, β, ξ and η without loss of generality:

rule 1 : If $\alpha(x,y,t)$ has the form $\alpha(x,y,t)=\alpha_0(x,y,t)+\beta(x,y,t)Q(\xi,\eta)$, then we can take $Q(\xi,\eta)=0$ [by substituting $P(\xi,\eta)\to P(\xi,\eta)-Q(\xi,\eta)$];

rule 2 : If $\beta(x,y,t)$ has the form $\beta(x,y,t)=\beta_0(x,y,t)Q(\xi,\eta)$, then we can take $Q(\xi,\eta)=1$ [by substituting $Q(\xi,\eta)\to P(\xi,\eta)/Q(\xi,\eta)$];

rule 3 : If $\xi(x,y,t)$ is determined by an equation of the form $Q(\xi,\eta)=\xi_0(x,y,t)$, where $Q(\xi,\eta)$ is any invertible function, then we can take $Q(\xi,\eta)=\xi_0$;

rule 4 : If $\eta(x,y,t)$ is determined by an equation of the form $Q(\xi,\eta)=\eta_0(x,y,t)$, where $Q(\xi,\eta)$ is any invertible function, then we can take $Q(\xi,\eta)=\eta_0$;

rule 5 : We reserve uppercase Greek letters for undetermined functions of z so that after performing operations (differentiation, integration, exponentiation, rescaling, etc.) the result can be denoted by the same letter for simplicity.

Case1 : $\xi_x\neq 0$;

We use the function $a\beta\xi_x^3$ as the normalizing coefficient and have

$$3a\beta\xi_x\eta_x^2=a\beta\xi_x^3\Gamma_1(\xi,\eta), \tag{5}$$

$$\beta^2\xi_x^2=a\beta\xi_x^3\Gamma_2(\xi,\eta), \tag{6}$$

$$4\beta\beta_x\xi_x + \beta^2\xi_{xx} = a\beta\xi_x^3\Gamma_3(\xi,\eta), \tag{7}$$

$$\alpha\beta\xi_x^2 + b\beta\xi_y^2 + 3a\beta_x\xi_x^2 + \beta\xi_x\xi_t + 3a\beta\xi_x\xi_{xx} = a\beta\xi_x^3\Gamma_4(\xi,\eta), \tag{8}$$

$$\beta^2\eta_x^2 = a\beta\xi_x^3\Gamma_5(\xi,\eta), \tag{9}$$

$$2\beta^2\xi_x\eta_x = a\beta\xi_x^3\Gamma_6(\xi,\eta), \tag{10}$$

$$4\beta\beta_x\eta_x + \beta^2\eta_{xx} = a\beta\xi_x^3\Gamma_7(\xi,\eta), \tag{11}$$

$$3a\beta\eta_x\xi_{xx} + 2\alpha\beta\xi_x\eta_x + 6a\beta_x\xi_x\eta_x = a\beta\xi_x^3\Gamma_8(\xi,\eta), \tag{12}$$

$$3a\beta\eta_x\xi_x^2 = a\beta\xi_x^3\Gamma_9(\xi,\eta), \tag{13}$$

$$a\beta\eta_x^3 = a\beta\xi_x^3\Gamma_{10}(\xi,\eta), \tag{14}$$

$$\beta_x^2 + \beta\beta_{xx} = a\beta\xi_x^3\Gamma_{11}(\xi,\eta), \tag{15}$$

$$\beta\alpha_{xx} + 2\alpha_x\beta_x + b\beta_{yy} + a\beta_{xxx} + \alpha\beta_{xx} + \beta_{xt} = a\beta\xi_x^3\Gamma_{12}(\xi,\eta), \tag{16}$$

$$3a\beta\eta_x\eta_{xx} + \alpha\beta\eta_x^2 + b\beta\eta_y^2 + \beta\eta_x\eta_t + 3a\beta_x\eta_x^2 = a\beta\xi_x^3\Gamma_{13}(\xi,\eta), \tag{17}$$

$$2b\beta\xi_y\eta_y + 3a\beta\xi_x\eta_{xx} + 3a\beta\xi_{xx}\eta_x + 2\alpha\beta\xi_x\eta_x + 6a\xi_x\eta_x\beta_x + \beta\xi_x\eta_t + \beta\eta_x\xi_t = a\beta\xi_x^3\Gamma_{14}(\xi,\eta), \tag{18}$$

$$\begin{aligned}&\beta\xi_{xt} + 3a\beta_x\xi_{xx} + 3a\beta_{xx}\xi_x + a\beta\xi_{xxx} + b\beta\xi_{yy} + \alpha\beta\xi_{xx} + 2b\beta_y\xi_y\\&+\beta_x\xi_t + \beta_t\xi_x + 2\alpha\beta_x\xi_x + 2\beta\alpha_x\xi_x = a\beta\xi_x^3\Gamma_{15}(\xi,\eta),\end{aligned} \tag{19}$$

$$\begin{aligned}&2\beta\alpha_x\eta_x + 2\alpha\beta_x\eta_x + \beta_t\eta_x + \beta_x\eta_t + 2b\beta_y\eta_y + \alpha\beta\eta_{xx} + b\beta\eta_{yy} + a\beta\eta_{xxx}\\&+3a\beta_{xx}\eta_x + 3a\beta_x\eta_{xx} + \beta\eta_{xt} = a\beta\xi_x^3\Gamma_{16}(\xi,\eta),\end{aligned} \tag{20}$$

$$\alpha_x^2 + a\alpha_{xxx} + \alpha\alpha_{xx} + b\alpha_{yy} + \alpha_{xt} = a\beta\xi_x^3\Gamma_{17}(\xi,\eta), \tag{21}$$

where $\Gamma_j(\xi,\eta), (j = 1,\ldots,17)$ are functions to be determined.

From Eq.(5), we have $\eta = \int \Gamma_1(\xi,\eta)\mathrm{d}\xi + \eta_0(y,t)$, thus we learn

$$\eta = \eta(y,t) \tag{22}$$

from *rule 4*. Using *rule 2* and (6), we have $\Gamma_2(\xi,\eta) = 1$ and

$$\beta = \xi_x. \tag{23}$$

Substituting Eq.(23) into Eq.(7), we have

$$\xi_{xx}/\xi_x + \xi_x\Gamma_3(\xi,\eta) = 0,$$

which upon integration gives

$$\Gamma_3(\xi,\eta) + \ln\xi_x = \theta(y,t).$$

Exponentiating and integrating again gives

$$\xi = x\theta(y,t) + \sigma(y,t), \tag{24}$$

by *rule 3* and the arbitrary functions $\theta(y,t), \sigma(y,t)$ are functions of integration.

Substituting Eqs.(22) and (24) into Eq.(8), we have

$$\alpha = -\frac{bx^2\theta_y^2 + 2bx\theta_y\sigma_y + b\sigma_y^2 + x\theta\theta_t + \theta\sigma_t}{\theta^2}, \tag{25}$$

and

$$\Gamma_4(\xi,\eta) = 0,$$

by *rule 1*.

Substituting Eqs.(22)-(24) into Eqs.(9)-(18) and (20), we have $\Gamma_j = 0, (j = 5, \cdots, 12, 16)$ and

$$\eta_y = \theta^{\frac{3}{2}}\Gamma_{13}(\eta),$$
$$\frac{\theta_y}{\theta} = \theta^{\frac{3}{2}}\Gamma_{14}(\eta),$$

which after integrating and using the freedom of η gives

$$\eta = f_1^3 y + f_2,\ \theta = f_1^2,\ \Gamma_{13}(\eta) = 1. \tag{26}$$

Substituting (26) into Eq.(18), we can obtain

$$\sigma = -\frac{3}{4b}y^2 f_1 f_{1t} - \frac{y f_{2t}}{2b f_1} + f_3,\ \Gamma_{14} = 0, \tag{27}$$

where f_2, f_3 are arbitrary functions of t. Then substituting (26)-(27) into Eqs.(19) and (21), we find that

$$f_1 = (-4t + c_1)^{-\frac{1}{4}},\ \Gamma_{15} = 1,\ \Gamma_{17} = \frac{1}{2a},$$

where c_1 is an arbitrary constant and the reduction equation is in the form

$$aP_{\xi\xi\xi} + PP_{\xi\xi} + P_\xi^2 + bP_{\eta\eta} - \frac{3}{2}P_\xi + \frac{1}{2} = 0.$$

When $\eta_y = 0$, we set $\eta = t$ without loss of generality. Then it is easily shown that

$$\beta = 1,\ \xi = x + y^2 f_4 + y f_5 + f_6,$$
$$\alpha = -(4bf_4^2 + f_{4t})y^2 - (4bf_4 f_5 + f_{5t})y - bf_5^2 - f_{6t},$$

and the corresponding reduction equation is expressed by

$$aP_{\xi\xi\xi} + PP_{\xi\xi} + P_\xi^2 + 2bf_4 P_\xi + P_{\xi t} - 8b^2 f_4^2 - 2bf_{4t} = 0, \tag{28}$$

where f_4, f_5, f_6 are arbitrary functions of t. When fixing $f_4 = 0$, we can obtain the exact solution of Eq.(28)

$$P(\xi, t) = 2aC_2 \tanh[C_2(\xi + C_3 t) + C_1] - C_3,$$

with C_1, C_2, C_3 being arbitrary constants.

Case2 : $\xi_x = 0$;

In this case, we set $\eta_x = 0$ and $\xi_y \neq 0$ for simplicity. Similar to the procedure of *Case1*, we have the similarity reduction

$$\alpha = x^2 f_7 + x f_8 + f_9,\ \beta = 1,\ \xi = y,\ \eta = t,$$

and the reduction equation is given in the following

$$bP_{yy} + 2f_7 P + f_{8t} + f_8^2 + 2f_7 f_9 + bf_{9yy} = 0,$$

with $f_9(y, t)$ being an arbitrary function and the functions $f_7(y, t), f_8(y, t)$ satisfy

$$6f_7^2 + bf_{7yy} = 0,\ 2f_{7t} + 6f_7 f_8 + bf_{8yy} = 0. \tag{29}$$

It is easy to find the solution of the first equation of Eq.(29) is a Weierstrass elliptic function $f_7 = -b\wp(y + F_1(t), 0, F_2(t))$ and $F_1(t), F_2(t)$ are two arbitrary functions.

3 Conclusion and discussion

As we all know, the study on the symmetry reductions of a given nonlinear system is of great importance to learn more integrable property in soliton theory. With the help of CK direct method, we obtain different symmetry reductions with several arbitrary functions or constants of (2+1)-dimensional DZK equation for $\xi_x \neq 0$ and $\xi_x = 0$. By selecting proper arbitrary functions, the exact solutions of the reduction equation are given out directly. Much more integrable properties such as the nonlocal symmetry, consistent tanh expansion and consistent Riccati expansion will be studied and reported elsewhere.

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