ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.23(2017) No.1, pp.11-21

Exact and Approximate Solutions of a System of Partial Differential Equations

Muna Al Ghabshi[1], E. V. Krishnan[1] *, Kamel Al-Khaled[1,2], Marwan Alquran[1,2]
[1] Department of Mathematics and Statistics, Sultan Qaboos University, PO Box 36, Al Khod 123, Muscat, Sultanate of Oman
[2] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
(Received 3 December 2015, accepted 24 November 2016)

Abstract: This paper deals with the exact and approximate solutions of a coupled wave equation. Mapping methods have been employed to derive Jacobi elliptic function solutions of the equation and their special cases when the modulus of the elliptic functions approach 0 and 1. The Adomian polynomial method has been used to find approximate solutions and they have been shown to converge to the exact solutions derived by the mapping methods.

Keywords: Mapping method; Jacobi elliptic function; solitary wave solution; shock wave solution; Adomian polynomial

Mathematics Subject Classification: 74J35, 34G20, 93C10.

1 Introduction

The nonlinear evolution equations (NLEEs) is one of the most important fields of research in applied mathematics and theoretical physics. There are several forms of NLEEs that arise in various branches of science and engineering [1-5]. Different aspects of these equations such as integrability, conservation laws, wave interactions, painleve analysis and a host of other subjects have been addressed in literature. Several methods such as tanh method [6-9], exponential function method [10], Jacobi elliptic function (JEF) method [11], mapping methods [12-16] etc. have been applied in the past couple of decades and the results have been reported. Also, many physical phenomena have been governed by systems of partial differential equations (PDEs) and there have been significant contributions in this area [17-19].

This paper addresses the integrability aspect of a coupled wave equation. We derive periodic wave solutions (PWSs) of the coupled wave equation in terms of JEFs [20] and deduce their infinite period counterparts in terms of hyperbolic functions such as solitary wave solutions (SWSs), shock wave solutions and singular wave solutions. We also derive certain trigonometric function solutions in a different limiting case. The mapping methods employed in this paper give a variety of solutions which other methods cannot. We also use the Adomian Decomposition Method (ADM) [21,22] to get an approximate solution and show its convergence to the exact solution.

The paper is organized as follows: In section 2, we give a mathematical analysis of the mapping methods, in section 3, we derive a variety of solutions using mapping methods, ADM has been applied to get an approximate solution in section 4, and section 5 is devoted for the conclusion. An appendix is given for the definition and properties of JEFs.

2 Mathematical Analysis of Mapping Methods

In this section, we give an analysis of mapping methods which will be employed in this paper.

Consider a nonlinear coupled PDE with two dependent variables u and v and two independent variables x and t given by

$$F(u, v, u_t, v_t, u_x, v_x, u_{xxx}, v_{xxx}, ...) = 0, \quad (1)$$

* **Corresponding author**. *E-mail address*: krish@squ.edu.om or drkrish52@gmail.com

IJNS.2017.02.15/941

where subscripts denote partial derivatives with respect to the corresponding independent variables and F is a polynomial function of the indicated variables.

Step 1: Assume that eq. (1) has a travelling wave solution (TWS) in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{l_1} A_i f^i(\xi), v(x,t) = v(\xi) = \sum_{i=0}^{l_2} B_i\, f^i(\xi), \tag{2}$$

where $\xi = x - \eta\, t$, A_i, B_i and η are arbitrary constants, l_1 and l_2 are integers and f^i represents integer powers of f.

The first derivative of f with respect to ξ denoted by f' can be expressed in powers of f in the form

$$f'^2 = p\, f^2 + \frac{1}{2} q f^4 + r, \tag{3}$$

where p, q and r are arbitrary constants.

The motivation for eq. (3) was that the squares of the first derivatives of JEFs can be expressed in even powers of themselves.

Step 2: Substituting eq. (2) into eq. (1), the PDE reduces to an ODE. Balancing the highest order derivative term and the highest order nonlinear term of the ODE, the values of l_1 and l_2 can be found.

Step 3: Substituting for u and v and using eq. (3), the ODE gives rise to a set of algebraic equations by setting the coefficients of various powers of f to zero.

Step 4: From the values of the parameters A_i, B_i, p, q and r, the solution of eq. (1) can be derived.

Thus a mapping relation is established through eq. (2) between the solution to eq. (3) and that of eq. (1).

It is to be noted that if the values of l_1 and l_2 are integers, we can use the method directly to get a variety of solutions in terms of hyperbolic functions or JEFs. If they are non integers, the equation may still have solutions as rational expressions involving hyperbolic functions or JEFs.

In the modified mapping method, we assume expansions in both positive and negative powers of f. In the extended mapping method, we have in our expansion a combination of two functions f and g where f satisfies eq. (3) and g^2 and g'' are expressed in terms of f and g.

The properties of JEFs are given in the appendix.

3 The Coupled Wave Equation

Consider the system of PDEs :

$$u_t + \alpha v^2 v_x + \beta u^2 u_x + \lambda u u_x + \gamma u_{xxx} = 0, \tag{4}$$

$$v_t + \sigma(uv)_x + \epsilon v v_x = 0, \tag{5}$$

where, $\alpha, \beta, \lambda, \gamma, \sigma$ and ϵ are constants.

We seek TWSs in the form $u = u(\xi), \quad v = v(\xi), \quad \xi = x - \eta t.$

Then eqs. (4) and (5) give:

$$-\eta u_\xi + \alpha v^2 v_\xi + \beta u^2 u_{\xi+\lambda} u u_\xi + \gamma u_{\xi\xi\xi} = 0, \tag{6}$$

$$-\eta v_\xi + \sigma(uv)_\xi + \epsilon v v_\xi = 0. \tag{7}$$

Integrating eq. (7) with respect to ξ ,

$$-\eta v + \sigma(uv) + \frac{\epsilon}{2} v^2 = k, \tag{8}$$

where k is the integration constant.

Dividing eq. (8) by v, we obtain

$$-\eta + \sigma u + \frac{\epsilon}{2} v = \frac{k}{v}. \tag{9}$$

So, for solutions to be uniformly valid, the integration constant k should be set equal to 0.

Therefore, eq. (9) can be written as

$$v = \frac{2(\eta - \sigma u)}{\epsilon}. \tag{10}$$

Substituting eq. (10) into eq. (6), we obtain

$$-\eta u_{\xi} - \frac{8\alpha\sigma}{\epsilon^3}(\eta^2 - 2\sigma\eta u + \sigma^2 u^2)u_{\xi} + \beta u^2 u_{\xi} + \lambda u u_{\xi} + \gamma u_{\xi\xi\xi} = 0. \tag{11}$$

Integrating eq. (11) with respect to ξ and assuming the boundary conditions $u, u_{\xi}, u_{\xi\xi} \longrightarrow 0$ as $|\xi| \longrightarrow \infty$, we have

$$\gamma u_{\xi\xi} - \left\{\eta + \frac{8\alpha\sigma\eta^2}{\epsilon^3}\right\}u + \left\{\frac{\lambda}{2} + \frac{8\alpha\sigma^2\eta}{\epsilon^3}\right\}u^2 + \left\{\frac{\beta}{3} - \frac{8\alpha\sigma^3}{3\epsilon^3}\right\}u^3 = 0. \tag{12}$$

We rewrite eq. (12) as

$$\gamma u_{\xi\xi} + Au + Bu^2 + Cu^3 = 0, \tag{13}$$

where

$$A = -\left\{\eta + \frac{8\alpha\sigma\eta^2}{\epsilon^3}\right\}, \ B = \left\{\frac{\lambda}{2} + \frac{8\alpha\sigma^2\eta}{\epsilon^3}\right\}, \ C = \left\{\frac{\beta}{3} - \frac{8\alpha\sigma^3}{3\epsilon^3}\right\}. \tag{14}$$

3.1 Mapping Method

Following the analysis of the method, we can easily see that we can assume the solution of eq. (13) in the form :

$$u(\xi) = A_0 + A_1 f, \tag{15}$$

where A_0 and A_1 are constants and f satisfies

$$f'' = pf + qf^3, \quad f'^2 = pf^2 + \frac{1}{2}qf^4 + r. \tag{16}$$

The prime denotes derivative with respect to ξ and p, q and r are parameters to be determined.

Substituting eq. (15) into eq. (13) and using eq. (16), we arrive at the following set of algebraic equations by equating the coefficients of like power of f to zero:

$$f^3: \quad \gamma q A_1 + CA_1^3 = 0 \tag{17}$$

$$f^2: \quad BA_1^2 + 3CA_0A_1^2 = 0 \tag{18}$$

$$f^1: \quad \gamma p A_1 + AA_1 + 2BA_1A_0 + 3CA_0^2A_1 = 0 \tag{19}$$

$$f^0: \quad AA_0 + BA_0 + CA_0^3 = 0 \tag{20}$$

From eqs. (17) - (20), it is found that:

$$A_0 = \frac{-B}{3C}, \ A_1 = \pm\sqrt{-\frac{\gamma q}{C}}, \tag{21}$$

with the constraint condition

$$B^2 = 9\gamma p C. \tag{22}$$

Therefore, the exact solution of eq. (13) is

$$u(\xi) = \frac{-B}{3C} \pm \sqrt{-\frac{\gamma q}{C}}\, f(\xi). \tag{23}$$

Case 1: $p = 2m^2 - 1, \ q = -2m^2, \ r = 1 - m^2$.

Now, eq. (16) has the solution $f(\xi) = \mathrm{cn}(\xi)$ and so the PWS of eq. (13) is

$$u(\xi) = \frac{-B}{3C} \pm \sqrt{\frac{2\gamma m^2}{C}}\mathrm{cn}(\xi). \tag{24}$$

As $m \to 1$, eq. (24) gives rise to the SWS

$$u(x,t) = \frac{-B}{3C} \pm \sqrt{\frac{2\gamma}{C}}\mathrm{sech}(x - \eta t), \tag{25}$$

which demands $\gamma < 0,\ C < 0$ or $\gamma > 0,\ C > 0$ and satisfies the condition $B^2 = 9\gamma C$.

Using eq. (10), we can write

$$v(x,t) = \frac{2\left(\eta - \sigma\left(\frac{-B}{3C} \pm \sqrt{\frac{2\gamma}{C}}\operatorname{sech}(x-\eta t)\right)\right)}{\epsilon}. \tag{26}$$

Case 2: $p = -(1+m^2),\ \ q = 2m^2,\ \ r = 1.$
Now, eq. (16) has the solutions $f(\xi) = \operatorname{sn}(\xi)$ and $f(\xi) = \operatorname{cd}(\xi)$ and so the PWSs of eq. (13) are

$$u(\xi) = \frac{-B}{3C} \pm \sqrt{-\frac{2\gamma m^2}{C}}\operatorname{sn}(\xi) \tag{27}$$

$$u(\xi) = \frac{-B}{3C} \pm \sqrt{-\frac{2\gamma m^2}{C}}\operatorname{cd}(\xi) \tag{28}$$

As $m \to 1$, eq. (27) gives rise to the kink solutions

$$u(x,t) = \frac{-B}{3C} \pm \sqrt{-\frac{2\gamma}{C}}\tanh(x-\eta t), \tag{29}$$

and

$$v(x,t) = \frac{2\left(\eta - \sigma\left(\frac{-B}{3C} \pm \sqrt{-\frac{2\gamma}{C}}\tanh(x-\eta t)\right)\right)}{\epsilon}, \tag{30}$$

which demands $\gamma < 0,\ C > 0$ or $\gamma > 0,\ C < 0$ and satisfies the condition $B^2 = -18\gamma C$.

Case 3: $p = -(1+m^2),\ \ q = 2,\ \ r = m^2.$
Now, eq. (16) has the solutions $f(\xi) = \operatorname{ns}(\xi)$ and $f(\xi) = \operatorname{dc}(\xi)$ and so the PWSs of eq. (13) are

$$u(\xi) = \frac{-B}{3C} \pm \sqrt{-\frac{2\gamma}{C}}\operatorname{ns}(\xi) \tag{31}$$

and

$$u(\xi) = \frac{-B}{3C} \pm \sqrt{-\frac{2\gamma}{C}}\operatorname{dc}(\xi) \tag{32}$$

As $m \to 1$, eq. (31) gives rise to the singular solutions along the line $x = \eta t$ given by

$$u(x,t) = \frac{-B}{3C} \pm \sqrt{-\frac{2\gamma}{C}}\coth(x-\eta t), \tag{33}$$

and

$$v(x,t) = \frac{2\left(\eta - \sigma\left(\frac{-B}{3C} \pm \sqrt{-\frac{2\gamma}{C}}\coth(x-\eta t)\right)\right)}{\epsilon}, \tag{34}$$

which demands $\gamma < 0,\ C > 0$ or $\gamma > 0,\ C < 0$ and satisfies the condition $B^2 = -18\gamma C$.

3.2 Modified Mapping Method

Letting $u = w + k$ in the expression $Au + Bu^2 + Cu^3$ and assuming the constant term and the coefficient of w^2 equal to zero, eq. (13) can be written as

$$\gamma w'' - \frac{1}{2}Aw + Cw^3 = 0, \tag{35}$$

with $k = -\dfrac{B}{3C}$ and $B^2 = \dfrac{9}{2}AC$.

In this case, we assume the solution of eq. (35) in the form

$$w(\xi) = B_1 f^{-1} + A_0 + A_1 f, \tag{36}$$

where f satisfes eq. (16).

Substituting eq.(36) into eq.(35) and using eq.(16) we arrive at the following set of algebraic equations by equating the coeffients of various powers of w to zero:

$$2r\gamma B_1 + CB_1^3 = 0 \tag{37}$$

$$3CB_1^2 A_0 = 0 \tag{38}$$

$$\gamma p B_1 - \frac{1}{2}AB_1 + 3CA_0^2 B_1 + 3CB_1^2 A_1 = 0 \tag{39}$$

$$-\frac{1}{2}AA_0 + CA_0^3 + 6CB_1 A_0 A_1 = 0 \tag{40}$$

$$\gamma p A_1 - \frac{1}{2}AA_1 + 3CA_0^2 A_1 + 3CB_1 A_1^2 = 0 \tag{41}$$

$$3CA_0 A_1^2 = 0 \tag{42}$$

$$q\gamma A_1 + CA_1^3 = 0 \tag{43}$$

From eqs.(37 - 43) it is found that

$$A_0 = 0, \tag{44}$$

$$A_1 = \pm\sqrt{-\frac{q\gamma}{C}}, \tag{45}$$

$$B_1 = \pm\sqrt{-\frac{2\gamma r}{C}}, \tag{46}$$

with the condition

$$\gamma p - \frac{1}{2}A + 3CB_1 A_1 = 0. \tag{47}$$

Using eq.(36), the value of k and the relation between u and w, the exact solution of eq. (13) can be written as

$$u(\xi) = -\frac{B}{3C} \pm \sqrt{-\frac{q\gamma}{C}} f(\xi) \pm \sqrt{\frac{-2\gamma r}{C}} f^{-1}(\xi). \tag{48}$$

Case 1: $p = -2,\ q = -2,\ r = 1.$

Now, eq. (16) has the solution $f(\xi) = \tanh(\xi)$ and so the solution of eq. (13) is

$$u(\xi) = \frac{-B}{3C} \pm \sqrt{-\frac{2\gamma}{C}} \{\tanh(\xi) + \coth(\xi)\} \tag{49}$$

which demands $\gamma < 0,\ C > 0$ or $\gamma > 0,\ C < 0$ and satisfies the condition $A + 16\gamma = 0$.

Case 2: $p = -(1 + m^2),\ q = 2m^2,\ r = 1.$

Now, eq. (16) has the solution $f(\xi) = \mathrm{sn}(\xi)$ and so the PWS of eq. (13) is

$$u(\xi) = \frac{-B}{3C} \pm \sqrt{-\frac{2m^2\gamma}{C}} \mathrm{sn}(\xi) \pm \sqrt{-\frac{2\gamma}{C}} \mathrm{ns}(\xi). \tag{50}$$

When $m \to 1$, eq. (50) will lead us to the same singular solution (49) which demands $\gamma < 0,\ C > 0$ or $\gamma > 0,\ C < 0$ and satisfies the condition $A + 16\gamma = 0$.

So, in both cases, in the infinite period limit we have the solutions

$$u(x,t)=\frac{-B}{3C}\pm\sqrt{-\frac{2\gamma}{C}}\left\{\tanh(x-\eta t)+\mathrm{cothh}(x-\eta t)\right\} \tag{51}$$

and

$$v(x,t)=\frac{2\left[\eta+\frac{\sigma B}{3C}\mp\sigma\sqrt{-\frac{2\gamma}{C}}\left\{\tanh(x-\eta t)+\coth(x-\eta t)\right\}\right]}{\epsilon}. \tag{52}$$

3.3 Extended Mapping Method

Here, we assume the solution of eq. (35) in the form :

$$w(\xi)=A_o+A_1f+B_1g, \qquad \xi=x-\eta t \tag{53}$$

where f and g satisfy the following equations:

$$f''=pf+qf^3, \quad f'^2=pf^2+\frac{1}{2}qf^4+r, \;\; g''=g(c_1+c_2f^2), \quad g^2=c_3+c_4f^2. \tag{54}$$

The prime denotes the derivative with respect to ξ and p,q and r are parameters to be determined.

Substituting eq. (53) into eq. (35) and using eq. (54), we arrive at a set of algebraic equations by equating the coefficients of like powers of w to zero:

$$f^3: \quad \gamma A_1q+3Cc_4B_1^2A_1+CA_1^3=0, \tag{55}$$

$$f^2g: \quad \gamma B_1c_2+CB_1^3c_4+3CB_1A_1^2=0, \tag{56}$$

$$f^2: \quad 3Cc_4A_0B_1^2+3CA_0A_1^2=0, \tag{57}$$

$$fg: \quad 6CA_0B_1A_1=0, \tag{58}$$

$$f: \quad \gamma pA_1-\frac{1}{2}AA_1+3Cc_3B_1^2A_1+3CA_0^2A_1=0, \tag{59}$$

$$g: \quad \gamma B_1c_1-\frac{1}{2}AB_1+Cc_3B_1^3+3CA_0^2B_1=0, \tag{60}$$

$$f^0: \quad -\frac{1}{2}AA_0+3Cc_3A_0B_1^2+CA_0^3=0. \tag{61}$$

From eqs.(55 -61) it is found that

$$A_0=0, \tag{62}$$

$$A_1=\pm\sqrt{\frac{(pc_4-qc_3)\gamma-\frac{1}{2}Ac_4}{Cc_3}}, \tag{63}$$

$$B_1=\pm\sqrt{\frac{\frac{1}{2}A-\gamma p}{3Cc_3}}, \tag{64}$$

with the two conditions

$$\gamma(3c_4p-3qc_3-c_1c_4+c_2c_3)-Ac_4=0, \tag{65}$$

$$\gamma(3c_1-p)-A=0. \tag{66}$$

Thus the exact solution of eq.(13) using eq. (53) and $k=-\frac{B}{3C}$ is

$$u(\xi)=\pm\sqrt{\frac{(pc_4-qc_3)\gamma-\frac{1}{2}Ac_4}{Cc_3}}f(\xi)\pm\sqrt{\frac{\frac{1}{2}A-\gamma p}{3Cc_3}}g(\xi)-\frac{B}{3C}. \tag{67}$$

Case 1: $p = -(1+m^2),\ q = 2, r = m^2,\ c_1 = -m^2,\ c_2 = 2,\ c_3 = -1,\ c_4 = 1$

Now, eq. (54) has the solutions $f(\xi) = \mathrm{ns}(\xi),\ g(\xi) = \mathrm{cs}(\xi)$ and so the PWS of eq. (13) is

$$u(\xi) = \pm\sqrt{\frac{(m^2-1)\gamma + \frac{1}{2}A}{C}}\,\mathrm{ns}(\xi) \pm \sqrt{-\frac{\frac{1}{2}A + (1+m^2)\gamma}{3C}}\,\mathrm{cs}(\xi) - \frac{B}{3C}. \tag{68}$$

When $m \to 1$, eq. (68) leads to

$$u(x,t) = \pm\sqrt{\frac{A}{2C}}\{\coth(x-\eta t) + \mathrm{csch}(x-\eta t)\} - \frac{B}{3C} \tag{69}$$

and

$$v(x,t) = \frac{2\left(\eta - \sigma\left(\pm\sqrt{\frac{A}{2C}}\{\coth(x-\eta t) + \mathrm{csch}(x-\eta t)\} - \frac{B}{3C}\right)\right)}{\epsilon}. \tag{70}$$

which demands that $A < 0,\ C < 0$ or $A > 0,\ C > 0$ and satisfies the condition $\gamma = -A$.

Case 2: $p = -(1+m^2),\ q = 2, r = m^2,\ c_1 = -1,\ c_2 = 2,\ c_3 = -m^2,\ c_4 = 1$

Now, eq. (54) has the solutions $f(\xi) = \mathrm{ns}(\xi),\ g(\xi) = \mathrm{ds}(\xi)$ and so the PWS of eq. (13) is

$$u(\xi) = \pm\sqrt{\frac{(1-m^2)\gamma + \frac{1}{2}A}{m^2C}}\,\mathrm{ns}(\xi) \pm \sqrt{-\frac{\frac{1}{2}A + (1+m^2)\gamma}{3Cm^2}}\,\mathrm{ds}(\xi) - \frac{B}{3C}. \tag{71}$$

When $m \to 1$, solutions given by eqs. (69) and (70) have been retained.

4 Approximate solution using ADM

4.1 Solution I

We use ADM to derive an approximate solution to eq. (13). For this purpose, we take $\beta = -2, \gamma = -1, \epsilon = 2, \eta = 1, \lambda = 4, \sigma = 1, \alpha = 1$ which gives $A = -2, B = 3, C = -1$.

Thus eq. (13) can be written as

$$u'' + 2u - 3u^2 + u^3 = 0. \tag{72}$$

We impose the initial conditions $u(0) = 1 + \sqrt{2}$ and $u'(0) = 0$.

Equation (72) can be written in the operator form as

$$u = 1 + \sqrt{2} - L^{-2}\left\{2u - 3u^2 + u^3\right\} \tag{73}$$

where the differential operator L is given by

$$L^{-2} = \int_0^\xi \int_0^\xi d\xi d\xi. \tag{74}$$

The ADM defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n,\ \ u^2 = \sum_{n=0}^{\infty} A_n,\ \ u^3 = \sum_{n=0}^{\infty} B_n \tag{75}$$

where the Adomian polynomials A_n and B_n are given by

$$A_0 = u_0^2,\ \ A_1 = 2u_0u_1,\ \ A_2 = 2u_0u_2 + u_1^2,\ \ A_3 = 2u_0u_3 + 2u_1u_2, ... \tag{76}$$

$$B_0 = u_0^3,\ \ B_1 = 3u_0^2u_1,\ \ B_2 = 3u_0^2u_2 + 3u_0u_1^2,\ \ B_3 = 3u_0^2u_3 + 6u_0u_1u_2 + u_1^3, ... \tag{77}$$

Thus we get

$$u_0 = 1 + \sqrt{2},$$

$$u_1 = -L^{-2}\Big\{2u_0 - 3A_0 + B_0\Big\} = -\frac{\xi^2}{\sqrt{2}}$$

,

$$u_2 = -L^{-2}\Big\{2u_1 - 3A_1 + B_1\Big\} = \frac{5\xi^4}{12\sqrt{2}},$$

$$u_3 = -L^{-2}\Big\{2u_2 - 3A_2 + B_2\Big\} = -\frac{61\xi^6}{360\sqrt{2}},$$

$$u_4 = -L^{-2}\Big\{2u_3 - 3A_3 + B_3\Big\} = \frac{277\xi^8}{4032\sqrt{2}}.$$

So, the solution of eq. (72) is,

$$u(\xi) = u_0 + u_1 + u_2 + u_3 + ... = 1 + \sqrt{2}\Big\{1 - \frac{\xi^2}{2} + \frac{5\xi^4}{24} - \frac{61\xi^6}{720} + \frac{277\xi^8}{8064} + ...\Big\}. \tag{78}$$

Thus the solutions $u(x,t)$ and $v(x,t)$ can be written as

$$u(x,t) = 1 + \sqrt{2}\,\mathrm{sech}(x-t), \tag{79}$$

$$v(x,t) = -\sqrt{2}\,\mathrm{sech}(x-t). \tag{80}$$

4.2 Convergence of Solution I

We establish the convergence of the approximate solution (78) to the exact solution (79) by a method proposed by Hosseini and Nasabzadeh [22].

We find the values of α_i where $\alpha_i = \dfrac{\|u_{i+1}\|}{\|u_i\|}$ and the solution converges when $\alpha_i < 1$. Here, $\|u_i\|$ is defined as the absolute value of the highest coefficient of powers of ξ. So, we get

$$\alpha_0 = \frac{\|u_1\|}{\|u_0\|} \approx 0.2929 < 1, \quad \alpha_1 = \frac{\|u_2\|}{\|u_1\|} \approx 0.4167 < 1$$

$$\alpha_2 = \frac{\|u_3\|}{\|u_2\|} \approx 0.4067 < 1, \quad \alpha_3 = \frac{\|u_4\|}{\|u_3\|} \approx 0.4054 < 1$$

This confirms that the obtained solution by using ADM converges to the exact solution.

4.3 Solution II

Let $\beta = 0,\ \gamma = \dfrac{1}{2},\ \epsilon = 2,\ \eta = -1,\ \lambda = 12,\ \sigma = 1,\ \alpha = 3$ which gives $A = -2,\ B = 3,\ C = -1$. Then eq. (13) can be written as

$$u'' - 4u + 6u^2 - 2u^3 = 0 \tag{81}$$

with the initial conditions $u(0) = 1$ and $u'(0) = 1$.

Equation (81) can be re-written in the operator form

$$u = 1 + x + L^{-2}\Big\{4u - 6u^2 + 2u^3\Big\} \tag{82}$$

where the differential operator L is defined as in Solution 1.

Following the ADM employed for solution 1, we get

$$u_0 = 1 + \xi, \tag{83}$$

$$u_1 = -L^{-2}\{-4u_0 + 6A_0 - 2B_0\} = -\frac{1}{3}\xi^3 + \frac{1}{10}\xi^5, \tag{84}$$

$$u_2 = -L^{-2}\{-4u_1 + 6A_1 - 2B_1\} = \frac{1}{30}\xi^5 - \frac{11}{210}\xi^7 + \frac{1}{120}\xi^9, \tag{85}$$

$$\begin{aligned} u_3 = -L^{-2}\{-4u_2 + 6A_2 - 2B_2\} = -\frac{143}{90090}\xi^7 + \frac{2431}{180180}\xi^9 \\ -\frac{3991}{600600}\xi^{11} + \frac{11}{15600}\xi^{13}, \end{aligned} \tag{86}$$

$$\begin{aligned} u_4 = -L^{-2}\{-4u_3 + 6A_3 - 2B_3\} = \frac{1}{22680}\xi^9 - \frac{101881}{45945900}\xi^{11} \\ +\frac{161177}{61261200}\xi^{13} - \frac{689061}{918918000}\xi^{15} + \frac{2321}{38896000}\xi^{17}. \end{aligned} \tag{87}$$

So, the solution of eq. (81) is,

$$u(\xi) = 1 + \xi - \frac{1}{3}\xi^3 + \frac{2}{15}\xi^5 - \frac{17}{315}\xi^7 + \frac{62}{2835}\xi^9 - ... \tag{88}$$

Thus the solutions $u(x,t)$ and $v(x,t)$ can be written as

$$u(x,t) = 1 + \tanh(x+t)) \tag{89}$$

$$v(x,t) = -2 - \tanh(x+t) \tag{90}$$

4.4 Convergence of Solution II

Now, we establish the convergence of the approximate solution (88) to the exact solution (89) by calculating α_i and if $\alpha_i < 1$, the approximate solution converges to the exact solution.

The values of α_i are:

$$\alpha_0 = \frac{\|u_1\|}{\|u_0\|} = 0.333 < 1, \quad \alpha_1 = \frac{\|u_2\|}{\|u_1\|} = 0.157 < 1$$

$$\alpha_2 = \frac{\|u_3\|}{\|u_2\|} = 0.258 < 1, \quad \alpha_3 = \frac{\|u_4\|}{\|u_3\|} = 0.195 < 1$$

which confirms that the obtained solution by using ADM is convergent to the exact solution.

5 Conclusion

The coupled wave equation was found to have a variety of PWSs in terms of JEFs which in the infinite period limit give rise to shock wave solutions, SWSs and singular wave solutions. For different values of the parameters, it was shown that the approximate solutions using ADM in terms of hyperbolic functions were found to be convergent to the exact solutions derived by mapping methods.

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Appendix

Consider the function

$$F(\phi, m) = \eta = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}}. \tag{91}$$

Letting $t = \sin\theta$, we obtain

$$\eta = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}. \tag{92}$$

This is called Legendre's standard elliptical integral of the first kind.

When $m = 0, \eta = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} x$, where $x = \sin\phi$.

When $m = 1, \eta = \displaystyle\int_0^x \frac{dt}{1-t^2} = \tanh^{-1} x$, where $x = \sin\phi$.

For $0 < m < 1$, we define η as the inverse of a function which is known as Jacobi Sine elliptic function, expressed in the form $\eta = \mathrm{sn}^{-1}x$ so that $x = \mathrm{sn}\eta$ or $\mathrm{sn}\eta = \sin\phi$. Here, m is known as the modulus of the JEFs (see [20]).

We define two other elliptic functions as

$$\mathrm{cn}\eta = \sqrt{1-x^2} = \sqrt{1-\mathrm{sn}^2\eta}, \tag{93}$$

$$\mathrm{dn}\eta = \sqrt{1-m^2x^2} = \sqrt{1-m^2\mathrm{sn}^2\eta}. \tag{94}$$

$\mathrm{cn}\eta$ is known as Jacobi cosine elliptic function and $\mathrm{dn}\eta$ is known as JEF of the third kind.

As $m \to 0, \mathrm{sn}\eta \to \sin\eta, \mathrm{cn}\eta \to \cos\eta$ and $\mathrm{dn}\eta \to 1$. As $m \to 1, \mathrm{sn}\eta \to \tanh\eta, \mathrm{cn}\eta \to \mathrm{sech}\eta$ and $\mathrm{dn}\eta \to \mathrm{sech}\eta$.

$\mathrm{ns}\eta$, $\mathrm{nc}\eta$ and $\mathrm{nd}\eta$ are the reciprocals of the three JEFs and $\mathrm{sc}\eta$, $\mathrm{cd}\eta$, $\mathrm{ds}\eta$, $\mathrm{cs}\eta$, $\mathrm{dc}\eta$ and $\mathrm{sd}\eta$ are the ratios of the corresponding JEFs.