ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.24(2017) No.3, pp.175-179

Fractal Interpolation Representation of A Class of Quadratic Functions

Chengbin Shen*, Zhigang Feng
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013. P.R. China
(Received 2 January 2017, accepted 15 May 2017)

Abstract: Fractal interpolation functions are important research contents in fractal theory. These functions generated by certain iterative function systems often have complex graphics and non-integer dimension. In general, it is difficult for us to get analytic expressions of fractal interpolation functions. In this paper, we investigate the representation of a class of special fractal interpolation functions. From the iterative process, we find that this fractal interpolation function is essentially a parabola when vertical scale factor is a given constant. Furthermore, we find that a class of quadratic functions can be represented as certain fractal interpolation functions, which paved the way for the further research of the analytic expressions of the fractal interpolation functions.

Keywords: Quadratic Function; self affine; fractal interpolation function.

1 Introduction

Before the emergence of fractal theory, we use straight lines, boxes, rounds and other tools to describe the usual objects. But for many extremely irregular and disorderly objects, we can do nothing. The American mathematician Barnsley [1] proposed the fractal interpolation function in 1986. The image of the function can approximately describe the object that the Euclidean geometry is not well described [2], such as the coast forest top fluctuant curve, the contours of the mountain, the cloud shape and so on. As long as the vertical scaling factor is required to transform, it can generate any curve whose dimension is between 1 and 2. Lebesgue [3] have given important ideas related to dimensions. On the basis of the measure, Hausdorff [4] introduced the Hausdorff dimension in 1919, which enhances the practicality. But Hausdorff dimension is difficult to calculate, and it has little actual background. This brought great obstacles to our follow-up study. Then Bouligand raised the box dimension in the last century. It can be calculated by experimental approximation, so its practical use more widely. As a new fitting method, the fractal interpolation has caused wide attention of applied mathematicians. In the numerical calculation, computational geometry, computer graphics and other fields, it has a very wide range of applications.

In the theory of functions, interpolation function approximation is one of the main contents. In Ref. [5], the construction of fractal interpolation function is introduced. In many cases, the fractal interpolation function (FIF) [6] is often generated by iterated function system (IFS). The smoothness and continuity of the fractal interpolation function have been discussed adequately in [7-9]. Li [10] depicted the smoothness of FIF and gave the relationship between the various parameters of IFS and the smoothness of FIF. Barnsley first gave the formula for calculating the box dimension, and enhanced the practical significance of fractal geometry. On this basis, Sha and Ruan [6] rejected the specific conditions, which further strengthening the application value. After introducing the concept of variability, the dimension formula of the continuous function image on the plane can be given directly by the variance estimation. Recently, the methods of image generation, the calculation of dimensions, and graphics simulation, have been the key research direction of the fractal theory and practical application[11-13]. However, few scholars have paid attention to the representation of fractal interpolation, because of its difficulty and complexity. It is such functions without obvious expressions, so that our research process is relatively slow. This paper attempts to give an expression to a special case of FIF, and extends to the general case.

*Corresponding author. *E-mail address*:949450768@qq.com

IJNS.2017.12.15/987

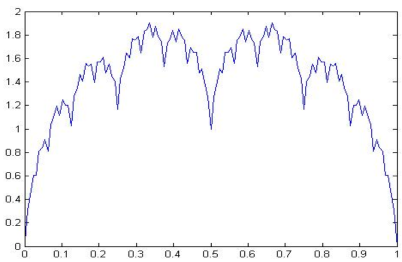

Figure 1: The stable region of $d_1 = d_2 = 2/3$.

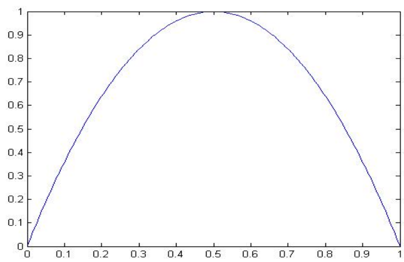

Figure 2: The stable region of $d_1 = d_2 = 1/4$.

2 Fractal Interpolation function

Let a set of interpolation nodes$\{(x_i, y_i) \in R_2, i = 0, 1, 2, \cdots, N\}$,$N \geq 2$ be given and $0 = x_0 < x_1 < \cdots < x_N = 1$. Define mapping $\omega_i : [0, 1] \times R \longrightarrow [0, 1] \times R$,

$$\omega_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x, y) \end{pmatrix}, i = 0, 1, 2, \cdots, N, \tag{2.1}$$

where $L_i(x)$ satisfies

$$L_i(x_0) = x_{(i-1)}, L_i(x_N) = x_i. \tag{2.2}$$

Let $-1 < d_i < 1, F_i(x, y) : [x_0, x_N] \times R \longrightarrow R$ satisfy

$$|F_i(x, y_1) - F_i(x, y_2)| \leq |d_i||y_1 - y_2|,$$

where $x \in [x_0, x_N], y_1, y_2 \in R$ and then

$$F_i(x_0, y_0) = y_{(i-1)}, F_i(x_N, y_N) = y_i, (i = 1, 2, \cdots, N). \tag{2.3}$$

By the formula (2.1), the function $f : [x_0, x_N] \longrightarrow R$ can be uniquely determined, and satisfies

$$f(L_i(x)) = F_i(x, f(x)), i = 1, 2, \cdots, N. \tag{2.4}$$

When $L_i(x), F_i(x, y)$ are linear functions, ω_i has the following form:

$$\omega_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, i = 1, 2, \cdots, N. \tag{2.5}$$

By the formulas (2.2) and (2.3), we can calculate the coefficient of iterated function system, where parameters d_i is vertical scale factor

$$\begin{cases} a_i = (x_i - x_{(i-1)})/(x_N - x_0), \\ e_i = (x_N x_{(i-1)} - x_0 x_1)/(x_N - x_0), \\ c_i = (y_i - y_{(i-1)})/(x_N - x_0) - d_i(y_N - y_0)/(x_N - x_0), \\ f_i = (x_N y_{(i-1)} - x_0 y_i)/(x_N - x_0) - d_i(x_N y_0 - x_0 y_N)/(x_N - x_0). \end{cases} \tag{2.6}$$

There is a continuous function whose image: $G = \{(x, f(x))|x \in [x_0, x_N]\}$ is the invariant set of iterated function system ω_i [6]. The selection of the vertical compression factor has a considerable influence on the shape of the image, as shown in $Fig.1$.

Ruan et.al [14] have proved that when $N = 2, x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, y_0 = y_2 = 0, y_1 = 1, d_1 = d_2 = \frac{1}{4}$, the FIF obtained by Iterated function system (2.5) is $f(x) = -4x^2 + 4x, x \in [0, 1]$, as shown in Fig.2.

3 Main Results

Theorem 1 *[15] : Let $x_0 < x_1 < \cdots < x_N$ be given, then $L_i(x)$ is an affine mapping and satisfies (2.2), where*

$$a_i = (x_i - x_{i-1})/(x_N - x_0), F_i(x, y) = d_i y + q_i(x), i = 1, 2, \cdots, N.$$

Let

$$F_{i,k}(x, y) = \frac{d_i y + q_i^k(x)}{a_i^k},$$

where

$$|d_i| < a_i^n, q_i \in C^n[x_0, x_N], i = 1, 2, \cdots, N,$$

$$q_i^{(0)} = q_i; y_{0,k} = \frac{q_1^k(x_0)}{a_1^k - d_1}, y_{N,k} = \frac{q_N^k(x_N)}{a_N^k - d_N}, k = 1, 2, \cdots, n.$$

If

$$F_{i-1,k}(x_N, y_{N,k}) = F_{i,k}(x_0, y_{0,k}),$$

then $F \in C^n[x_0, x_N]$ determined by $\{L_i(x), F_{i,k}(x, y)\}_{i=1}^N$ and f^k determined by $\{L_i(x), F_{i,k}(x, y)\}_{i=1}^N$ are both fractal interpolation functions.

Theorem 2 *When $N = 3$, let $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1, y_0 = y_3 = 0, y_1 = y_2 = 1, d_1 = d_2 = d_3 = \frac{1}{9}$, the FIF obtained by IFS (2.5) is $f(x) = -\frac{9}{2}x^2 + \frac{9}{2}x, x \in [0, 1]$.*

Proof. According to the conditions of the theorem, the concrete form of IFS can be obtained by formula (2.6):

$$\begin{cases} \omega_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 1 & 1/9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/3 \\ x + y/9 \end{pmatrix}, \\ \omega_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3x/3 \\ 1 + y/9 \end{pmatrix}, \\ \omega_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1/9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3x/3 \\ 1 - x + y/9 \end{pmatrix}. \end{cases} \tag{1}$$

Let

$$L_1(x) = x/3, L_2(x) = x/3 + 1/3, L_3(x) = x/3 + 2/3,$$
$$F_1(x, y) = y/9 + q_1(x), F_2(x, y) = y/9 + q_2(x), F_3(x, y) = y/9 + q_3(x),$$

where

$$q_1(x) = x, q_2(x) = 1, q_3(x) = x - 1.$$

On account of $L_i(x), F_i(x, y)$ satisfying $(2.2) and (2.3)$, $f : [x_0, x_N] \longrightarrow R$ based on $\{L_i(x), F_i(x, y)\}_{i=1}^3$ satisfies

$$f(L_i(x)) = F_i(x, f(x)), i = 1, 2, 3,$$

$$F_{i,1}(x, y) = \frac{d_i y + q_i'(x)}{a_i}, i = 1, 2, 3, y_{0,1} = \frac{q_1'(x_0)}{a_1 - d_1}, y_{3,1} = \frac{q_3'(x_3)}{a_3 - d_3}.$$

By calculation, we can get

$$F_{1,1}(x, y) = \frac{y}{3} + 3, F_{2,1}(x, y) = \frac{y}{3}, F_{3,1}(x, y) = \frac{y}{3} - 3, y_{0,1} = \frac{9}{2}, y_{3,1} = \frac{9}{2},$$

and

$$F_{1,1}(x_3, y_{3,1}) = F_{2,1}(x_0, y_{0,1}) = \frac{3}{2}, F_{2,1}(x_3, y_{3,1}) = F_{3,1}(x_0, y_{0,1}) = \frac{3}{2}.$$

Therefore according to Theorem 1 , a fractal interpolation function which satisfies

$$h(L_i(x)) = F_{i,1}(x, h(x)), i = 1, 2, 3, f' = h \tag{3.2}$$

can be determined by $\{L_i(x), F_i(x,y)\}_{i=1}^3$.
Then $r(x) = -9x + 9/2$ is a straight line equation, passing through two points:

$$(x_0, y_{0,1}) = (0, 9/2), (x_3, y_{3,1}) = (1, -9/2).$$

Due to

$$y_{1,1} = 3/2, y_{2,1} = -3/2,$$

thus the two points

$$(x_1, y_{1,1}) = (1/3, 3/2), (x_2, y_{2,1}) = (1/3, -3/2)$$

are on line $r(x) = -9x + 9/2$. That is,

$$y_{i,1} = r(x_i), i = 1, 2, 3.$$

For $\forall x \in [x_0, x_3]$, there are λ_1 and λ_2 satisfying

$$\lambda_1 + \lambda_2 = 1, \lambda_1 \geq 0, \lambda_2 \geq 0, x = \lambda_1 x_0 + \lambda_2 x_3, i = 1, 2, 3.$$

Therefore, we can get the following results:

$$r(L_i(x)) = r(L_i(\lambda_1 x_0 + \lambda_2 x_3)) = r(\lambda_1 x_{i-1} + \lambda_2 x_i) = \lambda_1 r(x_{i-1}) + \lambda_2 r(x_i),$$

$$\begin{aligned} F_{i,1}(x, r(x)) &= F_{i,1}(x, r(\lambda_1 x_0 + \lambda_2 x_3)) \\ &= F_{i,1}(x, \lambda_1 r(x_0) + \lambda_2 r(x_3)) \\ &= \lambda_1 F_{i,1}(x_0, r(x_0)) + \lambda_2 F_{i,1}(x_3, r(x_3)) \\ &= \lambda_1 y_{i-1,1} + \lambda_2 y_{i,1} \\ &= \lambda_1 r(x_{i-1}) + \lambda_2 r(x_i). \end{aligned}$$

Thus :

$$r(L_i(x)) = F_{i,1}(x, r(x)), x \in [x_0, x_3], i = 1, 2, 3.$$

According to formula (3.2), we can get

$$f'(x) = h(x) = r(x) = -9x + 9/2, f(0) = 0.$$

Thus

$$f(x) = -\frac{9}{2}x^2 + \frac{9}{2}x, x \in [0, 1].$$

This ends the proof. ■

Theorem 3 *About* $f(x) = -Ax^2 + Ax, x \in [0,1] (A > 0)$*, it always exists a definition mapping* $\{L_i(x), F_i(x,y)\}_{i=1}^N$, $\forall N \geq 2$*,and a set of interpolation nodes that can determine an iterated function system whose attractor G is the graph of this quadratic function.*

Proof. Let $x_i = \frac{i}{N}, y_i = \frac{A(iN - i^2)}{N^2}, i = 1, 2, 3, \cdots, N$ and give interpolation points:

$$(x_i, y_i) \in R^2, i = 1, 2, 3, \cdots, N, let\ d_1 = d_2 = \cdots = d_N = \frac{1}{N^2}.$$

It is easy to prove that:

$$L_i(x) = \frac{1}{N}x + \frac{i-1}{N}, F_i(x, y) = \frac{1}{N^2}y + q_i(x),$$

where

$$q_i(x) = \frac{A(N - 2i + 1)}{N^2} + A\frac{[(i-1)]N - (i-1)^2}{N^2}, i = 1, 2, 3, \cdots, N.$$

The following process is similar to Theorem 2. ■

4 Conclusions

In this paper, we investigate a class of special fractal interpolation functions obtained by the unique attractor of iterated function systems. From the iterative process, we find that this fractal interpolation function is essentially a parabola when vertical scale factor d=1/4. Then we find that all of the quadratic functions like $f(x) = -Ax^2 + Ax, x \in [0,1], A > 0$ can be represented by certain fractal interpolation functions which are separated by any isometry. Fractal interpolation functions can be used to get the basic line and the quadratic function, and provide a reference for the further study of other functions (such as polynomials, etc.) represented by fractal interpolations.

Ackowledgments

This paper is supported by the National Natural Science Foundation of China (No: 51276081, 91546118) and the Major Project of Natural Science Foundation of Jiangsu Province Colleges and Universities (14KJA110001).

References

[1] M. F. Barnsley. Fractal functions and interpolation. *Constructive Approximation*, 2(4)(1986):303-329.
[2] B. Mandelbrot. The fractal geometry of nature. New York: W. H. Freeman. 1983.
[3] H. Lebesgue. Sur la non-applicabilité de deux domaines appartenant respectivement à des espaces àn etn+p dimensions. *Mathematische Annalen*, 70(2)(1991): 166-168.
[4] F. Hausdorff. Dimension und äußeres Maß. Mathematische Annalen. 79(1)(1918): 157-179.
[5] M. F. Barnsley, J. H. Elton and D. P. Hardin. Recurrent interated function systems. *Constructive Approximation*, 5(1989): 3-31.
[6] Z. Sha and H. J. Ruan. Fractal and fitting. Hangzhou: Zhejiang University Press. 2005.
[7] M. F. Barnsley, J. H. Elton and D. P. Hardin. Hidden variable fractal interpolation functions. *Siam Journal on Mathematical Analysis*, 20(5)(1989):1218-1242.
[8] H. Y. Wang. On smoothness for a class of fractal interpolation surface. *Fractals*, 14(3)(2006): 223-230.
[9] H. Y. Wang. Sensitivity analysis for hidden variable fractal interpolation functions and their moments. *Fractals*, 17(2)(2009): 161-170.
[10] H. D. Li, Z. L. Ye and X. S. Gao. On the continuity and differentiability of fractal interpolation function. *Applied Mathematics and Mechanics*,23(4)(2002).
[11] P. Bouboulis, L. Dalla and V. Drakopoulos. Construction of recurrent bivariate fractal interpolation surfaces and computation of their box-counting dimension. *Journal of Approximation Theory*, 141(2006): 99-117.
[12] H. Y. Wang, S. Z. Yang and X. J. Li. Error analysis for bivariate fractal interpolation functions generated by 3D perturbed iterated function systems. *Computers Mathematics with Applications*, 56(2008): 1684-1692.
[13] Z. G. Feng and X. Q. Sun. Box-counting Dimensions of Fractal Interpolation Surfaces Derived from Fractal Interpolation Functions. *J. Math. Anal. Appl.*, 412(1)(2014): 416-425.
[14] H. J. Ruan, Z Sha and W. Y. Su. Parameter identification problem of the fractal interpolation functions. *Numerical Mathematics A Journal of Chinese Universities English Series*, 12(2)(2003): 205-213.
[15] M. F. Barnsley and A. N. Harrington. The calculus of fractal interpolation functions. *Journal of Approximation Theory*, 57(1)(1989): 14-34.