ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol. 32 (2021) No.2/3, pp.97-104

Controlling the Oscillation for a Stochastic Phytoplankton-Zooplankton Model

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(Received August 15 2021, accepted December 12 2021)

Abstract: In this paper, we investigate the problem of oscillation control for a stochastic phytoplankton-zooplankton model. Based on a deterministic differential equation model describing phytoplankton-zooplankton system with harvesting, a stochastic dynamic model is proposed by considering the influence of environmental fluctuations. For the stochastic dynamic model, we derive its ensemble mean response by using the orthogonal polynomial approximation method. It is found that the periodic oscillation of the phytoplankton-zooplankton model can be controlled by the catchability coefficients, and the control efficiency of the stochastic dynamic model is superior to that of the deterministic model.

Keywords: Stochastic phytoplankton-zooplankton model; Orthogonal polynomial approximation; Ensemble mean response; Oscillation control

1 Introduction

Plankton is an important part of the aquatic ecosystem, including phytoplankton and zooplankton. Phytoplankton is a kind of micro-phytoplankton living in water, which is usually referred to as planktonic algae[1]. As a primary producer in the aquatic ecosystem, it can produce organic matter through photosynthesis and provide nutrients and energy for other organisms. Zooplankton is a small animal that controls the quantity of the phytoplankton by preying on phytoplankton, and it is also preyed by fish and shellfish. In recent years, due to the disturbance of human activities in nature, red tides occur frequently[1]. Red tides not only cause serious harm to the marine environment, marine fisheries and aquaculture, but also have a great impact on human health and even life[2]. Therefore, it is of great significance to study the phytoplankton-zooplankton system.

Over the years, many scholars have constructed various mathematical models to describe the interaction between phytoplankton and zooplankton for different phytoplankton-zooplankton systems. Lv *et al.*[3] proposed and studied a phytoplankton-zooplankton model with harvesting and discussed the optimal harvesting strategy. Pei *et al.*[4] investigated a two-zooplankton one-phytoplankton model with harvesting and discussed the impact of harvesting on the coexistence and competitive exclusion of competitive predators. Chakraborty *et al.*[5] developed a simplified three-species model with non-toxic phytoplankton, toxic phytoplankton and predator zooplankton. They pointed the potential role of the toxin in terms of inducing an avoidance behavior in grazer zooplankton. Meanwhile, due to the growths of phytoplankton and zooplankton and the interaction between the two species usually depend on time and space, some researchers have proposed and discussed some phytoplankton-zooplankton models with delay[6–9] and diffusion[10, 11].

In fact, in the real aquatic ecosystem, environmental fluctuations will affect the growth of phytoplankton and zooplankton. So some researchers have constructed a number of stochastic plankton models[12–14]. In Ref.[12], Yu *et al.* proposed a stochastic nutrient-phytoplankton model with toxin-producing phytoplankton and revealed the effects of environmental fluctuations on the planktonic blooms. In Ref.[13], Yu *et al.* developed a stochastic phytoplankton-zooplankton model with Markov switching in an impulsive polluted environment, and discussed the influences of environmental fluctuations on the growth of plankton. Wang and Liu[14] considered a stochastic phytoplankton-zooplankton model with toxin-producing phytoplankton and regime-switching, and provided some sufficient conditions for the existence of unique ergodic stationary distribution. One can see[15–18] and the references cited therein for more results on the stochastic plankton models.

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IJNS.2021.12.15

In the literatures mentioned above, most of the authors used the method of constructing Lyapunov function to study the stochastic plankton models, and obtained a lot of significant results. Recently, another method, the orthogonal polynomial approximation method, has been successfully applied to the investigation of stochastic dynamic models[19–21]. In Ref.[19], Xu *et al.* analyzed a stochastic van der Pol system with nonlinear random controller, and obtained the ensemble mean response and the sample response of mean parameter system of the stochastic system by using the orthogonal polynomial approximation method. They found that to control the stochastic Hopf bifurcation, the random controller is superior to the deterministic one. In Ref.[20], with the help of the orthogonal polynomial approximation method, Ma explored the Hopf bifurcation in Brusselator system with random parameter. Besides, in Ref.[21], Yang and Ma investigated the stochastic Hopf bifurcation for a discrete coupling logistic system with symbiotic interaction.

Motivated by the above works, we will investigate the problem of periodic oscillation control for a stochastic phytoplankton-zooplankton model by using the orthogonal polynomial approximation method. The corresponding ensemble mean response is first derived, it is a deterministic equivalent system of the stochastic dynamic model. We find that the periodic oscillation of the stochastic phytoplankton-zooplankton model can be controlled by the catchability coefficients, and the control efficiency of the stochastic dynamic model is superior to that of the deterministic model.

The rest of this paper is organized as follows: In Section 2, we propose the stochastic phytoplankton-zooplankton model. The corresponding ensemble mean response of the stochastic model is derived in Section 3. In Section 4, we present some numerical simulations and discussions for the stochastic model.

2 Stochastic phytoplankton-zooplankton model

In Ref.[3], Lv *et al.* proposed the following deterministic phytoplankton-zooplankton model with harvesting:

$$\begin{cases} \frac{\mathrm{d}P}{\mathrm{d}t} = rP(1-\frac{P}{k}) - \frac{\beta PZ}{\alpha+P} - c_1EP, \\ \frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{\beta_1 PZ}{\alpha+P} - dZ - \frac{\rho PZ}{\alpha+P} - c_2EZ, \end{cases} \tag{1}$$

where P and Z denote the densities of toxin producing phytoplankton (TPP) population and zooplankton population at time t, r is the intrinsic growth rate and k is the environmental carrying capacity of TPP population, α is the half saturation constant for consumption of zooplankton on phytoplankton, β is the maximum uptake rate of zooplankton, β_1 denotes the ratio of biomass conversion, d is the natural death rate of zooplankton, ρ denotes the rate of toxic substances produced by per unit biomass of phytoplankton, E is the harvesting effort and the constants c_1 and c_2 are the catchability coefficients of the two species.

Lv *et al.*[3] established the stability conditions of equilibria and existence conditions of Hopf-bifurcation, and discussed the existence of bionomic equilibria and the optimal harvesting policy. They found that over exploitation would result in the extinction of the population and an appropriate harvesting strategy would ensure the sustainability of the population. Besides, they found that the zero discounting leads to the maximization of economic revenue and an infinite discount rate leads to complete dissipation of economic rent.

For model (1), we can propose the following equivalent version by taking some appropriate variable transformations:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = rx(1-\frac{x}{k})(a+x) - bxy - c_1(a+x)x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b_1xy - dy(a+x) - \rho xy - c_2(a+x)y, \end{cases} \tag{2}$$

where c_1 is equivalent to c_1E, c_2 is equivalent to c_2E. As we all know, there inevitably exist environmental fluctuations in the real aquatic ecosystem, and the catchability coefficients c_1 and c_2 are more susceptible to the disturbance from environmental fluctuations. Then we consider the catchability coefficients in model (2) are stochastic parameters, and suppose that the catchability coefficient can be written an average value plus an error term, i.e., c_1 and c_2 can be expressed as

$$\begin{cases} c_1 = \bar{c}_1 + \sigma_1 U, \\ c_2 = \bar{c}_2 + \sigma_2 U, \end{cases}$$

where $\bar{c}_1$ and $\bar{c}_2$ are the deterministic parts of c_1 and c_2 respectively, U is a random variable with a range of $[-1, 1]$ and an arch-like probability density function [19, 20, 22]

$$\rho_U(u) = \begin{cases} \frac{2}{\pi}\sqrt{1-u^2}, & |u| \le 1, \\ 0, & |u| > 1. \end{cases}$$

Nonnegative constants σ_1 and σ_2 are regarded as the intensities of random disturbance. Base on deterministic model (2), we can obtain the following stochastic phytoplankton-zooplankton model:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = rx(1-\frac{x}{k})(a+x) - bxy - (\bar{c}_1 + \sigma_1 u)(a+x)x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b_1 xy - dy(a+x) - \rho xy - (\bar{c}_2 + \sigma_2 u)(a+x)y. \end{cases} \tag{3}$$

We should point that the variables x and y in model (3) are functions of time t and sample point u.

3 Orthogonal polynomial approximation for stochastic model

It follows from the orthogonal polynomial approximation [23] that the responses of stochastic model (3) can be expressed by the following series:

$$\begin{cases} x(t,u) = \sum\limits_{i=0}^{M} x_i(t)P_i(u), \\ y(t,u) = \sum\limits_{i=0}^{M} y_i(t)P_i(u), \end{cases} \tag{4}$$

where $P_i(u)$ is the i th orthogonal polynomial, M represents the largest order of the polynomial, and

$$x_i(t) = \int_{-\infty}^{+\infty} \rho_U(u)x(t,u)P_i(u)\mathrm{d}u, \quad y_i(t) = \int_{-\infty}^{+\infty} \rho_U(u)y(t,u)P_i(u)\mathrm{d}u.$$

The orthogonality of polynomial [20] can be expressed as

$$\int_{-\infty}^{+\infty} \rho_U(u)P_i(u)P_j(u)du = \begin{cases} 1, & i=j, \\ 0, & i \neq j. \end{cases}$$

And the cycle recurrence formula of polynomials [20] is as follows

$$uP_i(u) = \frac{1}{2}P_{i+1}(u) + \frac{1}{2}P_{i-1}(u), \quad P_{-1}(u) = 0, \quad P_0(u) = 1. \tag{5}$$

Substituting (4) into (3), we have

$$\begin{cases} \frac{d}{dt}X_M(t) = -\frac{r}{k}X_M^3(t) + (r - \frac{ar}{k} - \bar{c}_1 - \sigma_1 u)X_M^2(t) - bX_M(t)Y_M(t) + a(r - \bar{c}_1 - \sigma_1 u)X_M(t), \\ \frac{d}{dt}Y_M(t) = (b_1 - d - \rho - \bar{c}_2 - \sigma_2 u)X_M(t)Y_M(t) - a(d + \bar{c}_2 + \sigma_2 u)Y_M(t), \end{cases} \tag{6}$$

where $X_M(t) = \sum\limits_{i=0}^{M} x_i(t)P_i(u)$, $Y_M(t) = \sum\limits_{i=0}^{M} y_i(t)P_i(u)$. The nonlinear terms $X_M^3(t)$, $X_M^2(t)$, and $X_M(t)Y_M(t)$ in equations (6) can be reduced into

$$\begin{aligned} X_M^3(t) &= S_0(t)P_0(u) + \cdots + S_{3M}(t)P_{3M}(u) = \sum_{i=0}^{3M} S_i(t)P_i(u), \\ X_M^2(t) &= R_0(t)P_0(u) + \cdots + R_{2M}(t)P_{2M}(u) = \sum_{i=0}^{2M} R_i(t)P_i(u), \\ X_M(t)Y_M(t) &= K_0(t)P_0(u) + \cdots + K_{2M}(t)P_{2M}(u) = \sum_{i=0}^{2M} K_i(t)P_i(u), \end{aligned} \tag{7}$$

where $S_i(t)(i = 0,1,2,\ldots,3M)$ is the coefficient of $P_i(u)$ in $X_M^3(t)$, $R_i(t)(i = 0,1,2,\ldots,2M)$ is the coefficient of $P_i(u)$ in $X_M^2(t)$, $K_i(t)(i = 0,1,2,\ldots,2M)$ is the coefficient of $P_i(u)$ in $X_M(t)Y_M(t)$, which can be calculated by the mathematical software. Besides, with the help of cycle recurrence formula (5), the stochastic terms $uX_M^2(t)$, $uX_M(t)$, $uX_M(t)Y_M(t)$ and $uY_M(t)$ in equations (6) can be written as

$$\begin{aligned} uX_M^2(t) &= \frac{1}{2}\left(\sum_{i=0}^{2M} P_i(u)[R_{i-1}(t)+R_{i+1}(t)]\right), \\ uX_M(t) &= \frac{1}{2}\left(\sum_{i=0}^{M} P_i(u)[x_{i-1}(t)+x_{i+1}(t)]\right), \\ uX_M(t)Y_M(t) &= \frac{1}{2}\left(\sum_{i=0}^{2M} P_i(u)[K_{i-1}(t)+K_{i+1}(t)]\right), \\ uY_M(t) &= \frac{1}{2}\left(\sum_{i=0}^{M} P_i(u)[y_{i-1}(t)+y_{i+1}(t)]\right). \end{aligned} \tag{8}$$

Then substituting equations (7) and (8) into equations (6), and taking the largest order of the polynomial $M = 4$, we can obtain

$$\begin{cases} \dfrac{d}{dt}X_4(t) = -\,b\left(\displaystyle\sum_{i=0}^{8} K_i(t)P_i(u)\right) - \dfrac{\sigma_1}{2}\left(\displaystyle\sum_{i=0}^{8} P_i(u)[R_{i-1}(t)+R_{i+1}(t)]\right) \\ \qquad - \dfrac{a\sigma_1}{2}\left(\displaystyle\sum_{i=0}^{4} P_i(u)[x_{i-1}(t)+x_{i+1}(t)]\right) - \dfrac{r}{k}\left(\displaystyle\sum_{i=0}^{12} S_i(t)P_i(u)\right) \\ \qquad + (r - \dfrac{ar}{k} - \bar{c}_1)\left(\displaystyle\sum_{i=0}^{8} R_i(t)P_i(u)\right) + a(r-\bar{c}_1)\left(\displaystyle\sum_{i=0}^{4} x_i(t)P_i(u)\right), \\ \dfrac{d}{dt}Y_4(t) = (b_1 - d - \rho - \bar{c}_2)\left(\displaystyle\sum_{i=0}^{8} K_i(t)P_i(u)\right) - \dfrac{\sigma_2}{2}\left(\displaystyle\sum_{i=0}^{8} P_i(u)[K_{i-1}(t)+K_{i+1}(t)]\right) \\ \qquad - \dfrac{a\sigma_2}{2}\left(\displaystyle\sum_{i=0}^{4} P_i(u)[y_{i-1}(t)+y_{i+1}(t)]\right) - a(d+\bar{c}_2)\left(\displaystyle\sum_{i=0}^{4} y_i(t)P_i(u)\right), \end{cases} \tag{9}$$

where $X_4(t) = \sum_{i=0}^{4} x_i(t)P_i(u)$, $Y_4(t) = \sum_{i=0}^{4} y_i(t)P_i(u)$. Then multiplying both sides of equations (9) by $P_i(u)$ $(i = 0, 1, \ldots, 4)$ in sequence and taking expectation with respect to the random variable U, we can obtain the equivalent deterministic equations

$$\begin{cases} \dot{x_0} = -bK_0 - \frac{\sigma_1}{2}R_1 - \frac{a\sigma_1}{2}x_1 - \frac{r}{k}S_0 + (r - \frac{ra}{k} - \bar{c}_1)R_0 + a(r-\bar{c}_1)x_0, \\ \dot{y_0} = (b_1 - d - \rho - \bar{c}_2)K_0 - \frac{\sigma_2}{2}K_1 - \frac{a\sigma_2}{2}y_1 - a(d+\bar{c}_2)y_0, \\ \dot{x_1} = -bK_1 - \frac{\sigma_1}{2}(R_0+R_2) - \frac{a\sigma_1}{2}(x_0+x_2) - \frac{r}{k}S_1 + (r - \frac{ra}{k} - \bar{c}_1)R_1 + a(r-\bar{c}_1)x_1, \\ \dot{y_1} = (b_1 - d - \rho - \bar{c}_2)K_1 - \frac{\sigma_2}{2}(K_0+K_2) - \frac{a\sigma_2}{2}(y_0+y_2) - a(d+\bar{c}_2)y_1, \\ \cdots \\ \dot{x_4} = -bK_4 - \frac{\sigma_1}{2}(R_3+R_5) - \frac{a\sigma_1}{2}x_3 - \frac{r}{k}S_4 + (r - \frac{ra}{k} - \bar{c}_1)R_4 + a(r-\bar{c}_1)x_4, \\ \dot{y_4} = (b_1 - d - \rho - \bar{c}_2)K_4 - \frac{\sigma_2}{2}(K_3+K_5) - \frac{a\sigma_2}{2}y_3 - a(d+\bar{c}_2)y_4. \end{cases} \tag{10}$$

In addition, the random response (4) and the corresponding ensemble mean response can be approximately expressed as

$$\begin{cases} x(t,u) \approx \sum\limits_{i=0}^{4} x_i(t)P_i(u), \\ y(t,u) \approx \sum\limits_{i=0}^{4} y_i(t)P_i(u), \end{cases}$$

$$\begin{cases} E[x(t,u)] \approx \sum\limits_{i=0}^{4} x_i(t)E[P_i(u)] = x_0(t), \\ E[y(t,u)] \approx \sum\limits_{j=0}^{4} y_j(t)E[P_j(u)] = y_0(t). \end{cases}$$

4 Numerical analysis and discussion

In [24], Saha and Bandyopadhyay pointed that in the plankton ecosystem, the most common features of the phytoplankton population is rapid increase of biomass due to rapid cell proliferation and almost equally rapid decrease in population, separated by some fixed time period. This type of rapid change in phytoplankton population density is known as 'bloom'. Some of these blooms more adequately called "harmful algal blooms" [25], due to the accumulation of high biomass and the presence of toxicity are noxious to marine ecosystems or to human health. Thus, in what follows, we will discuss the control of periodic oscillation for the stochastic phytoplankton-zooplankton model.

If $\sigma_1 = \sigma_2 = 0$, then stochastic model (3) is transformed into the following deterministic model:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = rx(1-\frac{x}{k})(a+x) - bxy - \bar{c}_1(a+x)x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b_1xy - dy(a+x) - \rho xy - \bar{c}_2(a+x)y, \end{cases} \tag{11}$$

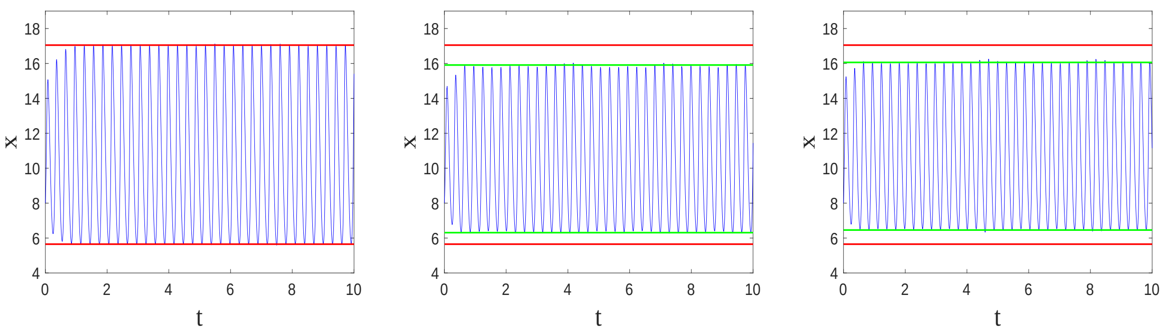
which can determine the deterministic response of the stochastic model.

For the deterministic response and ensemble mean response of stochastic model (3), we mainly analyse the amplitude changes with the change of catchability coefficients and noise intensities, and compare their control efficiencies. Some of parameters in models (10) and (11) are fixed as

$$r = 8, \quad k = 28, \quad a = 4, \quad b = 6.9, \quad b_1 = 4, \quad d = 2, \quad \rho = 1.$$

The initial conditions of model (11) is taken as $x(0) = 8$, $y(0) = 8$. The initial conditions of model (10) is taken as $x_0(0) = 8, y_0(0) = 8, x_i(0) = y_i(0) = 0, i = 1, 2, 3, 4.$

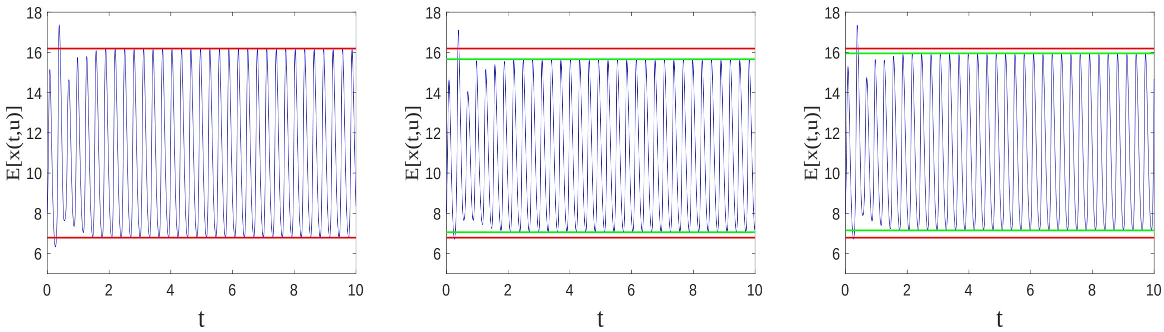
Figure 1 shows the deterministic response $x(t)$ with different catchability coefficients. If we take $\bar{c}_1 = 0.52$ and $\bar{c}_2 = 0.17$, then the deterministic response exhibits oscillation behavior, which is shown in Figure 1(a). By increasing catchability coefficient $\bar{c}_1$ or $\bar{c}_2$, we find that the deterministic response also exhibits oscillation behavior but the amplitude will decrease. The numerical results are reported in Figure 1(b) and Figure 1(c).



(a) $\bar{c}_1 = 0.52, \bar{c}_2 = 0.17$ (b) $\bar{c}_1 = 0.62, \bar{c}_2 = 0.17$ (c) $\bar{c}_1 = 0.52, \bar{c}_2 = 0.18$

Figure 1: Deterministic response $x(t)$ with different catchability coefficients.

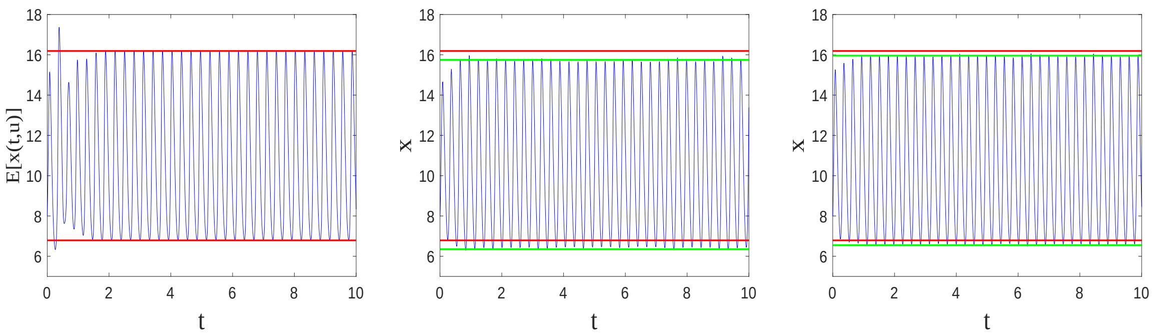
Taking noise intensities $\sigma_1 = 0.3$, $\sigma_2 = 0.2$ in model (11), we show the ensemble mean response with different catchability coefficients in Figure 2, which indicates that the ensemble mean response expectedly exhibits oscillation behavior, and the amplitude of the oscillation is decreased with the increase of catchability coefficient $\bar{c}_1$ or $\bar{c}_2$. Besides, by comparing Figure 1(a) with Figure 2(a) (Figure 1(b) with Figure 2(b) or Figure 1(c) with Figure 2(c)), one can see that the amplitude of ensemble mean response is less than that of deterministic response for the same catchability coefficients.



(a) $\bar{c}_1 = 0.52, \bar{c}_2 = 0.17$ (b) $\bar{c}_1 = 0.62, \bar{c}_2 = 0.17$ (c) $\bar{c}_1 = 0.52, \bar{c}_2 = 0.18$

Figure 2: Ensemble mean response $E[x(t,u)]$ with different catchability coefficients.

Figure 3 shows the ensemble mean response $E[x(t,u)]$ and the deterministic response $x(t)$ with same oscillation behavior. For the ensemble mean response, we take $\bar{c}_1 = 0.52$, $\bar{c}_2 = 0.17$ and obtain its oscillation behavior in Figure 3(a). However, for the deterministic response, we need to increase the capture parameter $\bar{c}_1$ or $\bar{c}_2$ to get the same oscillation behavior, which are shown in Figure 3(b) and Figure 3(c). We can find that the catchability coefficients of ensemble mean response is less than that of deterministic response for the same oscillation behavior.



(a) $\bar{c}_1 = 0.52, \bar{c}_2 = 0.17$ (b) $\bar{c}_1 = 0.631, \bar{c}_2 = 0.17$ (c) $\bar{c}_1 = 0.52, \bar{c}_2 = 0.181$

Figure 3: Ensemble mean response $E[x(t,u)]$ and deterministic response $x(t)$ with same oscillation behavior.

In this paper, we investigate the problem of oscillation control for a stochastic phytoplankton-zooplankton model by considering the influence of environmental fluctuations. The ensemble mean response of the stochastic model was first derived through the orthogonal polynomial approximation method. By comparing with the amplitude changes of the ensemble mean response $E[x(t,u)]$ and the deterministic response $x(t)$, we found that it is feasible to control the periodic oscillation of phytoplankton-zooplankton model by changing the values of catchability coefficients $\bar{c}_1$ and $\bar{c}_2$. Specifically, the amplitude of oscillation will decrease with the increase of $\bar{c}_1$ and $\bar{c}_2$, and the control efficiency of stochastic model is superior to that of deterministic model through comparison.

Acknowledgments

This work was jointly supported by the grants from National Natural Science Foundation of China (No. 11801224) and Natural Science Foundation of Jiangsu Province (No. BK20180856).

Appendices

$$
\begin{aligned}
S_0 =& x_0^3 + 3x_0x_1^2 + 3x_0x_2^2 + 3x_0x_3^2 + 3x_0x_4^2 + 3x_1^2x_2 + 6x_1x_2x_3 + 6x_1x_3x_4 + x_2^3 + 3x_2^2x_4 + \\
& 3x_2x_3^2 + 3x_2x_4^2 + 3x_3^2x_4 + x_4^3, \\
S_1 =& 3x_0^2x_1 + 6x_0x_1x_2 + 6x_0x_2x_3 + 6x_0x_3x_4 + 2x_1^3 + 3x_1^2x_3 + 6x_1x_2^2 + 6x_1x_2x_4 + 6x_1x_3^2 + \\
& 6x_1x_4^2 + 6x_2^2x_3 + 12x_2x_3x_4 + 2x_3^3 + 6x_3x_4^2, \\
S_2 =& 3x_0^2x_2 + 3x_0x_1^2 + 6x_0x_1x_3 + 3x_0x_2^2 + 6x_0x_2x_4 + 3x_0x_3^2 + 3x_0x_4^2 + 6x_1^2x_2 + 3x_1^2x_4 + \\
& 12x_1x_2x_3 + 12x_1x_3x_4 + 3x_2^3 + 6x_2^2x_4 + 9x_2x_3^2 + 9x_2x_4^2 + 9x_3^2x_4 + 3x_4^3, \\
S_3 =& 3x_0^2x_3 + 6x_0x_1x_2 + 6x_0x_1x_4 + 6x_0x_2x_3 + 6x_0x_3x_4 + x_1^3 + 6x_1^2x_3 + 6x_1x_2^2 + 12x_1x_2x_4 + \\
& 6x_1x_3^2 + 6x_1x_4^2 + 9x_2^2x_3 + 18x_2x_3x_4 + 4x_3^3 + 12x_3x_4^2, \\
S_4 =& 3x_0^2x_4 + 6x_0x_1x_3 + 3x_0x_2^2 + 6x_0x_2x_4 + 3x_0x_3^2 + 3x_0x_4^2 + 3x_1^2x_2 + 6x_1^2x_4 + 12x_1x_2x_3 + \\
& 12x_1x_3x_4 + 2x_2^3 + 9x_2^2x_4 + 9x_2x_3^2 + 9x_2x_4^2 + 12x_3^2x_4 + 5x_4^3, \\
R_0 =& x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2, \\
R_1 =& 2x_0x_1 + 2x_2x_1 + 2x_3x_2 + 2x_3x_4, \\
R_2 =& 2x_0x_2 + x_1^2 + 2x_1x_3 + x_2^2 + 2x_4x_2 + x_3^2 + x_4^2, \\
R_3 =& 2x_0x_3 + 2x_2x_1 + 2x_4x_1 + 2x_3x_2 + 2x_3x_4, \\
R_4 =& 2x_0x_4 + 2x_1x_3 + x_2^2 + 2x_4x_2 + x_3^2 + x_4^2, \\
K_0 =& x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4, \\
K_1 =& x_0y_1 + x_1y_0 + x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2 + x_3y_4 + x_4y_3, \\
K_2 =& x_0y_2 + x_1y_1 + x_1y_3 + x_2y_0 + x_2y_2 + x_2y_4 + x_3y_1 + x_3y_3 + x_4y_2 + x_4y_4, \\
K_3 =& x_0y_3 + x_1y_2 + x_1y_4 + x_2y_1 + x_2y_3 + x_3y_0 + x_3y_2 + x_3y_4 + x_4y_1 + x_4y_3, \\
K_4 =& x_0y_4 + x_1y_3 + x_2y_2 + x_2y_4 + x_3y_1 + x_3y_3 + x_4y_0 + x_4y_2 + x_4y_4, \\
K_5 =& x_1y_4 + x_2y_3 + x_3y_2 + x_3y_4 + x_4y_1 + x_4y_3.
\end{aligned}
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