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Existence of Peakons for the Generalized Camassa-Holm Type Equation with Cubic Nonlinearity

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Abstract: In this paper, we study the following generalized Camassa-Holm equation with both cubic and quadratic nonlinearities:

$$m_t + k_1(3uu_x m + u^2 m_x) + k_2(2mu_x + m_x u) = 0, \quad m = u - u_{xx},$$

which is presented as a linear combination of the Novikov equation and the Camassa-Holm equation with constants k_1 and k_2. The model is a cubic generalization of the Camassa-Holm equation. In this paper it is shown that the equation admits single-peaked soliton and periodic peakons.

Keywords: Generalization of CH equation, Peakons, Periodic peakons

1 Introduction

The well-known Camassa-Holm(CH) equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \tag{1}$$

which was proposed by Camassa and Holm as a nonlinear model for the unidirectional propagation of the shallow water waves over a flat bottom with $u(x,t)$ representing the water's free surface [6, 17, 18]. It has attracted much attention in the past decades. In addition, the CH equation (1) has several nice geometrical structures, for example, its description about a geodesic flow on the diffeomorphism group on the circle [19] and its derivation from a non-stretching invariant planar curve flow in the centro-equiaffine geometry [16]. Moreover, well-posedness theory and wave breaking phenomenon of the CH equation were studied extensively, and many interesting results have been deduced, see [2, 7–9, 26]. The stability and interaction of peakons were discussed in several references [10, 11, 25]. Among these properties, a remarkable one is that it admits the single peakons and periodic peakons in the following forms

$$\varphi_c(x,t) = ce^{-|x-ct|}, \quad c \in \mathbb{R}, \tag{2}$$

and

$$u_c(x,t) = \frac{c}{\mathrm{sh}(1/2)}\,\mathrm{ch}(\frac{1}{2} - (x-ct) + [x-ct]), \quad c \in \mathbb{R}, \tag{3}$$

where the notation $[x]$ denotes the largest integer part of the real number $x \in \mathbb{R}$.

In addition to the CH equation being an integrable model with peakons, other integrable peakon models, which include the Degasperis-Procesi equation and the cubic nonlinear peakon equations [3, 20], have been found. Indeed, two integrable CH-type equations with cubic nonlinearity have been discovered recently. The first one is mCH equation:

$$m_t + \left[(u^2 - u_x^2)m\right]_x = 0, \;\; m = u - u_{xx}, \tag{4}$$

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and the second one is the so-called Novikov equation:

$$u_t - u_{txx} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \quad t > 0, \quad x \in \mathbb{R}. \tag{5}$$

The perturbative symmetry approach [5] yielded necessary conditions for PDEs to admit infinitely many symmetries. Using this approach, Novikov [20] was able to isolate Eq.(5) in a symmetry classification and also found its first few symmetries. He subsequently found a scalar Lax pair for it, and also proved that the equation is integrable. By defining a new dependent variable m, Eq.(5) can be written as

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}. \tag{6}$$

Analogous to the Camassa-Holm equation, the Novikov equation has a bi-Hamiltonian structure and an infinite sequence of conserved quantities. In addition, the single-peaked solutions of the Novikov equation was obtained by Hone and Wang in [3], which takes the form

$$u(t,x) = \pm\sqrt{c}e^{-|x-ct|}, \quad c > 0,$$

and the periodic peakons

$$u_c(x,t) = \sqrt{c}\frac{\mathrm{ch}(\frac{1}{2} - (x-ct) + [x-ct])}{\mathrm{ch}(1/2)}, \quad c > 0.$$

Afterwards, Liu, Liu and Qu [23] proved the single peakons are orbital stable. Wang, Tian also proved the existence and orbital stability of the periodic peakons.

On the other hand, applying tri-Hamiltonian duality to the modified Korteweg-de Vries (mKdV) equation leads to the modified Camassa-Holm (mCH) equation with cubic nonlinearity. More generally, applying tri-Hamiltonian duality to the bi-Hamiltonian Gardner equation

$$Gu_t + u_{xxx} + k_1 u^2 u_x + k_2 uu_x = 0, \tag{7}$$

the resulting dual system is the following generalized modified Camassa-Holm (gmCH) equation with both cubic and quadratic nonlinearities [12]:

$$m_t + k_1\left[(u^2 - u_x^2)m\right]_x + k_2(2u_x m + um_x) = 0, \quad m = u - u_{xx}. \tag{8}$$

Recently, it was found that in [27] , for $k_1 \neq 0$, the gmCH equation (8) admits a single peakon of the form

$$\varphi_c(t,x) = ae^{-|x-ct|}, \ \ c \in \mathbb{R},$$

with

$$a = \frac{3}{4}\frac{-k_2 \pm \sqrt{k_2^2 + \frac{8}{3}k_1 c}}{k_1}, \ \ k_2^2 + \frac{8}{3}k_1 c \geq 0,$$

and also found that, for $k_1 \neq 0$, the gmCH equation (8) admits periodic peakons of the form

$$u_c(t,x) = a\,\mathrm{ch}(\frac{1}{2} - (x-ct) + [x-ct]),$$

where

$$a = \frac{3}{4}\frac{-k_2\,\mathrm{ch}(1/2) \pm \sqrt{k_2^2\,\mathrm{ch}^2(1/2) + \frac{4}{3}k_1 c(1 + 2\,\mathrm{ch}^2(1/2))}}{k_1(1 + 2\,\mathrm{ch}^2(1/2))}$$

and

$$k_2^2\,\mathrm{ch}^2(1/2) + \frac{4}{3}k_1 c(1 + 2\,\mathrm{ch}^2(1/2)) \geqslant 0.$$

The existence of (periodic) peakons is of interest for the nonlinear integrable equations since they are relatively new solitary waves. Inspired by [27], we focus on the following generalized Camassa-Holm equation with both cubic and quadratic nonlinearities:

$$m_t + k_1(3uu_x m + u^2 m_x) + k_2(2mu_x + m_x u) = 0, \quad m = u - u_{xx}, \tag{9}$$

where k_1 and k_2 are arbitrary constants. It is clear that equation (9) reduces to the CH equation for $k_1 = 0, k_2 = 1$ and the Novikov equation for $k_1 = 1, k_2 = 0$, respectively. Equation (9) is actually a linear combination of CH equation (1) and cubic nonlinear equation (6). Therefore, we may view equation (9) as a generalization of the CH equation, or simply call equation (9) a generalized CH equation. Like the Camassa-Holm and Novikov equations, the new equation also admits peaked soliton solutions. We will show the detailed proof in the paper.

2 Preliminaries

In this paper, we are concerned with the Cauchy problem for the generalized CH equation on both line and the unit circle:

$$\begin{cases} m_t + k_1(3uu_x m + u^2 m_x) + k_2(2mu_x + m_x u) = 0, \quad t > 0,\ x \in X = \mathbb{R} \text{ or } \mathcal{S}, \\ m(t,x) = u(t,x) - u_{xx}(t,x), \\ u(0,x) = u_0(x), \quad x \in X. \end{cases} \tag{10}$$

First, we will present the definition of strong (or classical) solutions as follows:

Definition 1 *Let $u \in C([0,T); H^s(X)) \cap C^1([0,T); H^{s-1}(X))$ with $s > \frac{5}{2}$ and some $T > 0$ satisfies (10), then u is called a strong solution on $[0,T)$. If u is a strong solution on $[0,T)$ for every $T > 0$, then it is called a global strong solution.*

The following local well-posedness result and properties for strong solutions on the line and unit circle can be established using the same approach as in [14].

Proposition 1 *Let $u_0 \in H^s(X)$ with $s > \frac{5}{2}$. Then there exists a time $T > 0$ such that the initial value problem (10) has a unique strong solution $u \in C([0,T); H^s(X)) \cap C^1([0,T); H^{s-1}(X))$ and the map $u_0 \to u$ is continuous from a neighborhood of u_0 in $H^s(X)$ into $u \in C([0,T); H^s(X)) \cap C^1([0,T); H^{s-1}(X))$.*

If $m = u - u_{xx}$ is substituted in terms of u into the generalized CH equation (10), then the resulting fully nonlinear partial differential equation takes the following form:

$$\begin{aligned} u_t + k_1 u^2 u_x + \frac{1}{2}k_1(1-\partial_x^2)^{-1}u_x^3 + k_1(1-\partial_x^2)^{-1}\partial_x(u^3 + \frac{3}{2}uu_x^2) \\ + k_2 uu_x + k_2\partial_x(1-\partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2) = 0. \end{aligned} \tag{11}$$

Taking the convolution with the Green's function for the Helmholtz operator $(1 - \partial_x^2)$, equation (11) can be rewritten as

$$\begin{aligned} u_t + k_1 u^2 u_x + \frac{1}{2}k_1 G(x) * u_x^3 + k_1 G(x) * \partial_x(u^3 + \frac{3}{2}uu_x^2) \\ + k_2 uu_x + k_2 G(x) * \partial_x(u^2 + \frac{1}{2}u_x^2) = 0. \end{aligned} \tag{12}$$

Note that u can be formulated by the Green function $G(x)$ as

$$u = (1-\partial_x^2)^{-1}m = G * m, \tag{13}$$

where $G(x) = \frac{1}{2}e^{-|x|}$ for the non-periodic case, $G(x) = \dfrac{\text{ch}(1/2 - x + [x])}{2\,\text{sh}(1/2)}$ for the periodic case, and $*$ denotes the convolution product on X, defined by

$$(f * g)(x) = \int_X f(y)g(x-y)dy.$$

Next, we can derive the single solutions of equation (9).

Theorem 2 *(Single peakons) For the wave speed c satisfying $k_2^2 + 4k_1 c \geq 0$, equation (9) with $k_1 \neq 0$ admits the single peakons of the form:*

$$u = Ae^{-|x-ct|}, \tag{14}$$

where $A = \dfrac{-k_2 \pm \sqrt{k_2^2 + 4k_1 c}}{2k_1}$.

The above formulation (11) allows us to define the periodic weak solutions as follows.

Definition 2 *Given initial data $u_0 \in W^{1,3}(\mathcal{S})$, the function $u \in L^\infty_{loc}([0,T), W^{1,3}_{loc}(\mathcal{S}))$ is called a periodic weak solution to the initial value problem (10) if it satisfies the following identity:*

$$\begin{aligned} \int_0^T \int_{\mathcal{S}} \Big[u\partial_t\phi + \frac{k_1}{3}u^3\partial_x\phi + k_1 G(x) * \left(u^3 + \frac{3}{2}uu_x^2\right)\partial_x\phi - k_1 G(x) * \left(\frac{u_x^3}{2}\right)\phi \\ + \frac{k_2}{2}u^2\partial_x\phi + k_2 G(x) * (u^2 + \frac{1}{2}u_x^2)\partial_x\phi \Big] dxdt + \int_{\mathcal{S}} u_0(x)\phi(0,x)dx = 0, \end{aligned} \tag{15}$$

for any smooth test function $\phi(t,x) \in C_c^\infty([0,T) \times \mathcal{S})$*. If* u *is a weak solution on* $[0,T)$ *for every* $T>0$*, then it is called a global periodic weak solution.*

The following theorem shows the existence of periodic peakons for the generalized CH equation (9).

Theorem 3 *(Periodic peakons) For the wave speed c satisfying* $k_2^2 \operatorname{ch}^2(1/2) + 4k_1 c\left(1+\operatorname{sh}^2(1/2)\right) \geq 0$*, equation (9) with* $k_1 \neq 0$ *possesses the periodic peaked traveling-wave solutions of the form:*

$$u_c(x,t) = a\operatorname{ch}(\zeta), \;\; \zeta = \frac{1}{2} - (x-ct) + [x-ct], \tag{16}$$

where

$$a = \frac{-k_2 \operatorname{ch}(1/2) \pm \sqrt{k_2^2 \operatorname{ch}^2(1/2) + 4k_1 c\left(1+\operatorname{sh}^2(1/2)\right)}}{2k_1\left(1+\operatorname{sh}^2(1/2)\right)} \tag{17}$$

as the global periodic weak solutions to (10) in the sense of Definition 2.2.

3 Proof of Existence

In this section, we offer the detailed proof of existence of single-peakon solutions and periodic peakons for equation (9).

3.1 Proof of Theorem 2

Proof. Firstly, let us suppose the single-peakon solution of equation (9) in the form of

$$u = Ae^{-|x-ct|}, \tag{18}$$

where A is to be determined. The derivatives of expression (18) do not exist at $x = ct$, thus (18) can not satisfy equation (9) in the classical sense. However, in the weak sense, we can write out the expressions of u_x and u_t with help of distribution:

$$u_x = -Asgn(x-ct)e^{-|x-ct|}, \;\; u_t = cAsgn(x-ct)e^{-|x-ct|}. \tag{19}$$

Next, we need consider two cases (i) $x > ct$ and (ii) $x < ct$.
For $x > ct$, we calculate from (18) and (19) that

$$u_t + k_1 u^2 u_x + k_2 u u_x = Ace^{-(x-ct)} - k_1 A^3 e^{-3(x-ct)} - k_2 A^2 e^{-2(x-ct)}. \tag{20}$$

Note that for the non-periodic case, the Green function $G(x) = \frac{1}{2}e^{-|x|}$, it is thus deduced that

$$\begin{aligned}
&\frac{1}{2}k_1 G(x) * u_x^3 + k_1 G(x) * \partial_x(u^3 + \frac{3}{2}uu_x^2) \\
&= -\frac{1}{4}k_1 A^3 \int_R sgn(y-ct)e^{-|x-y|-3|y-ct|}dy \\
&\qquad -\frac{15}{4}k_1 A^3 \int_R sgn(y-ct)e^{-|x-y|-3|y-ct|}dy \\
&= -4k_1 A^3 \left(-\int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{+\infty}\right) e^{-|x-y|-3|y-ct|}dy \\
&= -k_1 A^3 e^{-(x-ct)} + k_1 A^3 e^{-3(x-ct)},
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&k_2 G(x) * \partial_x(u^2 + \frac{1}{2}u_x^2) \\
&= -\frac{3}{2}k_2 A^2 \int_R sgn(y-ct)e^{-|x-y|-2|y-ct|}dy \\
&= -\frac{3}{2}k_2 A^2 \left(-\int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{+\infty}\right) e^{-|x-y|-2|y-ct|}dy \\
&= -k_2 A^2 e^{-(x-ct)} + k_2 A^2 e^{-2(x-ct)}.
\end{aligned} \tag{22}$$

The case $x < ct$ is similar to $x > ct$, here we do not compute in detail.
Plugging (20), (21) and (22) into (12), we deduce that

$$\begin{aligned} u_t + k_1 u^2 u_x + \frac{1}{2} k_1 G(x) * u_x^3 + k_1 G(x) * \partial_x (u^3 + \frac{3}{2} u u_x^2) \\ + k_2 u u_x + k_2 G(x) * \partial_x (u^2 + \frac{1}{2} u_x^2) \\ = (Ac - k_1 A^3 - k_2 A^2) e^{-(x-ct)} \\ = 0. \end{aligned} \tag{23}$$

Therefore, we are able to conclude from (23) that A should satisfy

$$k_1 A^2 + k_2 A - c = 0. \tag{24}$$

In general, we may obtain

$$A = \frac{-k_2 \pm \sqrt{k_2^2 + 4k_1 c}}{2k_1} \tag{25}$$

where $k_2^2 + 4k_1 c \geq 0$ with $k_1 \neq 0$. The proof of Theorem 2 is completed.

Remark 4 *In particular,* $k_1 = 0$*,* $k_2 \neq 0$*, we obtain* $A = \frac{c}{k_2}$*. In general, for* $k_1 \neq 0$*, we can derive*

$$A = \frac{-k_2 \pm \sqrt{k_2^2 + 4k_1 c}}{2k_1}. \tag{26}$$

If $k_2^2 + 4k_1 c \geq 0$*, then* A *is a real number. In particular, when* $k_1 = 1$*,* $k_2 = 1$ *and* $c = \frac{3}{4}$*, we can obtain the figure of single-peakons in Fig.1. If* $k_2^2 + 4k_1 c \leq 0$*, then* A *is a complex number, which means the peakon solution with complex coefficient is obtained.*

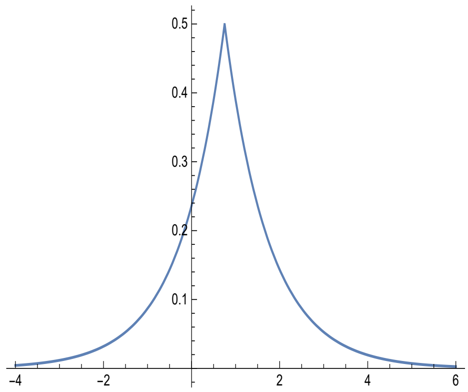


Figure 1: Single-peakons with $k_1 = 1$,$k_2 = 1$,$c = \frac{3}{4}$,$(t = 1)$.

3.2 Proof of Theorem 3

Proof. Firstly, we identify $\mathcal{S} = [0, 1)$ and regard $u_c(t, x)$ as spatial periodic function on $\mathcal{S}$ with period one. On one hand, it is noted that u_c is continuous on $\mathcal{S}$ with peak at $x = 0$. On the other hand, u_c is smooth on $(0, 1)$ and for all $t \in \mathbb{R}^+$,

$$\partial_x u_c(t, x) = -a\,\mathrm{sh}(\zeta) \in L^\infty(\mathcal{S}). \tag{27}$$

Hence, if one denotes $u_{c,0}(x) = u_c(0,x),\ x \in \mathcal{S}$, then it holds that

$$\lim_{t\to 0^+} \|u_c(t,\cdot) - u_{c,0}(\cdot)\|_{W^{1,\infty}(\mathcal{S})} = 0. \tag{28}$$

As in (27), it is found that

$$\partial_t u_c(x,t) = ac\,\mathrm{sh}(\zeta) \in L^\infty(\mathcal{S}),\ t \geq 0. \tag{29}$$

A direct computation gives the following identity:

$$u_c^2 \partial_x u_c = -a^3\,\mathrm{ch}^2(\zeta)\,\mathrm{sh}(\zeta) = -a^3\,\mathrm{sh}(\zeta) - a^3\,\mathrm{sh}^3(\zeta). \tag{30}$$

Using (27)-(29) and integration by parts, it is thus deduced that, for every test function $\phi(t,x) \in C_c^\infty([0,\infty)\times\mathcal{S})$,

$$\begin{aligned}
&\int_0^\infty \int_{\mathcal{S}} \left(u_c \partial_t \phi + \frac{k_1}{3} u_c^3 \partial_x \phi + \frac{k_2}{2} u_c^2 \partial_x \phi \right) dxdt + \int_{\mathcal{S}} u_{c,0}(x)\phi(x,0)dx \\
&= -\int_0^\infty \int_{\mathcal{S}} \phi(\partial_t u_c + k_1 u_c^2 \partial_x u_c + k_2 u_c \partial_x u_c) dxdt \\
&= \int_0^\infty \int_{\mathcal{S}} \phi\left((-ac + k_1 a^3)\,\mathrm{sh}(\zeta) + k_1 a^3\,\mathrm{sh}^3(\zeta) + k_2 a^2\,\mathrm{sh}(\zeta)\,\mathrm{ch}(\zeta)\right) dxdt.
\end{aligned} \tag{31}$$

It follows from (27), (29) and the proof of Theorem 4.1 in [13] that

$$\begin{aligned}
&\int_0^\infty \int_{\mathcal{S}} \left[k_1 G(x) * \left(u_c^3 + \frac{3}{2} u_c (\partial_x u_c)^2 \right) \partial_x \phi - \frac{k_1}{2} G(x) * (\partial_x u_c)^3 \phi \right] dxdt \\
&= -k_1 \int_0^\infty \int_{\mathcal{S}} \phi G(x) * \left(3u_c^2 \partial_x u_c + \frac{1}{2} (\partial_x u_c)^3 \right) dxdt \\
&\quad -\frac{3}{2} k_1 \int_0^\infty \int_{\mathcal{S}} \phi G_x(x) * \left(u_c (\partial_x u_c)^2 \right) dxdt.
\end{aligned} \tag{32}$$

We calculate from (27) and (30) that

$$3u_c^2 \partial_x u_c + \frac{1}{2}(\partial_x u_c)^3 = -3a^3\,\mathrm{ch}^2(\zeta)\,\mathrm{sh}(\zeta) - \frac{1}{2} a^3\,\mathrm{sh}^3(\zeta) = -3a^3\,\mathrm{sh}(\zeta) - \frac{7}{2} a^3\,\mathrm{sh}^3(\zeta)$$

and

$$u_c(\partial_x u_c)^2 = a^3\,\mathrm{ch}(\zeta)\,\mathrm{sh}^2(\zeta),$$

which together with (32), we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathcal{S}} \left[k_1 G(x) * \left(u_c^3 + \frac{3}{2} u_c (\partial_x u_c)^2 \right) \partial_x \phi - \frac{k_1}{2} G(x) * (\partial_x u_c)^3 \phi \right] \\
&= k_1 a^3 \int_0^\infty \int_{\mathcal{S}} \phi G(x) * \left(3\,\mathrm{sh}(\zeta) + \frac{7}{2}\,\mathrm{sh}^3(\zeta) \right) dxdt \\
&\quad -\frac{3}{2} k_1 a^3 \int_0^\infty \int_{\mathcal{S}} \phi G_x(x) * \left(\mathrm{ch}(\zeta)\,\mathrm{sh}^2(\zeta) \right) dxdt.
\end{aligned} \tag{33}$$

On the other hand, noticing from the explicit form of the Green function $G(x)$ for the periodic case that

$$G(x) = \frac{\mathrm{ch}(1/2 - x + [x])}{2\,\mathrm{sh}(1/2)} \ \ and\ \ G_x(x) = -\frac{\mathrm{sh}(1/2 - x + [x])}{2\,\mathrm{sh}(1/2)},\ x \in \mathbb{R},$$

we obtain

$$\begin{aligned}
&G(x) * \left(3\,\mathrm{sh}(\zeta) + \frac{7}{2}\,\mathrm{sh}^3(\zeta) \right)(x,t) \\
&= \frac{1}{2\,\mathrm{sh}(1/2)} \int_{\mathcal{S}} \mathrm{ch}(1/2 - (x-y) + [x-y]) \cdot \Big(3\,\mathrm{sh}(1/2 - (y-ct) + [y-ct]) \\
&\quad + \frac{7}{2}\,\mathrm{sh}^3(1/2 - (y-ct) + [y-ct]) \Big) dy
\end{aligned} \tag{34}$$

and

$$
\begin{aligned}
&G_x(x)*(\mathrm{ch}(\zeta)\,\mathrm{sh}^2(\zeta))(x,t)\\
&=-\frac{1}{2\,\mathrm{sh}(1/2)}\int_{\mathcal{S}}\mathrm{sh}(1/2-(x-y)+[x-y])\cdot\Big(\mathrm{ch}(1/2-(y-ct)+[y-ct])\\
&\quad\cdot\mathrm{sh}^2(1/2-(y-ct)+[y-ct])\Big)dy.
\end{aligned}
\tag{35}
$$

To proceed, we consider two cases: (i) $x > ct$ and (ii) $x < ct$. When $x > ct$, we split the right-hand side of (34) into the following three parts:

$$
\begin{aligned}
&G(x)*\left(3\,\mathrm{sh}(\zeta)+\frac{7}{2}\,\mathrm{sh}^3(\zeta)\right)(x,t)\\
&=\frac{1}{2\,\mathrm{sh}(1/2)}\left(\int_0^{ct}+\int_{ct}^{x}+\int_x^1\right)\mathrm{ch}(1/2-(x-y)+[x-y])\\
&\quad\cdot\Big(3\,\mathrm{sh}(1/2-(y-ct)+[y-ct])+\frac{7}{2}\,\mathrm{sh}^3(1/2-(y-ct)+[y-ct])\Big)dy\\
&=I_1+I_2+I_3.
\end{aligned}
\tag{36}
$$

Using the identity $\mathrm{sh}(3x)=4\,\mathrm{sh}^3(x)+3\,\mathrm{sh}(x)$, a direct calculation gives rise to

$$
\begin{aligned}
I_1&=\frac{1}{2\,\mathrm{sh}(1/2)}\int_0^{ct}\mathrm{ch}(1/2-x+y)\\
&\quad\cdot\left(3\,\mathrm{sh}(-1/2+ct-y)+\frac{7}{2}\,\mathrm{sh}^3(-1/2+ct-y)\right)dy\\
&=\frac{1}{2\,\mathrm{sh}(1/2)}\Big(\int_0^{ct}\frac{3}{8}\,\mathrm{ch}(1/2-x+y)\,\mathrm{sh}(-1/2+ct-y)dy\\
&\qquad+\int_0^{ct}\frac{7}{8}\,\mathrm{ch}(1/2-x+y)\,\mathrm{sh}(-3/2+3ct-3y)dy\Big)\\
&\quad+\frac{7}{64\,\mathrm{sh}(1/2)}\Big(-\mathrm{ch}(1+x-ct)+\mathrm{ch}(1+x-3ct)\\
&\qquad-\frac{1}{2}\,\mathrm{ch}(2-x+ct)+\frac{1}{2}\,\mathrm{ch}(2-x-3ct)\Big)\\
&=\frac{1}{64\,\mathrm{sh}(1/2)}\Big(-6ct\,\mathrm{sh}(x-ct)-3\,\mathrm{ch}(1-x+ct)+3\,\mathrm{ch}(1-x-ct)\\
&\quad-7\,\mathrm{ch}(1+x-ct)+7\,\mathrm{ch}(1+x-3ct)-\frac{7}{2}\,\mathrm{ch}(2-x+ct)+\frac{7}{2}\,\mathrm{ch}(2-x-3ct)\Big).
\end{aligned}
\tag{37}
$$

In a similar manner,

$$
\begin{aligned}
I_2&=\frac{1}{2\,\mathrm{sh}(1/2)}\int_{ct}^{x}\mathrm{ch}(1/2-x+y)\\
&\quad\cdot\left(3\,\mathrm{sh}(1/2+ct-y)+\frac{7}{2}\,\mathrm{sh}^3(1/2+ct-y)\right)dy\\
&=\frac{1}{2\,\mathrm{sh}(1/2)}\int_0^{ct}\mathrm{ch}(1/2-x+y)\\
&\quad\cdot\left(\frac{3}{8}\,\mathrm{sh}(1/2+ct-y)+\frac{7}{8}\,\mathrm{sh}(3/2+3ct-3y)\right)dy\\
&=\frac{1}{64\,\mathrm{sh}(1/2)}\Big(6(x-ct)\,\mathrm{sh}(1-x+ct)-7\,\mathrm{ch}(2-3x+3ct)+7\,\mathrm{ch}(2-x+ct)\\
&\qquad-\frac{7}{2}\,\mathrm{ch}(1-3x+3ct)+\frac{7}{2}\,\mathrm{ch}(1+x-ct)\Big)
\end{aligned}
\tag{38}
$$

and

$$\begin{aligned}
I_3 &= \frac{1}{2\,\mathrm{sh}(1/2)}\int_x^1 \mathrm{ch}(1/2-x+y) \\
&\quad\cdot\left(3\,\mathrm{sh}(1/2+ct-y)+\frac{7}{2}\,\mathrm{sh}^3(1/2+ct-y)\right)dy \\
&= \frac{1}{2\,\mathrm{sh}(1/2)}\int_x^1 \mathrm{ch}(1/2-x+y) \\
&\quad\cdot\left(\frac{3}{8}\,\mathrm{sh}(1/2+ct-y)+\frac{7}{8}\,\mathrm{sh}(3/2+3ct-3y)\right)dy \\
&= \frac{1}{64\,\mathrm{sh}(1/2)}\Big(-6(1-x)\,\mathrm{sh}(x-ct)-3\,\mathrm{ch}(1-x-ct)+3\,\mathrm{ch}(1-x+ct) \\
&\quad -7\,\mathrm{ch}(1+x-3ct)+7\,\mathrm{ch}(1-3x+3ct) \\
&\quad -\frac{7}{2}\,\mathrm{ch}(2-x-3ct)+\frac{7}{2}\,\mathrm{ch}(2-3x+3ct)\Big).
\end{aligned} \tag{39}$$

Plugging (37), (38) and (39) into (36), we deduce that for $x > ct$,

$$\begin{aligned}
&G(x)*\left(3\,\mathrm{sh}(\zeta)+\frac{7}{2}\,\mathrm{sh}^3(\zeta)\right)(x,t) \\
&= \frac{1}{64\,\mathrm{sh}(1/2)}\Big(6(x-ct)\,\mathrm{sh}(1-x+ct)-6(1-x+ct)\,\mathrm{sh}(x-ct) \\
&\quad -\frac{7}{2}\,\mathrm{ch}(1+x-ct)+\frac{7}{2}\,\mathrm{ch}(2-x+ct)-\frac{7}{2}\,\mathrm{ch}(2-3x+3ct)+\frac{7}{2}\,\mathrm{ch}(1-3x+3ct)\Big).
\end{aligned} \tag{40}$$

On the other hand, when $x > ct$, the right-hand side of (35) can be split into

$$\begin{aligned}
&G_x(x)*\left(\mathrm{ch}(\zeta)\,\mathrm{sh}^2(\zeta)\right)(x,t) \\
&= -\frac{1}{2\,\mathrm{sh}(1/2)}\left(\int_0^{ct}+\int_{ct}^{x}+\int_x^1\right)\Big(\mathrm{sh}(1/2-(x-y)+[x-y]) \\
&\qquad \cdot\mathrm{ch}(1/2-(y-ct)+[y-ct])\cdot\mathrm{sh}^2(1/2-(y-ct)+[y-ct])\Big)dy \\
&= J_1+J_2+J_3.
\end{aligned} \tag{41}$$

For J_1, due to the identity $2\,\mathrm{sh}^2(x)=\mathrm{ch}(2x)-1$, a direct calculation gives rise to

$$\begin{aligned}
J_1 &= -\frac{1}{2\,\mathrm{sh}(1/2)}\int_0^{ct}\mathrm{sh}(1/2-x+y)\cdot\mathrm{ch}(1/2-ct+y)\cdot\mathrm{sh}^2(1/2-ct+y)dy \\
&= -\frac{1}{4\,\mathrm{sh}(1/2)}\int_0^{ct}\mathrm{sh}(1/2-x+y)\cdot\mathrm{ch}(1/2-ct+y)\cdot(\mathrm{ch}(1-2ct+2y)-1)\,dy \\
&= -\frac{1}{32\,\mathrm{sh}(1/2)}\Big(2ct\,\mathrm{sh}(x-ct)+\frac{1}{2}\,\mathrm{ch}(2-x+ct)-\frac{1}{2}\,\mathrm{ch}(2-x-3ct) \\
&\quad -\mathrm{ch}(1-x+ct)+\mathrm{ch}(1-x-ct)-\mathrm{ch}(1+x-ct)+\mathrm{ch}(1+x-3ct)\Big).
\end{aligned} \tag{42}$$

Similarly, we also obtain

$$
\begin{aligned}
J_2 &= -\frac{1}{2\,\mathrm{sh}(1/2)} \int_{ct}^{x} \mathrm{sh}(1/2 - x + y) \cdot \mathrm{ch}(1/2 + ct - y) \cdot \mathrm{sh}^2(1/2 + ct - y) dy \\
&= -\frac{1}{4\,\mathrm{sh}(1/2)} \int_{ct}^{x} \mathrm{sh}(1/2 - x + y) \cdot \mathrm{ch}(1/2 + ct - y) \cdot (\mathrm{ch}(1 + 2ct - 2y) - 1)\, dy \\
&= -\frac{1}{32\,\mathrm{sh}(1/2)} \Big(-2(x - ct)\,\mathrm{sh}(1 - x + ct) - \frac{1}{2}\,\mathrm{ch}(1 + x - ct) + \frac{1}{2}\,\mathrm{ch}(1 - 3x + 3ct) \\
&\qquad - \mathrm{ch}(2 - 3x + 3ct) + \mathrm{ch}(2 - x + ct) \Big)
\end{aligned}
\tag{43}
$$

and

$$
\begin{aligned}
J_3 &= -\frac{1}{2\,\mathrm{sh}(1/2)} \int_{x}^{1} \mathrm{sh}(-1/2 - x + y) \cdot \mathrm{ch}(1/2 + ct - y) \cdot \mathrm{sh}^2(1/2 + ct - y) dy \\
&= -\frac{1}{32\,\mathrm{sh}(1/2)} \Big(2(1 - x)\,\mathrm{sh}(x - ct) + \frac{1}{2}\,\mathrm{ch}(2 - x - 3ct) - \frac{1}{2}\,\mathrm{ch}(2 - 3x + 3ct) \\
&\qquad - \mathrm{ch}(1 + x - 3ct) + \mathrm{ch}(1 - 3x + 3ct) - \mathrm{ch}(1 - x - ct) + \mathrm{ch}(1 - x + ct) \Big).
\end{aligned}
\tag{44}
$$

Plugging (42), (43) and (44) into (41), we deduce that for $x > ct$,

$$
\begin{aligned}
&G_x(x) * \left(\mathrm{ch}(\zeta)\,\mathrm{sh}^2(\zeta) \right)(x, t) \\
&= -\frac{1}{32\,\mathrm{sh}(1/2)} \Big(2(1 - x + ct)\,\mathrm{sh}(x - ct) - 2(x - ct)\,\mathrm{sh}(1 - x + ct) \\
&\qquad -\frac{3}{2}\,\mathrm{ch}(1 + x - ct) + \frac{3}{2}\,\mathrm{ch}(2 - x + ct) \\
&\qquad -\frac{3}{2}\,\mathrm{ch}(2 - 3x + 3ct) + \frac{3}{2}\,\mathrm{ch}(1 - 3x + 3ct) \Big).
\end{aligned}
\tag{45}
$$

It follows from (32), (35), (40) and (45) that

$$
\begin{aligned}
&\int_0^\infty \int_{ct}^1 \left[k_1 G(x) * \left(u_c^3 + \frac{3}{2} u_c (\partial_x u_c)^2 \right) \partial_x \phi - \frac{k_1}{2} G(x) * (\partial_x u_c)^3 \phi \right] dxdt \\
&= \frac{k_1 a^3}{8\,\mathrm{sh}(1/2)} \int_0^\infty \int_{ct}^1 \phi \Big(2\,\mathrm{sh}(3/2) \cdot \mathrm{sh}(1/2 - x + ct) - 2\,\mathrm{sh}(1/2) \cdot \mathrm{sh}(3/2 - 3x + 3ct) \Big) dxdt \\
&= k_1 a^3 \int_0^\infty \int_{ct}^1 \phi \left(\mathrm{sh}^2(1/2) \cdot \mathrm{sh}(1/2 - x + ct) - \mathrm{sh}^3(1/2 - x + ct) \right) dxdt.
\end{aligned}
\tag{46}
$$

In a similar manner, for the case of $x < ct$, we have

$$
\begin{aligned}
&\int_0^\infty \int_0^{ct} \left[k_1 G(x) * \left(u_c^3 + \frac{3}{2} u_c (\partial_x u_c)^2 \right) \partial_x \phi - \frac{k_1}{2} G(x) * (\partial_x u_c)^3 \phi \right] dxdt \\
&= k_1 a^3 \int_0^\infty \int_0^{ct} \phi G(x) * \left(3\,\mathrm{sh}(\zeta) + \frac{7}{2}\,\mathrm{sh}^3(\zeta) \right) - \frac{3}{2} \phi G_x(x) * \Big(\mathrm{ch}(\zeta) \cdot \mathrm{sh}^2(\zeta) \Big) dxdt \\
&= \frac{k_1 a^3}{8\,\mathrm{sh}(1/2)} \int_0^\infty \int_0^{ct} \phi \Big(-\mathrm{ch}(2 + x - ct) - \mathrm{ch}(1 - x + ct) \\
&\qquad - \mathrm{ch}(1 + 3x - 3ct) - \mathrm{ch}(2 + 3x - 3ct) \Big) dxdt \\
&= k_1 a^3 \int_0^\infty \int_0^{ct} \phi \left(-\mathrm{sh}^2(1/2) \cdot \mathrm{sh}(1/2 + x - ct) + \mathrm{sh}^3(1/2 + x - ct) \right) dxdt.
\end{aligned}
\tag{47}
$$

Hence, associated with (46), we obtain

$$\begin{aligned}&\int_0^\infty \int_{\mathcal{S}} \left[k_1 G(x) * \left(u_c^3 + \frac{3}{2} u_c (\partial_x u_c)^2\right) \partial_x \phi - \frac{k_1}{2} G(x) * (\partial_x u_c)^3 \phi\right] dxdt \\ &= k_1 a^3 \int_0^\infty \int_{\mathcal{S}} \phi \left(\mathrm{sh}^2(1/2) \cdot \mathrm{sh}(\zeta) - \mathrm{sh}^3(\zeta)\right) dxdt.\end{aligned} \tag{48}$$

Now we compute directly that

$$\begin{aligned}&\int_0^\infty \int_{\mathcal{S}} k_2 G(x) * \left(u_c^2 + \frac{1}{2}(\partial_x u_c)^2\right) \partial_x \phi dxdt \\ &= \int_0^\infty \int_{\mathcal{S}} k_2 \phi G(x) * \partial_x \left(u_c^2 + \frac{1}{2}(\partial_x u_c)^2\right) dxdt \\ &= -\frac{3k_2}{2} a^2 \int_0^\infty \int_{\mathcal{S}} \phi G(x) * \mathrm{sh}(2\zeta) dxdt.\end{aligned} \tag{49}$$

When $x > ct$, a direct calculation gives rise to

$$\begin{aligned}&G(x) * \mathrm{sh}(2\zeta)(t,x) \\ &= \frac{1}{2\,\mathrm{sh}(1/2)} \int_{\mathcal{S}} \mathrm{ch}(1/2 - (x-y) + [x-y]) \cdot \mathrm{sh}\left(1 - 2(y-ct) + 2[y-ct]\right) dy \\ &= \frac{1}{2\,\mathrm{sh}(1/2)} \Big[\int_0^{ct} \mathrm{ch}(1/2 - x + y) \cdot \mathrm{sh}(-1 - 2y + 2ct) dy \\ &\qquad + \int_{ct}^{x} \mathrm{ch}(1/2 - x + y) \cdot \mathrm{sh}(1 - 2y + 2ct) dy \\ &\qquad + \int_x^1 \mathrm{ch}(1/2 + x - y) \cdot \mathrm{sh}(1 - 2y + 2ct) dy \Big] \\ &= \frac{2}{3} \left[\mathrm{ch}(1/2)\,\mathrm{sh}(1/2 - (x-ct)) - \mathrm{sh}(1/2 - (x-ct))\,\mathrm{ch}\left(1/2 - (x-ct)\right)\right].\end{aligned} \tag{50}$$

In a similar manner, for $x < ct$,

$$\begin{aligned}&G(x) * \mathrm{sh}(2\zeta)(t,x) \\ &= \frac{2}{3} \left[-\mathrm{ch}(1/2)\,\mathrm{sh}(1/2 + (x-ct)) + \mathrm{sh}(1/2 + (x-ct))\,\mathrm{ch}\left(1/2 + (x-ct)\right)\right].\end{aligned} \tag{51}$$

Plugging (50) and (51) into (49), it is deduced by a straightforward computation that

$$\begin{aligned}&\int_0^\infty \int_{\mathcal{S}} k_2 G(x) * \left(u_c^2 + \frac{1}{2}(\partial_x u_c)^2\right) \partial_x \phi dxdt \\ &= -k_2 a^2 \int_0^\infty \int_{\mathcal{S}} \phi \left(\mathrm{sh}(\zeta)\,\mathrm{ch}(\zeta) - \mathrm{ch}(1/2)\,\mathrm{sh}(\zeta)\right) dxdt.\end{aligned} \tag{52}$$

In view of (31), (48) and (52), we have

$$\begin{aligned}&\int_0^\infty \int_{\mathcal{S}} [u_c \partial_t \phi + \frac{k_1}{3} u_c^3 \partial_x \phi + \frac{k_2}{2} u_c^2 \partial_x \phi \\ &\qquad + k_1 G(x) * \left(u_c^3 + \frac{3}{2} u_c (\partial_x u_c)^2\right) \partial_x \phi - k_1 G(x) * \left(\frac{(\partial_x u_c)^3}{2}\right) \phi \\ &\qquad + k_2 G(x) * (u_c^2 + \frac{1}{2}(\partial_x u_c)^2) \partial_x \phi] dxdt + \int_{\mathcal{S}} u_{c,0}(x) \phi(0,x) dx \\ &= \int_0^\infty \int_{\mathcal{S}} \phi a \Big[k_1 (1 + \mathrm{sh}^2(1/2)) a^2 + k_2\,\mathrm{ch}(1/2) a - c\Big] \mathrm{sh}(\zeta) dxdt.\end{aligned} \tag{53}$$

If a takes value as (17), then

$$k_1(1+\mathrm{sh}^2(1/2))a^2+k_2\,\mathrm{ch}(1/2)a-c=0,$$

which implies that

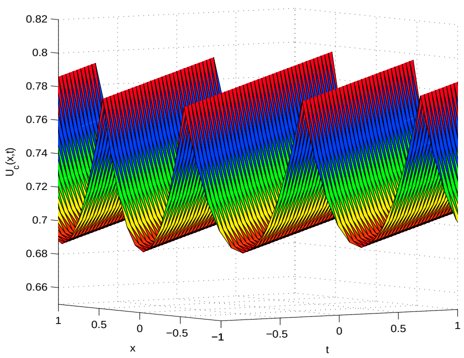
$$\begin{aligned}\int_0^\infty\int_{\mathcal{S}}&[u_c\partial_t\phi+\frac{k_1}{3}u_c^3\partial_x\phi+\frac{k_2}{2}u_c^2\partial_x\phi\\&+k_1G(x)*\left(u_c^3+\frac{3}{2}u_c(\partial_xu_c)^2\right)\partial_x\phi-\frac{k_1}{2}G(x)*(\partial_xu_c)^3\phi\\&+k_2G(x)*(u_c^2+\frac{1}{2}(\partial_xu_c)^2)\partial_x\phi]dxdt+\int_{\mathcal{S}}u_{c,0}(x)\phi(0,x)dx=0,\end{aligned}\tag{54}$$

for any test function $\phi(x,t)\in C_c^\infty([0,\infty)\times\mathcal{S})$. Thus the theorem is proved.

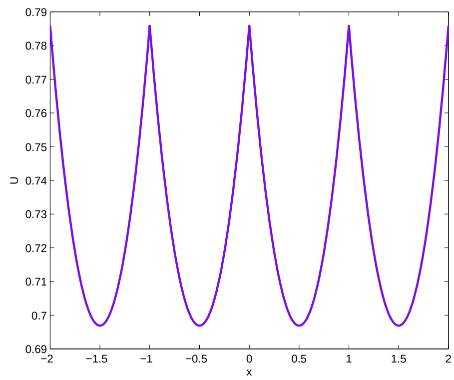
Remark 5 *In particular, when* $k_1=0$, $k_2\neq0$, *we obtain* $a=\dfrac{c}{k_2\,\mathrm{ch}(1/2)}$. *In general, if* $k_1\neq0$, *then we can derive*

$$a=\frac{-k_2\,\mathrm{ch}(1/2)\pm\sqrt{k_2^2\,\mathrm{ch}^2(1/2)+4k_1(1+\mathrm{sh}^2(1/2))c}}{2k_1(1+\mathrm{sh}^2(1/2))}.\tag{55}$$

If $k_2^2\,\mathrm{ch}^2(1/2)+4k_1(1+\mathrm{sh}^2(1/2))c\geq0$, *then a is a real number. If* $k_2^2\,\mathrm{ch}^2(1/2)+4k_1(1+\mathrm{sh}^2(1/2))c\leq0$, *then a is a complex number, which means that the periodic peakons with complex coefficient are found. The graph 2(a) and 2(b) show the shape of periodic peakons.*



(a) $k_1=k_2=1,\ c=1$



(b) $k_1=k_2=1,\ c=1,\ t=1$

Figure 2: The graph of periodic peakons for Novikov-CH equation.

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