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Existence of Normalized Solutions for the Coupled Elliptic System

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Abstract: In this paper, we focus on the existence of normalized solutions to the elliptic with quadratic nonlinearity under the Sobolev critical case

$$\begin{cases} -\Delta u = \lambda_1 u + uv, \text{ in } \mathbb{R}^N, \\ -\Delta v = \lambda_2 v + \frac{1}{2}u^2, \text{ in } \mathbb{R}^N, \\ \left(\int_{\mathbb{R}^N} u^2 dx\right)^{\frac{1}{2}} = a_1, \left(\int_{\mathbb{R}^N} v^2 dx\right)^{\frac{1}{2}} = a_2, \end{cases}$$

where $a_i > 0$ $(i = 1, 2)$ and λ_1, λ_2 are Lagrangian multipliers. We can use the minimizing argument on the Pohozaev manifold to obtain the existence of normalized solutions. When $N = 3$, we could obtain a normalized grate state. When $N = 4$, there is no normalized solution on the Pohozaev manifold. When $N = 5$, we could obtain an excited state on the Pohozaev manifold.

Keywords: Normalized solution; Pohozaev manifold; Minimizing argument

1 Introduction

The aim of this paper is to investigate the existence of the normalized solutions to the following quadratically coupled Schrödinger systems of the form

$$\begin{cases} -\Delta u = \lambda_1 u + uv, \text{ in } \mathbb{R}^N, \\ -\Delta v = \lambda_2 v + \frac{1}{2}u^2, \text{ in } \mathbb{R}^N, \\ \left(\int_{\mathbb{R}^N} u^2 dx\right)^{\frac{1}{2}} = a_1, \left(\int_{\mathbb{R}^N} v^2 dx\right)^{\frac{1}{2}} = a_2, \end{cases}$$

Here $3 \leq N \leq 4$ and $a_1, a_2 > 0$ are fixed. Since the constraint imposes a normalization on the L^2-masses of u and v, λ_1 and λ_2 cannot be determined a prior, but are part of the unknown.

The problem under consideration is associated to the research of elliptic equations with quadratic nonlinearity. In [19], second-order (or quadratic) nonlinearities of the so-called $\chi^{(2)}$ materials were usually discussed in the theory of the second-hamonic generation, where optical $\chi^{(2)}$ materials provided one of the fastest electronic nonlinearities among those that are available at that moment. It is well known and investigated in many branches of physics, where more detailed information can be referred to [8].

Since the pioneer work [11] proved the existence of solutions with a prescribed norm for a semilinear elliptic equation. From the physical viewpoint, the prescribed L^2-norm is a preserved quantity of evolution. Therefore, it has particular significance for finding normalized solutions (i.e., solutions with the prescribed L^2-norm). In recent decades, many outstanding results about normalized solutions of the elliptic equation [12, 13, 15, 18, 20, 21, 23] and the elliptic system [1–5, 7] have appeared. What is worth to mention is [16]. For $N = 1, 2$, Liang and Liu research the system

$$\begin{cases} -\Delta u = \mu_1 u + 2\alpha uv, \text{ in } \mathbb{R}^N, \\ -\Delta v = \mu_2 v + \alpha u^2, \text{ in } \mathbb{R}^N, \\ \left(\int_{\mathbb{R}^N} u^2 dx\right)^{\frac{1}{2}} = a, \left(\int_{\mathbb{R}^N} v^2 dx\right)^{\frac{1}{2}} = b, \end{cases}$$

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where $\alpha > 0$ and $a, b > 0$ are fixed. In the system, μ_1 and μ_2 are unknown. They heavily based on the rearrangement techniques to prove that there exists a solution $(\mu_1, \mu_2, \tilde{u}, \tilde{v})$ to the system with $\mu_1 < 0, \mu_2 < 0$, and $\tilde{u}, \tilde{v} \in C^2(\mathbb{R}^N)$ being positive, radially symmetric, and decreasing with $r = |x|$. Inspired by the literature above, it is natural to research the existence of normalized solutions for the following system

$$\begin{cases} -\Delta u = \lambda_1 u + uv, \text{ in } \mathbb{R}^N, \\ -\Delta v = \lambda_2 v + \frac{1}{2}u^2, \text{ in } \mathbb{R}^N, \\ \left(\int_{\mathbb{R}^N} u^2 dx\right)^{\frac{1}{2}} = a_1, \left(\int_{\mathbb{R}^N} v^2 dx\right)^{\frac{1}{2}} = a_2, \end{cases} \tag{1}$$

where $3 \le N \le 5$ and λ_1, λ_2 act as the Lagrange multipliers . As we know, the solution of the system can be obtained as the critical points of the energy functional $I(u, v) :\ H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$, where

$$I(u, v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 v dx \tag{2}$$

constrained on $S_{a_1} \times S_{a_2}$. Inspired by much of the work of mathematicians, we can introduce a fiber translation that also conserves the L^2-norms. Let $s \in \mathbb{R}$ and $(u, v) \in S_{a_1} \times S_{a_2}$, we define the function

$$s \star (u, v) := \left(e^{\frac{N}{2}s} u(e^s x), e^{\frac{N}{2}s} v(e^s x)\right).$$

One can easily check that $s \star (u, v) \in S_{a_1} \times S_{a_2}$. Hence, we can introduce an auxiliary function

$$\Psi_{(u,v)}(s) := E(s \star (u, v)) = \frac{e^{2s}}{2}\left[\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx\right] - \frac{e^{\frac{N}{2}s}}{2}\int_{\mathbb{R}^N} u^2 v dx, \tag{3}$$

where $\gamma_p = \frac{N(p-2)}{2p}$. The Pohozaev identity is given by

$$\Psi'_{(u,v)}(0) = P(u, v) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{N}{4}\int_{\mathbb{R}^N} u^2 v dx. \tag{4}$$

Thus, we can define the Pohozave set,

$$\mathcal{P} := \{(u, v) \in S_{a_1} \times S_{a_2} :\ P(u, v) = 0\}. \tag{5}$$

Based on the nature constraint $P(u, v) = 0$ and combining with the mass constraint $(u, v) \in S_{a_1} \times S_{a_2}$, we can deduce that $\int_{\mathbb{R}^N} u^2 v dx > 0$, which is essential to prove the existence of normalized solution.

For $N = 3$, the normalized solution of (1) minimizes the functional $I(u, v)$, that is,

$$m(a_1, a_2) := \inf_{(u,v)\in S_{a_1}\times S_{a_2}} I(u, v) = \inf_{(u,v)\in\mathcal{P}} I(u, v) < 0. \tag{6}$$

We can prove the existence of normalized grate state for (1).

Theorem 1 *Assume that $N = 3$. Then $m(a_1, a_2)$ is achieved by a normalized ground state of* (1) *with $\lambda_i < 0, a_i > 0 (i = 1, 2)$. Both components of solution are positive, radially symmetric and radially decreasing.*

For $N = 4$, it results that

$$\inf_{(u,v)\in S_{a_1}\times S_{a_2}} I(u, v) = \inf_{(u,v)\in\mathcal{P}} I(u, v) = 0. \tag{7}$$

Thus, we obtain a nonexistence result.

Theorem 2 *Assume $N = 4$ and let*

$$C_1 := \frac{1}{3}a_1^{(1-\gamma_3)3} + \frac{1}{6}a_2^{(1-\gamma_3)3} < \frac{1}{2C_{4,3}^3}. \tag{8}$$

The problem (1) *has no solution at all.*

Assume $N=5$. In order to overcome the difficulty in the proof of convergence, we introduce one mass constraint $S_a := \left\{(u,v)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N}(u^2+v^2)=a^2\right\}$ and $S_r=\{(u,v)\in S_a:\ u,v \text{ is radial.}\}$. Thus, there is one λ acting as the Lagrange multiplier. The normalized solution of (1) minimizes the functional $I(u,v)$ on $\mathcal{P}\cap S_r$, that is,

$$\sigma(a) := \inf_{(u,v)\in\mathcal{P}} I(u,v) > 0. \tag{9}$$

We can prove the existence of normalized excite state for (1.6).

Theorem 3 *Assume that $N=5$. Then $\sigma(a)$ is achieved by a normalized mountain pass type solution of* (1) *with $\lambda<0, a>0$. Both components of solution are positive, radially symmetric and radially decreasing.*

The structure of the paper is organized as follows. In section 2, some preliminaries are listed for later use. In section 3, the proof of Theorem 1 can be finished. Section 4 is devoted to studying the nonexistence to Theorem 2. Finally, we finish the proof of Theorem 3.

2 Preliminaries

In this paper, otherwise mentioned, we use the following notations:

- $|\cdot|_p$ is the norm of $L^p(\mathbb{R}^N)$ defined by $|u|_p=(\int_{\mathbb{R}^N}|u|^p dx)^{\frac{1}{p}}$ for $1\le p<\infty$;
- Sobolev space $H^1(\mathbb{R}^N)$ be endowed with the norm $\|u\|=[\int_{\mathbb{R}^N}(|\nabla u|^2+|u|^2)dx]^{\frac{1}{2}}$;
- $H^1_r(\mathbb{R}^N)=\{u\in H^1(\mathbb{R}^N):\ u \text{ is radial}\}$ and $H^1_r(\mathbb{R}^N)\hookrightarrow L^p(\mathbb{R}^N)$ compactly for $2<p<2^*$;
- $S_{a_1,r}\times S_{a_2,r}$ means that $(u,v)\in S_{a_1}\times S_{a_2}$ is radially symmetric respect to 0.

We also recall the Gagliardo-Nirenberg-Sobolev inequality, which is important to research the structure of $E(u,v)$. For every $q\in[2,2^*)$, there is

$$|u|_q^q \le C^q_{N,q}|u|_2^{(1-\gamma_q)q}|\nabla u|_2^{\gamma_q q},\ \forall u\in H^1(\mathbb{R}^N) \text{ and } \gamma_q=\frac{N(q-2)}{2q}, \tag{10}$$

where $C^q_{N,q}$ is the smallest constant that makes the inequality satisfied, that by invariance under dilations of (10) is

$$C^q_{N,q}=\sup_{u\in H^1(\mathbb{R}^N)}\frac{|u|_q^q}{|u|_2^{(1-\gamma_q)q}|\nabla u|_2^{\gamma_q q}}.$$

Based on the definition of $\Psi_{(u,v)}(s)$ and $\mathcal{P}$, we can obtain a crucial role for yielding a Palais-Smale sequence of $E|_{S_{a_1}\times S_{a_2}}$ with the elements from the Pohozaev set.

Proposition 4 *Let $(u,v)\in S_{a_1}\times S_{a_2}$. Then: $s\in\mathbb{R}$ is a critical point for $\Phi_{(u,v)}(s)$ if and only if $s_{(u,v)}\star(u,v)\in\mathcal{P}$.*

In particular, $(u,v)\in\mathcal{P}$ if and only if 0 is a critical point of $\Phi_{(u,v)(s)}$. For future convenience, we recall the continuity results in the following two lemmas.

Lemma 5 *([5, Lemma 3.5]) Let $\{u_n\}\subset H^1(\mathbb{R}^N)$, $s_n\subset\mathbb{R}$, and assume that $u_n\to u$ strongly in $H^1(\mathbb{R}^N)$, $s_n\to s$ in $\mathbb{R}$. Then $s_n\star u_n\to s\star u$ strongly in $H^1(\mathbb{R}^N)$.*

Lemma 6 *([5, Lemma 3.6]) For $(u,v)\in S_{a_1}\times S_{a_2}$ and $s\in\mathbb{R}$, the map*

$$T_uS_{a_1}\times T_vS_{a_2}\to T_{s\star u}S_{a_1}\times T_{s\star v}S_{a_2},\ \ (\varphi,\phi)\to s\star(\varphi,\phi)$$

is linear isomorphism with inverse $(\bar\varphi,\bar\phi)\to(-s)\star(\bar\varphi,\bar\phi)$.

The following lemma is an Brézis-Lieb type result, which is necessary to obtain the strong convergence in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

Lemma 7 *([16, Lemma 3.2]) Assume $(u_n,v_n)\rightharpoonup(u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N}\left(u_n^2v_n-(u_n-u)^2(v_n-v)\right)dx=\int_{\mathbb{R}^N}u^2vdx+o_n(1),$$

where $o_n(1)\to 0$ as $n\to\infty$.

3 Proof of Theorem 1

In this section, we devote to proving Theorem 1, which is based on the minimizing techniques and energy estimates argument. In order to obtain the critical point of $E|_{S_{a_1}\times S_{a_2}}(u,v)$, by (10) and Young inequlity, we can first obtain its below estimates,

$$\begin{aligned} I(u,v) =&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)dx-\frac{1}{2}\int_{\mathbb{R}^3}u^2vdx \\ \geq&\frac{1}{2}(|\nabla u|_2^2+|\nabla v|_2^2)-\frac{1}{3}C_{3,3}^3a_1^{(1-\gamma_3)3}|\nabla u|_2^{\gamma_3 3}-\frac{1}{6}C_{3,3}^3a_2^{(1-\gamma_3)3}|\nabla v|_2^{\gamma_3 3} \\ \geq&\frac{1}{2}(|\nabla u|_2^2+|\nabla v|_2^2)-C_{3,3}^3C_1\left(|\nabla u|_2^2+|\nabla v|_2^2\right)^{\frac{\gamma_3 3}{2}} \\ :=&H(|\nabla u|_2^2+|\nabla v|_2^2), \end{aligned} \tag{11}$$

where

$$C_1=\frac{1}{3}a_1^{(1-\gamma_3)3}+\frac{1}{6}a_2^{(1-\gamma_3)3}.$$

We know that $H(t):(0,+\infty)\to\mathbb{R}$ is defined by

$$H(t)=\frac{1}{2}t-C_{3,3}^3C_1t^{\frac{3}{4}}.$$

We can research the specific structure of $H(t)$.

Lemma 8 *When $N=3, a_i>0$ $(i=1,2)$, there exactly exists a strict minimizer at a negative level for $H(t)$. Moreover, there exists $0<T_0$ (relating to the parameters $a_i,(i=1,2)$) such that $H(t)>0$ if and only if $t\in(T_0,+\infty)$.*

Proof. The proof is omitted since it is easy to find the stationary point of simple function $H(t)$ ■

Lemma 9 *$\mathcal{P}_0$ is empty and $\mathcal{P}$ is a a C^1-submanifold of codimension 3 in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.*

Proof. *Step 1.* Since $\mathcal{P}$ and for the sake of completeness, we devote to prove $\mathcal{P}_0=\emptyset$. When $N=3,5$, we assume that there exists $(u,v)\in\mathcal{P}_0$. Thus, $P(u,v)=0$ and $\Psi''_{(u,v)}(0)=0$, that is

$$|\nabla u|_2^2+|\nabla v|_2^2-\frac{N}{4}\int_{\mathbb{R}^N}u^2vdx=0,$$

$$2|\nabla u|_2^2+2|\nabla v|_2^2-\frac{N^2}{8}\int_{\mathbb{R}^N}u^2vdx=0.$$

By calculating, we would obtain that $|\nabla u|_2^2+|\nabla v|_2^2=0$ contradicts $(u,v)\in S_{a_1}\times S_{a_2}\subset H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ when $N=3,5$.

Step 2. The fact $\mathcal{P}$ is a smooth manifold of codimension 3 in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ can be established. We know that $\mathcal{P}=\{(u,v)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N):P(u,v)=0, I(u)=0, G(v)=0\}$, where $I(u)=|u|_2^2-a_1^2, G(v)=|v|_2^2-a_2^2$ and P,I,G of class C^1 in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Thus, we can check that

$$d(P(u,v),I(u),G(v)):H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\to\mathbb{R}^N \text{ is surjective,}$$

for $(u,v)\in(I^{-1}(0)\times G^{-1}(0))\cap P^{-1}(0)$. The proof can refer to [17, Lemma 4.3] and [25, Lemma 3.7], which are omitted here. ■

The following lemma mainly research the structure of $\Phi_{(u,v)}(s)$, which is the key to obtain the necessary elements in the Pohozaev set. From (3) and when $N=3$, we have

$$\Psi_{(u,v)}(s):=E(s\star(u,v))=\frac{e^{2s}}{2}\left[\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)dx\right]-\frac{e^{\frac{3}{2}s}}{2}\int_{\mathbb{R}^3}u^2vdx. \tag{12}$$

Lemma 10 *For every $(u,v)\in S_{a_1}\times S_{a_2}$, we can obtain:*

(i) The function $\Phi_{(u,v)}(s)$ has an unique critical point $s_{(u,v)}$, which is a strict minimum point at negative level.

(ii) For every $(u,v) \in S_{a_1} \times S_{a_2}$, there exists a unique $s_{(u,v)} \in \mathbb{R}$ such that $s_{(u,v)} \star (u,v) \in \mathcal{P}$. Furthermore, $\mathcal{P} = \mathcal{P}_+$.

(iii) For every $(u,v) \in S_{a_1} \times S_{a_2}$ and $s \in \mathbb{R}$, there exists

$$E(s_{(u,v)} \star (u,v)) = \min\{E(s \star (u,v)) : s \in \mathbb{R}\} < 0.$$

(iv) The map $(u,v) \in S_{a_1} \times S_{a_2} \to s_{(u,v)} \in \mathbb{R}$ is of class C^1.

Proof.

(i) Based on Lemma 9 and as the demonstration in the first section, there exists $\int_{\mathbb{R}^N} u^2 v dx > 0$. We first prove that $\Psi_{(u,v)}(s)$ has a unique critical points. To this end, we recall that $\Psi'_{(u,v)}(s) = 0$. From (12) and by calculating, we obtain that

$$s_{(u,v)} = \left[\frac{3\int_{\mathbb{R}^N} u^2 v dx}{4\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx}\right]^2 \text{ and } \Psi_{(u,v)}(s_{(u,v)}) < 0. \tag{13}$$

(ii) From (13) and Proposition 4, there is an unique $s_{(u,v)} \star (u,v) \in \mathcal{P}$. From the structure of $\Psi_{(u,v)}(s)$ and combining with $\mathcal{P}_0 = \emptyset$, we can obtain $\Psi''_{(u,v)}(s_{(u,v)}) > 0$. Thus, $\mathcal{P} = \mathcal{P}_+$.

(iii) Since (i) and $\Psi_{(u,v)}(s) := E(s \star (u,v))$, it is obvious to obtain (iii).

(iv) It remains to show that $(u,v) \to s_{(u,v)} \in \mathbb{R}$ is of class C^1. We apply the implicit function theorem to the C^1 function $\Upsilon(s,(u,v)) = \Psi'_{(u,v)}(s)$. From $\Upsilon(s,(u,v)) = 0$, $\partial_s \Upsilon(s,(u,v)) = \Psi''_{(u,v)}(s) > 0$ and $\mathcal{P}_0 = \emptyset$, the map is C^1.

■

Furthermore, we define the infimum of $I(u,v)$.

Lemma 11 *It results that*

$$m(a_1,a_2) := \inf_{S_{a_1}\times S_{a_2}} I(u,v) = \inf_{\mathcal{P}} I(u,v) = \inf_{\mathcal{P}_+} I(u,v) \in (-\infty, 0)$$

Proof. For $(u,v) \in S_{a_1} \times S_{a_2}$,

$$E(u,v) \geq H(t) \geq \min_{t\in[0,T_0]} H(t).$$

Thus, $m(a_1,a_2) > -\infty$. Moreover, $s \star (u,v) \in S_{a_1} \times S_{a_2}$ and $e^{2s}(|\nabla u|_2^2 + |\nabla v|_2^2) < T_0$ as $s \to -\infty$, we have $E(s \star (u,v)) < 0$ and then $m(a_1,a_2) < 0$. For every $(u,v) \in \mathcal{P}_+ \subset S_{a_1} \times S_{a_2}$, we can obtain $m(a_1,a_2) \leq \inf_{\mathcal{P}} I(u,v)$. On the other hand, if $u \in S_{a_1} \times S_{a_2}$, then $s_{(u,v)} \star (u,v) \in P_+$, and

$$E(s_{(u,v)} \star (u,v)) = \min\{E(s \star (u,v) : s_{(u,v)} \in \mathbb{R}\} \leq I(u,v),$$

which implies that $\inf_{\mathcal{P}_+} I(u,v) \leq m(a_1,a_2)$. From Lemma 10 (i), we have $\inf_{\mathcal{P}_+} I(u,v) = \inf_{\mathcal{P}_+} I(u,v)$. Then, the proof is finished. ■

The following lemma indicates that we can choose the nonnegative minimizing sequence.

Lemma 12 *Let $(u_n,v_n) \in S_{a_1} \times S_{a_2}$ be a minimizing sequence for $I(u_n,v_n) \to m(a_1,a_2) := c$. Then $(|u_n|,|v_n|) \in S_{a_1} \times S_{a_2}$ is also a minimizing sequence for $E(u_n,v_n)$.*

Proof. Suppose that $(u_n,v_n) \in S_{a_1} \times S_{a_2}$ is a minimizing sequence for $I(u,v)$ with $I(u_n,v_n) \to c$. Since $(|u_n|,|v_n|) \in S_{a_1} \times S_{a_2}$ and $I(|u_n|,|v_n|) \leq I(u_n,v_n)$, there holds

$$c \leq \lim_{n\to\infty} I(|u_n|,|v_n|) \leq \lim_{n\to\infty} I(u_n,v_n) = c.$$

Then $(|u_n|,|v_n|) \in S_{a_1} \times S_{a_2}$ is also a minimizing sequence for $I(u,v)$ with $I(|u_n|,|v_n|) \to c$. ■

In the next lemma, we prove the existence of the Palais-Smale sequence of $m(a_1,a_2)$.

Lemma 13 *There is a radial Palais-Smale sequence $(u_n,v_n) \subset S_{a_1} \times S_{a_2}$ for $I(u,v)|_{S_{a_1}\times S_{a_2}}$ at level $m(a_1,a_2)$ with additional properties $P(u_n,v_n) \to 0$ and $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^N$.*

Proof. From Lemma 11, there holds that $(\bar{u}_n, \bar{v}_n)$ is the minimizing sequence on $S_{a_1} \times S_{a_2}$. We know that $(|\bar{u}_n|, |\bar{v}_n|)$ be the minimizing sequence on $S_{a_1} \times S_{a_2}$ from Lemma 12. Let (u_n, v_n) be the radially symmetric rearrangement of $(|\bar{u}_n|, |\bar{v}_n|)$, which still belongs to $S_{a_1} \times S_{a_2}$. Furthermore, from Lemma 10, for every n we can take $s_{(u_n,v_n)} \star (u_n, v_n) \in \mathcal{P}_+$ such that $s_{(u_n,v_n)} \star (u_n, v_n) \in S_{a_1} \times S_{a_2}$ and

$$E(s_{(u_n,v_n)} \star (u_n, v_n)) = \min\{E(s \star (u_n, v_n)) : \ s \in \mathbb{R}\} \leq E(u_n, v_n).$$

Thus we obtain a new minimizing sequence $(\omega_n, \sigma_n) = (s_{(u_n,v_n)} \star u_n, s_{(u_n,v_n)} \star v_n)$ for $I(u, v)$ with $(\omega_n, \sigma_n) \in S_{a_1} \times S_{a_2} \cap \mathcal{P}_+$, which is radially nonnegative. By Lemma 10 and then adopting Ekeland's variational principle in a standard way, we obtain the existence of a new radial minimizing sequence $(u_n, v_n) \in S_{a_1} \times S_{a_2} \cap A_{T_0}$ for $I(u, v) \to m(a_1, a_2)$ with the property that $\|(u_n, v_n) - (\omega_n, \sigma_n)\| \to 0$ as $n \to \infty$, which is also a Palais-Smale sequence for $m(a_1, a_2)$ on $S_{a_1} \times S_{a_2}$. By Lemma 7 and Lemma 10, we have $P(u_n, v_n) = P(\omega_n, \sigma_n) \to 0$ and $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^N$. ■

In what follows, we prove the convergence of Palais-Smale sequence in Lemma 13.

Lemma 14 *The Palais-Smale sequence obtained in Lemma 13 with*

$$m(a_1, a_2) < 0, \tag{14}$$

up to a subsequence, is strongly convergent to (u, v) *in* $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$.

Proof. The proof can be divided into three steps. Step 1: we would prove that (u_n, v_n) is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and the corresponding sequences $(\lambda_{1,n}, \lambda_{2,n})$ are bounded in $\mathbb{R}^2$. Since (u_n, v_n) is Palais-Smale sequence and from $P(u_n, v_n) \to 0$, we can obtain

$$\begin{aligned} c + 1 &:= m(a_1, a_2) + 1 \geq I(u_n, v_n) \\ &= \frac{1}{2}(|\nabla u_n|_2^2 + |\nabla v_n|_2^2) - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 v_n dx \\ &= \frac{1}{2}(|\nabla u_n|_2^2 + |\nabla v_n|_2^2) - \frac{1}{2}C_1 C_{3,3}^3 (|\nabla u_n|_2^2 + |\nabla v_n|_2^2)^{\frac{3}{4}}. \end{aligned}$$

Hence, we know that the Palais-Smale sequence is bounded from the coerciveness of $E(u_n, v_n)$. Moreover, there exists $(u, v) \in H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N), \ (u_n, v_n) \to (u, v) \text{ in } L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), \ p \in (2, 2^*),$$

$$(u_n, v_n) \to (u, v) \text{ a.e. in } \mathbb{R}^N.$$

From [24, Proposition 5.12], there exists $\lambda_{1,n}, \lambda_{2,n}$ such that

$$\begin{aligned} o_n(1)\|(\varphi, \phi)\| = &\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + \nabla v_n \nabla \phi) dx - \int_{\mathbb{R}^N} (\lambda_{1,n} u_n \varphi + \lambda_{2,n} v_n \phi) dx \\ &- \int_{\mathbb{R}^N} (u_n v_n \varphi + \frac{1}{2} u_n^2 \phi) dx \text{ in } \mathbb{R}^N, \end{aligned} \tag{15}$$

for every $(\varphi, \phi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Let $(\varphi, \phi) = (u_n, 0)$ and $(\varphi, \phi) = (0, v_n)$ act as the test functions, we have

$$\begin{cases} \lambda_{1,n} a_1^2 = |\nabla u_n|_2^2 - \int_{\mathbb{R}^N} u_n^2 v_n, \text{ in } \mathbb{R}^N, \\ \lambda_{2,n} a_2^2 = |\nabla v_n|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 v_n, \text{ in } \mathbb{R}^N. \end{cases} \tag{16}$$

Since the boundedness of (u_n, v_n) in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$, the strongly convergence of (u_n, v_n) in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), p \in (2, 2^*)$, and combining with

$$0 \leq \int_{\mathbb{R}^N} u_n^2 v_n dx \leq C|u_n|_3^2 |v_n|_3 < \tilde{C}, \ (\tilde{C} \text{ is a necessary finite constant}),$$

we can obtain that $\lambda_{i,n}, (i = 1, 2)$ is bounded. So, there exists $\lambda_i \in \mathbb{R}$ $(i = 1, 2)$, such that $\lambda_{1,n} \to \lambda_1$ and $\lambda_{2,n} \to \lambda_2$ in $\mathbb{R}^N$. Passing to the weak convergence of (15), we deduce that (u, v) and $u, v \geq 0$ satisfying the weak limit equations in $\mathbb{R}^N$,

$$\begin{cases} -\Delta u = \lambda_1 u + uv, \\ -\Delta v = \lambda_2 v + \frac{1}{2}u^2. \end{cases} \tag{17}$$

Therefore, we have $P(u, v) = 0$.

Therefore, we have $P(u, v) = 0$ and

$$\lambda_1|u|_2^2 + \lambda_2|v|_2^2 = \frac{(N-6)}{4}\int_{\mathbb{R}^N} u^2 v dx. \tag{18}$$

We also claim that $\lambda_i < 0, (i = 1, 2)$. From (16) and $P(u_n, v_n) \to 0$, there is

$$\lambda_{1,n}a_1^2 + \lambda_{2,n}a_2^2 = \frac{(N-6)}{4}\int_{\mathbb{R}^N} u_n^2 v_n dx, \tag{19}$$

which implies at least one of $\lambda_{1,n}$ and $\lambda_{2,n}$ is negative. Then we deduce that $\lambda_1 \leq$ or $\lambda_2 \leq 0$, with equality holds only if $u \equiv 0$ or $v \equiv 0$. This is impossible, since we can claim $u \neq 0$ and $v \neq 0$ and then from maximum principle, $u > 0, v > 0$. So, we need to eliminate the following two cases:

Case 1: If $u = 0$ and $v = 0$, we have

$$0 > m(a_1, a_2) = \lim_{n\to\infty} E(u_n, v_n) \geq \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx \geq 0,$$

which is a contradiction.

Case 2: If $u = 0$ and $v \neq 0$, we have

$$0 > m(a_1, a_2) = \lim_{n\to\infty} E(u_n, v_n) \geq \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx \geq 0,$$

which is a contradiction. Thus, at least one of λ_1 and λ_2 is negative. Next, we prove that if only if λ_1 and λ_2 is negative, then the other one is also negative, i.e. if $\lambda_2 < 0$ (or $\lambda_1 < 0$), then $\lambda_1 < 0$ (or $\lambda_2 < 0$). Making a contradiction that $\lambda_2 < 0$ while $\lambda_1 \geq 0$, then there exists

$$-\Delta u = \lambda_1 u + uv \geq 0, u \in L^2(\mathbb{R}^N).$$

However, the inequality does not have any positive classical solution for $3 \leq N \leq 4$ and $N = 5, p \in (2, \frac{2N-2}{N-2}]$. When $N = 3, 4$, we can adopt [10, Lemma A.2] and $u \in L^2(\mathbb{R}^N)$ to obtain $u \equiv 0$. Above all, we obtain $u \equiv 0$. In particular, this implies that v satisfies that

$$\begin{cases} -\Delta v = \lambda_2 v, \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} v^2 dx = a_2^2. \end{cases}$$

Then we have $P(0, \bar{v}) = 0$ and $J_{p,\mu_2,a_2}(\bar{v}) = m_\beta(0, a_2)$. Therefore,

$$\begin{aligned} 0 > m(a_1, a_2) &= \lim_{n\to\infty} E(u_n, v_n) \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 v_n dx\right] \\ &= \lim_{n\to\infty}\left[\left(\frac{N}{8} - \frac{1}{2}\right)\int_{\mathbb{R}^N} u_n^2 v_n dx\right] = o_n(1), \end{aligned}$$

which is a contradiction. Hence, $\lambda_1 < 0$ and $\lambda_2 < 0$. In the second case, we argue by contradiction and assume that $\lambda_1 < 0$ while $\lambda_2 \geq 0$, then we can use the same argument above to obtain $v \equiv 0$. Since the structure of our system (1), we obtain $u \equiv 0$, which is impossible. So $\lambda_1 < 0$ and $\lambda_2 < 0$.

Finally, we can prove the convergence of (u_n, v_n) in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. We know $(u_n, v_n) \rightharpoonup (\bar{u}, \bar{v})$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ and $u \neq 0, v \neq 0$. Let $\tilde{u}_n = u_n - u$ and $\tilde{v}_n = v_n - v$, where $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (0, 0)$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ and $(\tilde{u}_n, \tilde{v}_n) \to (0, 0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$. From Lemma 5 and Brézis-Lieb Lemma in [24], we can get

$$0 = P(u_n, v_n) + o_n(1) = P(\tilde{u}_n, \tilde{v}_n) + P(u, v) + o_n(1) = P(\tilde{u}_n, \tilde{v}_n) + o_n(1).$$

Since $\int_{\mathbb{R}^N} u^2 dx \leq a_1^2$ and $\int_{\mathbb{R}^N} v^2 dx \leq a_2^2$, from (18)-(19), we have

$$\lambda_1\left(\int_{\mathbb{R}^N} u^2 dx - a_1^2\right) + \lambda_2\left(\int_{\mathbb{R}^N} v^2 dx - a_2^2\right) = 0.$$

From $\lambda_i < 0,\ (i = 1, 2)$, we obtain $\int_{\mathbb{R}^N} u^2 dx = a_1^2$ and $\int_{\mathbb{R}^N} v^2 dx = a_2^2$, that is $(u, v) \in S_{a_1} \times S_{a_2}$. Hence, we also have

$$\begin{aligned} m(a_1, a_2) &= \lim_{n\to\infty} I(u_n, v_n) = \lim_{n\to\infty} I(\tilde{u}_n, \tilde{v}_n) + I(u, v) \\ &= \lim_{n\to\infty} \left[\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \tilde{u}_n|^2 + |\nabla \tilde{v}_n|^2) dx\right] + I(u, v) \geq I(u, v) \geq m(a_1, a_2). \end{aligned}$$

Thus, $m(a_1, a_2) = I(u, v)$ and hence $(u_n, v_n) \to (\bar{u}, \bar{v})$ strongly in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$, where (u, v) is global minimizer $I(u, v)|_{S_{a_1}\times S_{a_2}}$. ∎

4 Proof of Theorem 2

In this section, we begin to prove Theorem 2. Based on (8), we have

$$\begin{aligned} I(u, v) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 v dx \\ &\geq \left(\frac{1}{2} - C_{4,3}^3 C_1\right)\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx \geq 0, \end{aligned} \tag{20}$$

for every $(u, v) \in S_{a_1} \times S_{a_2}$ and $C_1 = \frac{1}{3}\beta a_1 + \frac{1}{6}\beta a_2$. Thus, we can define $m(a_1, a_2) := \inf_{S_{a_1}\times S_{a_2}} I(u, v) \geq 0$. Suppose that there exists a solution $(u, v) \in S_{a_1} \times S_{a_2}$ for (1) such that $m(a_1, a_2) = I(u, v) \geq 0$. We have the Pohozaev identity $P(u, v) = 0$, i.e.

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \int_{\mathbb{R}^N} u^2 v dx = 0.$$

Thus, we have

$$0 = \frac{1}{2}P(u, v) = I(u, v) \geq \left(\frac{1}{2} - C_{4,3}^3 C_1\right)\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx \geq 0.$$

Thus, we obtain $(|\nabla u|_2^2 + |\nabla v|_2^2) = 0$. Combining with $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, it results $(u, v) = 0$, which contradicts $(u, v) \in S_{a_1} \times S_{a_2}$. So, the problem (1) has no normalized solution at all.

5 Proof of Theorem 3

In this section, we would like to prove the existence of mountain pass solutions when $N = 5$. Here, we will solve the problem by using the thought derived from Section 7 of [22]. First, we have the following estimates:

$$\begin{aligned} I(u, v) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 v dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - C_{5,3}^3 C_1 (\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx)^{\frac{5}{4}}. \end{aligned} \tag{21}$$

where $C_1 = \frac{1}{2} a^{(1-\gamma_3)3}$. From the definite of $\Psi(s)$,

$$\begin{aligned} \Psi_{(u,v)}(s) &:= \tilde{E}(s, (u, v)) = E(s \star (u, v)) \\ &= \frac{1}{2}\left(\int_{\mathbb{R}^N} |\nabla(s \star u)|^2 + |\nabla(s \star v)|^2 dx\right) - \frac{1}{2}\int_{\mathbb{R}^N} (s \star u)^2 (s \star v) dx \\ &= \frac{e^{2s}}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx\right) - \frac{e^{\frac{5}{2}s}}{2}\int_{\mathbb{R}^N} u^2 v dx, \end{aligned} \tag{22}$$

Lemma 15 *For every $(u, v) \in S_a$, there exists a unique $t_{(u,v)} \in \mathbb{R}$ such that $t_{(u,v)} \star (u, v) \in \mathcal{P}$. $t_{(u,v)}$ is the unique critical point of the function $\Psi_{(u,v)}(s)$, and is a strict maximum point at positive level. Moreover:*

(i) $\mathcal{P} = \mathcal{P}_-$.

(ii) $\Phi_{(u,v)}(s)$ *is strictly increasing and convex on* $(T_0, +\infty)$*, and* $s_{(u,v)}$ *if and only if* $P(u,v) < 0$.

(iii) The maps $(u,v) \in S_a \to s_{(u,v)} \in \mathbb{R}$ *is of class* C^1

Proof. The proof is very similar to the one of Lemma 10, and hence is omitted. ■

Lemma 16 *It results that*

$$\sigma(a) := \inf_{(u,v)\in\mathcal{P}\cap S_r} I(u,v) > 0$$

Proof. If $(u,v) \in \mathcal{P} \cap S_r$, we have

$$|\nabla u|_2^2 + |\nabla v|_2^2 = \frac{5}{4}\int_{\mathbb{R}^N} u^2 v dx \le \frac{5C_{5,3}^3 C_1}{4}(|\nabla u|_2^2 + |\nabla v|_2^2)^{\frac{5}{4}},$$

which implies

$$(|\nabla u|_2^2 + |\nabla v|_2^2) \ge \left(\frac{4}{5C_{5,3}^3 C_1}\right)^4 := \zeta > 0.$$

For any $(u,v) \in \mathcal{P} \cap S_r$, we have

$$\begin{aligned} I(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - \frac{\beta}{2}\int_{\mathbb{R}^N} u^2 v dx \\ &= \frac{1}{10}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx \ge \frac{1}{10}\zeta > 0. \end{aligned}$$

Then $\sigma(a) := \inf_{(u,v)\in\mathcal{P}\cap S_r} I(u,v) > 0$. ■

Lemma 17 *There exists* $\delta > 0$ *sufficiently small, such that*

$$0 < \sup_{\bar{A}_\delta} I < \sigma(a) \text{ and } E(u,v) > 0,\ P(u,v) > 0 \text{ for } (u,v) \in \bar{A}_\delta$$

where $\bar{A}_\delta := \{(u,v) \in S_a : |\nabla u|_2^2 + |\nabla v|_2^2 < \delta\}$

Proof. By Gagliardo-Nirenberg's inequality,

$$\begin{aligned} I(u,v) &\ge \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - C_{5,3}^3 C_1 \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx\right]^{\frac{5}{4}}, \\ P(u,v) &\ge \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - \frac{5}{4}C_{5,3}^3 C_1 \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx\right]^{\frac{5}{4}}, \end{aligned}$$

choosing the necessary δ which is small enough such that $E(u,v) > 0,\ P(u,v) > 0$.

Recalling Lemma 16 and replacing δ with a smaller quantity to achieve

$$E(u,v) \le \frac{1}{2}(|\nabla u|_2^2 + |\nabla v|_2^2) < \sigma(a).$$

Hence, we prove $0 < \sup_{\bar{A}_\delta} E < \sigma(a)$. ■

Since the tendency of $\Psi(s)$ described in (22), we can present the tendency of $E(s \star (u,v))$:

$$\begin{aligned} E(s \star (u,v)) \to -\infty \text{ as } |\nabla(s \star u)|_2^2 + |\nabla(s \star v)|_2^2 \to +\infty, \\ E(s \star (u,v)) \to 0^+ \text{ as } |\nabla(s \star u)|_2^2 + |\nabla(s \star v)|_2^2 \to -\infty. \end{aligned}$$

Existence of a critical point of mountain pass-type. Combining Lemma 17 with Lemma 10-11 and denoting $E^0 = \{(u,v) \in S_{a_1} \times S_{a_1} : E(u,v) < 0\}$, we consider a minimax class for $\tilde{E}(s,(u,v)) := E(s \star (u,v)) : \mathbb{R} \times (H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)) \mapsto \mathbb{R}$:

$$\bar{\sigma}(a) := \inf_{\gamma\in\Gamma} \max_{(s,(u,v))\in\gamma([0,1])} \tilde{E}(s,(u,v)),$$

and

$$\Gamma := \left\{\gamma = (\theta, (\alpha, \beta)) \in C([0,1], \mathbb{R} \times S_r) : \ \gamma(0) \in (0, \bar{A}_c), \ \gamma(1) \in (0, E^0)\right\}.$$

Let $(u, v) \in S_r$, there exists $s_0 \ll -1, s_1 \gg 1$, such that

$$\gamma_{(u,v)} : \tau \in [0,1] \mapsto (0, ((1-\tau)s_0 + \tau s_1) \star (u, v)) \in \mathbb{R} \times S_r, \tag{23}$$

is a path in Γ(the continuity follows from (2.5)), then $\sigma(a)$ is a real number.

Claim 1: For every $\gamma = (\theta(\tau_\gamma), (\alpha(\tau_\gamma), \beta(\tau_\gamma))) \in \Gamma$, there exists

$$\tau_\gamma \in (0,1), \ s.t. \ \theta(\tau_\gamma) \star (\alpha(\tau_\gamma), \beta(\tau_\gamma)) \in \mathcal{P}, \ \text{that is } P(\tau_\gamma) = P(\theta(\tau_\gamma) \star (\alpha(\tau_\gamma), \beta(\tau_\gamma))) = 0 \in \mathbb{R}. \tag{24}$$

Proof. Since $s_0 \ll -1$, i.e. $|\nabla(s \star u)|_2^2 + |\nabla(s \star v)|_2^2 < \delta$, and by Lemma 17, we have $P(0) = P_{\theta(0)\star(\alpha(0),\beta(0))} > 0$. By Lemma 15, $P(1) = P_{\theta(1)\star(\alpha(1),\beta(1))} < 0$ if $t_{(\alpha(1),\beta(1))} < 0$. Since the map is of class C^1, (24) can be established.

It is necessary to prove $t_{(\alpha(1),\beta(1))} < 0$. We know that $\Psi_{(\alpha(1),\beta(1))}(s) > 0$ for $s \in (-\infty, t_{(\alpha(1),\beta(1))}]$, and $\Psi_{(\alpha(1),\beta(1))}(0) = E(\alpha(1), \beta(1)) < 0 < \Psi_{(\alpha(1),\beta(1))}(s)$, according to the decreasing property of $\Psi(s)$ in $(t_{(u,v)}, +\infty)$, we have $t_{(\alpha(1),\beta(1))} < 0$. ■

Claim 2: Denoting $S_r = \{(u, v) \in S_a : \ u, v \text{ is radial}\}$, we have

$$\bar{\sigma}(a) = \inf_{(u,v)\in\mathcal{P}\cap S_r} E(u, v). \tag{25}$$

Proof. On the one hand, for $(u, v) \in \mathcal{P} \cap S_r$,

$$\max_{\gamma([0,1])} \tilde{E} \geq \tilde{E}(\gamma(\tau_\gamma)) = E(\theta(\tau_\gamma) \star (\alpha(\tau_\gamma), \beta(\tau_\gamma))) \geq \inf_{\mathcal{P}\cap S_r} E(u, v) \Rightarrow \bar{\sigma}(a) \geq \inf_{\mathcal{P}\cap S_r} E(u, v).$$

On the other hand, for $(u, v) \in \mathcal{P} \cap S_r$, and let $\gamma_{(u,v)}$ defined in **Claim 1**. It result that

$$E(u, v) = \max_{\gamma_{(u,v)}([0,1])} \tilde{E} \geq \bar{\sigma}(a),$$

then (25) holds. ■

Combining with Lemma 17, we infer that

$$\bar{\sigma}(a) = \inf_{(u,v)\in\mathcal{P}\cap S_r} E(u, v) > 0 \geq \sup_{(\bar{A}_c\cup E^0)\cap S_r} E = \sup_{(0,\bar{A}_\delta)\cup(0,E^0)\cap S_r} \tilde{E}. \tag{26}$$

Using the concepts in section 5 of [9], we can notarize $\{\gamma([0,1]) : \gamma \in \Gamma\}$ is a homotopy stable family of compact subsets of $\mathbb{R} \times S_r$, its extend closed boundary is $(0, \bar{A}_c) \cup (0, E^0)$, and the super-level $\{\tilde{E} \geq \bar{\sigma}(a)\}$ respects to the location of duel set. Then after satisfying the assumptions in Theorem 5.2 of [9], we obtain the existence of a radial Palais-Smale sequence (u_n, v_n) for $E\big|_{\mathcal{P}\cap S_r}$, and hence a radial nonnegative Palais-Smale sequence for $E\big|_{S_a}$.

Claim 3: The radial nonnegative Palais-Smale sequence (u_n, v_n) converges to (u, v) in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. **Proof.** We only describe the sketch. We can obtain the boundedness of (u_n, v_n) in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ as Lemma 14. There exist

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N), \ (u_n, v_n) \to (u, v) \text{ in } L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), \ p \in (2, 2^*),$$

$$(u_n, v_n) \to (u, v) \text{ a.e. in } \mathbb{R}^N.$$

Thus, we can obtain $\int_{\mathbb{R}^N} u_n^2 v_n dx = \int_{\mathbb{R}^N} u^2 v dx + o_n(1)$. Next, from $P(u_n, v_n) = o_n(1)$ and $P(u, v) = 0$, we have

$$\lambda_n a^2 = -\frac{1}{5}[|\nabla u_n|_2^2 + |\nabla v_n|_2^2] = -\frac{1}{5}[|\nabla u|_2^2 + |\nabla v|_2^2] + o_n(1) = \lambda |u|_2^2 + o_n(1),$$

where $\lambda_n \to \lambda$ in $\mathbb{R}$. So, we have $(u_n, v_n) \to (u, v)$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. ■

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