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Evolutionary Dynamics of Donation Game on Two-Layer Network with Different Degree Distributions

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Abstract: In this paper, the donation game is researched under a two-layer network model. Each layer of the network follows a different degree distribution. Individuals play games with each other in the interaction layer network and evolve strategies in the dispersal layer network. This study defines a parameter on the initial strategy configuration and then investigates the effect of small changes in the initial strategy configuration on the tendency of populations to cooperate. The results show that, as the parameter increases, the population become more likely to cooperate. Furthermore, the influence of the parameter on the cooperative behavior of populations is significant when networks with heterogeneous degree distributions are used as interaction layers.

Keywords: Donation game; Small-word network; Initial strategy configuration; Cooperative behavior

1 Introduction

Evolutionary game theory is an effective mathematical tool for modeling and studying cooperative behavior among unrelated individuals, which is puzzling [1–5]. Nodes and connected edges in complex networks can describe the spatial structure of a population effectively [6–8].

As the most common game model, the Prisoner's Dilemma game(PDG) is two-person game with two strategies, i.e. cooperate and to defect. Cooperation is hard to exist in PDG [9, 10]. Donation game (DG) is a special case of PDG. In DG, the cooperators bring benefits b to other players while paying costs c, and $0 < c < b$ [11]. Betrayers pay no costs and brings no benefits to other players. There has been much work on DG. Ohtsuki et al. found that natural selection favors cooperation if the benefit-cost ratio $\frac{b}{c}$ exceeds the average number of neighbors [7]. Ohtsuki et al. creat a two-layer network with interaction layer and dispersal layer to study spatial DG, they found that the higher the overlap of these two layers, the more favorable the group cooperation [12, 13]. Allen et al. provided a solution for the critical threshold of the benefit-cost ratio $\left(\frac{b}{c}\right)^*$ under weak selection applicable to any network, by calculating the merging time of random walking, and evaluated the propensity of different structural groups to cooperate [11]. Then they further studied the evolutionary game on the isotherm diagram [18]. McAvoy et al. obtained weakly selected perturbation expansions with fixed probabilities for mutants from arbitrary initial configurations of mutants and resident types, and the results apply to a large class of stochastic evolutionary models in which size and spatial structure are arbitrary (but fixed) [14]. The actual large network often presents the "small world" characteristics, which can portray the true state of the population very well [15–17].

The individual whose strategy choice under the initial strategy configuration of the group is different from all other individuals is called the initially mutated individual [14]. In previous studies, the selection of the initially mutated individuals is often random [11]. However, does the choice for initially mutated individuals affect cooperative behavior of the population? In order to analyze the effect of the choice of initially mutated individuals on the cooperative behavior of the population under weak selection, we first defines the parameter d with respect to initially mutated individuals. Then we consider a model of two-layer networks to describe the spatial structure of DG. The two layers of the network in the model have different degree distributions. Specifically, one layer is the RR network, and the other layer is the Small-world network constructed based on the RR network [15]. By using the method of calculating the coalescence times of random walks, we get the threshold of benefit-cost ratio $\left(\frac{b}{c}\right)^*$ corresponding to different d theoretically, and verify it by numerical simulation [11].

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2 Model and results

The process of the spatial donation game is represented as the model shown in Figure 1(a). The population structure is represented as a two-layer network, the interaction layer G_I and the dispersal layer G_R [12, 13, 22]. Each layer is a connected undirected weighted network. Each layer of the network has N nodes, and nodes with the same number in both layers of the network represent same individual. Edges in G_I represent interaction events between two individuals, and edges in G_R represent strategy dispersal events between two individuals. To simplify the situation, the weight of each edge is specified as 1. Individuals have two strategy types, cooperation and betrayal, abbreviated as C and D, respectively.

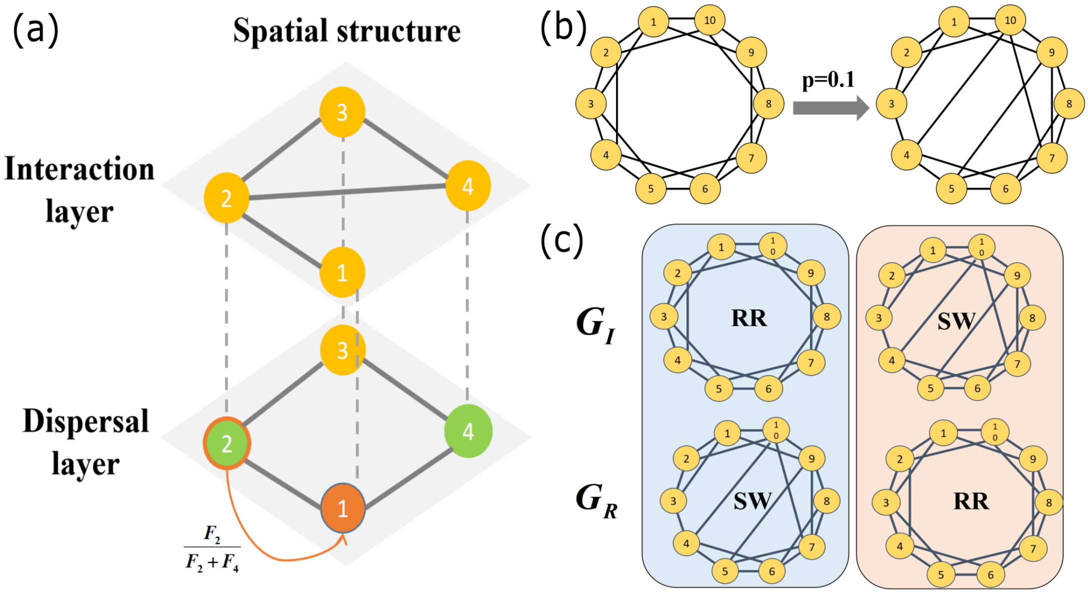

Figure 1: (a) Schematic diagram of a two-layer dependent network model of the donation game. The interaction layer(G_I) and dispersal layer(G_R) regulate the spatial structure of the game. Nodes with the same ID represent the same individual. In each time step, one random individual(orange node) updates strategy, and takes the strategy of its neighbor with probability proportional to the fitness of the neighbor. (b) Schematic diagram of generate a small-world network, with $p = 0.1$. (c) Schematic diagram of creating the model of DG. G_I and G_R have different degree distributions.

In each step of the evolutionary game, payoff of the two-body determined by the payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} b-c & -c \\ b & c \end{pmatrix} \end{array}$$

where b represents the benefit that a cooperator provides to one of its neighbors in G_I in each round of the game, and c represents the cost paid by that cooperator. In each step, the individual i receives cumulative payoff f_i, which determines the fitness $F_i = 1 + \delta f_i$. $\delta > 0$ is the selection intensity [11, 19–21]. Here we consider the weak selection setting, which implies that $\delta \ll 1$. We choose the classical DB update rule (death-birth update) [7]. This means that in each time step, an individual is selected uniformly at random for replacement, and its neighbors produce offspring to replace that individual with probability proportional to its own fitness F_i. The offspring inherit the parents' strategy types. According to the fixation axiom, over time the population will reach the state of all cooperators or all betrayers [14].

Base on the method of calculating the coalescence times of random walks, the benefit-to-cost ratio $(\frac{b}{c})^*$ is calculated, which characterizes the propensity of populations to cooperate. This method is applicable to any two-layer networks and any Initial strategy configuration [11]. Here, the population size N is 100. G_I and G_R are networks with different degree distributions. Small-world network model(SW) is used to construct networks. The parameter K means that each node is connected to K nodes adjacent to its left and right.In this paper, we always take $K = 5$. The parameter p represents the probability of random reconnection of each edge in the network. Figure 1(b) shows $p = 0.1$ as an example. It is an RR network When $p = 0$. As shown in Figure 1(c), we use two different methods to build the final model. The left panel shows the case where G_I is RR and G_R is SW. The left panel shows the opposite scenario. In the two cases above, G_I and G_R has different degree distributions. In Figure 2(a), as p changes, the degree distribution of the network also changes. When $p = 1$, the degree distribution is Poisson distribution.

Parameter d is defined as the sum of the shortest distances from the initially mutated node to the other nodes in the small-world network. Figure 2(b) shows the distribution of d when $p = 0.1$ and $p = 0.2$.

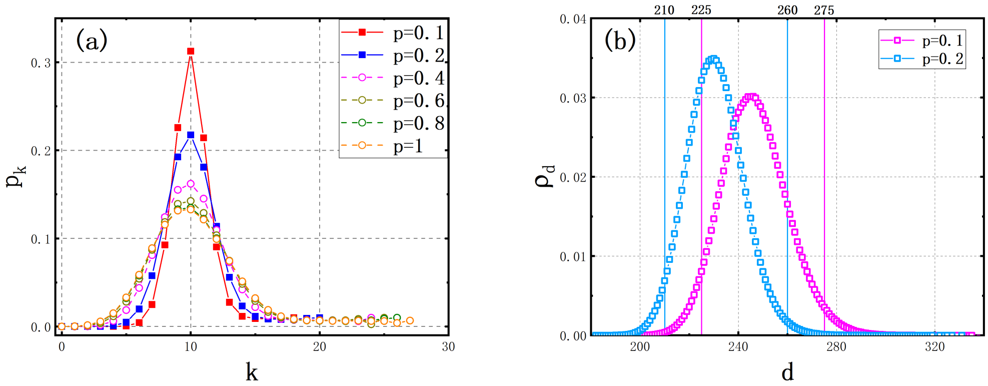

Figure 2: (a) The change of degree distribution with the change of p. The result is the average of 10^4 networks. (b) The distribution of d when $p = 0.1$ and $p = 0.2$. This is the result after averaging 10^5 networks.

2.1 The case when $p = 0.1$ and $p = 0.2$

We choose the case of $p = 0.1$ and $p = 0.2$ for further studies, calculating the benefit-to-cost ratio $(\frac{b}{c})^*$ with different d. The range of values of d is from 225 to 275 when $p = 0.1$ and from 210 to 260 when $p = 0.2$. As found by Figure 2(b), this range covers the vast majority of d corresponding to p.

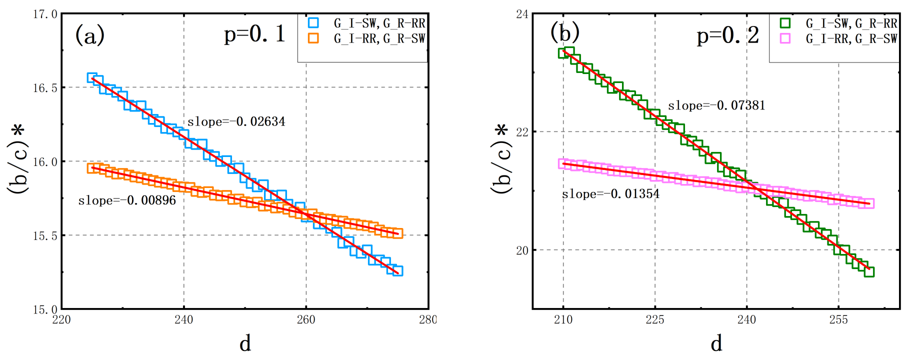

Figure 3: $(\frac{b}{c})^*$ changes with d. (a) The case of $p = 0.1$. (b) The case of $p = 0.2$.

Figure 3(a) shows the theoretical results for the benefit-to-cost ratio $(\frac{b}{c})^*$ with different d when $p = 0.1$. As the increase of d, whether G_I is SW or RR (and G_R is RR or SW), the population is easier to cooperate. Notice that when G_I is SW and G_R is RR, the change of d has a greater effect on $(\frac{b}{c})^*$. Figure 3(b) shows the theoretical results for the benefit-to-cost ratio $(\frac{b}{c})^*$ with different d when $p = 0.2$. The conclusion is the same as in the case of $p = 0.2$. In order to eliminate the effect of randomness, the above graphs are averaged after 2000 repetitions of the calculation.

2.2 Simulation in a single model

When $K = 5$, we randomly generated two networks with $p = 0$ (RR) and $p = 0.1$ (SW). One of the networks is G_I, the other is G_R. Two individuals are randomly selected for simulation to verify the theoretical results.

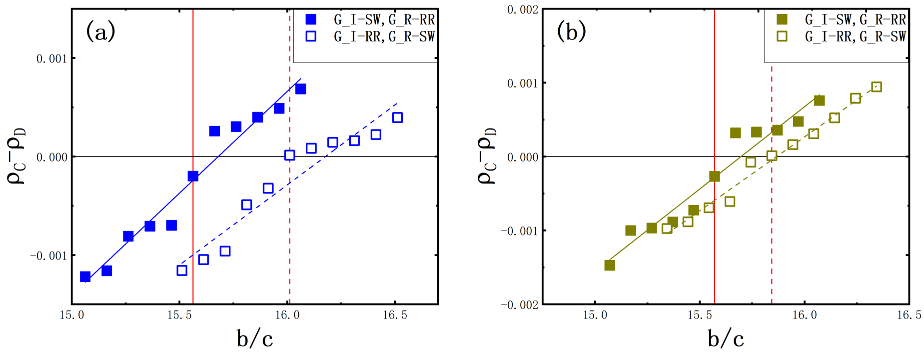

Figure 4: Simulation of two individuals in a model of same SW and RR. $K = 5$ and $p = 0.1$. The simulation was performed with 10^6 repetitions with $\delta = 0.005$. Red lines mark the position of different benefit-to-cost ratios. (a) An individual whose $d = 238$. (b) An individual whose $d = 252$.

The simulation results are shown in Figure 4. Figure 4(a) and Figure 4(b) show the difference between the fixed probability of cooperators and the fixed probability of betrayers, which is $\rho_C - \rho_D$. Red lines mark the position of different $\left(\frac{b}{c}\right)^*$, with $K = 5$ and $p = 0.1$. The solid points represent cases where G_I is SW and G_R is RR. The hollow points represent cases where G_I is RR and G_R is SW. The simulation was performed with 10^6 repetitions with $\delta = 0.005$. The findings demonstrate the numerical simulations agree well with the theoretical results.

3 Conclusions

In this paper, the cooperative behavior of the population under weak selection with different initial strategy configuration is investigated. The parameter d is used to quantify the difference between initially mutated individuals. In the model of the spatial structure of DG, the two layers of the network have different degree distributions. One layer is the RR network, and the other layer is the small-world network constructed based on the RR network. Whether $p = 0.1$ or $p = 0.2$, G_I is SW or G_R is SW, results show that as d increases, the average of $(\frac{b}{c})^*$ decreases. This indicates that population are more inclined to cooperate when d is larger. In addition, when G_I is SW and G_R is RR, the change of d has a greater effect on $(\frac{b}{c})^*$.

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