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Distribution Properties of the Weighted Evolving Network Including Activity and Inactivity

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Abstract: In real life, there are many complex network models, such as global transportation network, metabolic network and social relationship network. It is well known that not all receptors are susceptible during transmission. And various iterative methods have been widely used to construct complex networks. Inspired by the idea and the evolving network, one weighted evolving network is introduced by using the modified corona product. The weighted evolving network focuses on the weight distribution, the statistical feature and the iterative process. To elaborate on this model, we first distinguish the active and inactive meanings. Inactive node, created in the last iteration, will stop iterating, active node otherwise. Meanwhile in each step the number of generating new nodes depends on the active degree of active nodes. Then we derive analytically relevant properties of this weighted evolving network, including the degree distribution and the weight distribution.

Keywords: Modified corona product; Weighted evolving network; Active and inactive; Degree distribution; Weight distribution

1 Introduction

Complex networks play more and more important role in real world [1, 2]. Many researchers were devoted to describing and understanding the characteristic of complex network in the last two decades [3]. And many previous studies show that complex networks exhibit the small-world properties [1] or scale-free properties [4] in real-life networks, including World Wide Web [5], Internet [6], the protein interaction networks [7], the collaboration networks [8] and the transportation networks [9], etc.

Many researchers use stochastic algorithms to build networks [10, 11]. A very typical and representative example was published by Erdös and Rényi in 1960, called ER random graph [12]. As a transition from a completely regular network to a completely random network, Watts and Strogatz put forward an interesting WS model [1] by randomly reconnecting the edges. And then, Newman and Watts proposed NW world model [13] by randomly adding edges. On the one hand, the degree distribution of the network can be approximately represented by Poisson distribution which is a common feature of ER random graph and WS world model. On the other hand, the degree distribution of many actual networks follows a power-law distribution. For example, the scale-free network model proposed by Barabási and Albert [4, 14]. For the better description of the complicated network environment in the real world, realistic models can be studied by using graph products, including corona product [15, 16], hierarchical product [17–19] and Kronecker product [20–23]. And, many studies [24, 25] have extended the binary graph products to weighted graph products. Qi et. al[26] defined an extended corona product for weighted graph and explored the corresponding properties.

Rumours are an important form of social communications, and their spreading plays a significant role in a variety of human affairs. An standard model of rumour spreading, called DK model, was introduced many years ago by Daley and Kendall [27, 28]. In the DK model a closed and homogeneously mixed population is subdivided into three groups, those who are ignorant of the rumour, those who have heard it and actively spread it, and those who have heard the rumour but have ceased to spread it. In this paper, we divide the population into two groups, those who have heard the rumour and

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actively spread it, those who have heard the rumour and ceased to spread it. In the construction of the existed weighted network, many iterations are related to the degree of nodes in the network, not all degrees work, however. We consider the rumour dissemination network over the Internet. The more well-known a website is, the more likely it is to spread the rumour.

In this paper, inactive node, which is created in the last iteration and will stop iterating, is introduced. Inactive node can be used to refer to people who have heard the rumour and ceased to spread it. Consequently, active node, active edge, active degree, inactive edge and inactive degree are defined. Then, a weighted evolving network is introduced using the modified corona product in Sec.2. This model is different from the previous model because the weighted evolving network combines the heterogeneity weight and the statistical evolution. The total number of degrees, nodes, edges and weights are obtained in Sec.3. In Sec.4, we derive analytically relevant properties of this weighted evolving network, including the degree distribution and the weight distribution. In Sec.5, we give some conclusion.

2 Preliminaries

Let $G(V, E, w)$ be a simple connected weighted evolving network. Let $|V|$ and $|E|$ denote the number of nodes and edges of G, respectively. The weight of nodes i and j is denoted by w_{ij}. The neighbor set of node i is denoted by $\Gamma(i)$. The strength of node i is defined as $s_i = \sum_j w_{ij}$.

In the construction of the existed weighted network, many iterative processes are related to the degree of nodes. In this paper, a weighted evolving network including activity and inactivity , which is constructed by the active degree of active nodes of graph, is introduced as follows.

Definition 2.1 The active and inactive meanings of node, edge and degree.

1. The set of nodes of weighted evolving network is classified as two subsets. One subset is the set of inactive nodes, another subset is the set of active nodes. Inactive node means that the new node has no offspring at the next iteration, active node otherwise.
2. The edge e_{ij} linking one inactive node is called inactive edge, active edge otherwise.
3. The degree of an inactive node is inactive degree. The degree of one active node i includes inactive and active degree. The inactive degree of active node i is the number of inactive edges $(e_{ij})_{j\in\Gamma(i)}$, denoted by d_i^{IA}. The active degree of active node i is the number of active edges $(e_{ij})_{j\in\Gamma(i)}$, denoted by d_i^{A}.

Let H_1, H_2 and H_3 be three weighted networks. Let p be a positive real number, satisfying $0 < p < 1$. The active degree of node i in H_1 is labeled as d_i^A, d_i^{IA} otherwise. Meanwhile, all the nodes in H_2 are inactive nodes and all the nodes in H_3 are active nodes.

Definition 2.2 The modified corona product of three weighted networks, denoted by $H_1 \circledcirc (H_2 \cdot H_3)$, is constructed as follows.

1. For each active node i in H_1 with active degree d_i^A, take pd_i^A copies of H_2 and all the nodes of these copies link to node i with an unit weight edge.
2. Meanwhile, for the active node i take $(1-p)d_i^A$ copies of H_3 and all the nodes on these copies link to node i with an unit weight edge. The weight w of each edge in H_1 becomes rw ($r \geq 1$ is a integer).

Then, we can construct the weighted evolving network G_n.

Definition 2.3 Let G be the initial graph. Let K_1 be an isolated inactive node and K_2 be an isolated active node. Then the weighted evolving network G_n ($n \geq 0$) is constructed as follows.

1. For $n = 0$, $G_0 = G$.
2. $G_{n-1} \circledcirc (K_1 \cdot K_2)$ that is G_n constructed by $G_{n-1}, K_1 and K_2)$ applying the modified corona product.

3 The total number of degrees, nodes, edges and weights

By Definition 2.3, each iteration will create active nodes and inactive nodes, and the number of active and inactive nodes depend on active degrees of the last iteration. Let V_n, E_n and W_n denote the set of nodes, edges and weights in G_n,

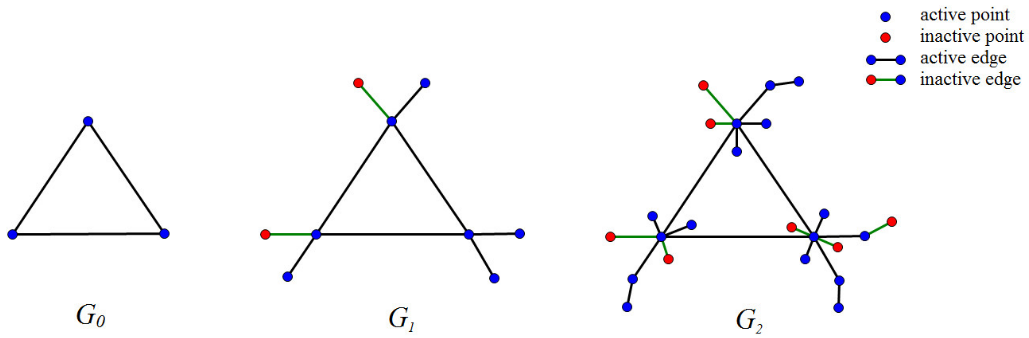

Figure 1: Illustrate of the construction for G_0, G_1 and G_2 with the probability $p = \frac{1}{2}$.

respectively. We use $D^A(G_n)$ and $D^{IA}(G_n)$ denote the total active degree and inactive degree of G_n, respectively. Then $D^A(G_n) = \sum_{i \in V_n} d_i^A$ and $D^{IA}(G_n) = \sum_{i \in V_n} d_i^{IA}$. The total degree of G_n equals to the sum of $D^A(G_n)$ and $D^{IA}(G_n)$, denoted by $D(G_n)$. That is,

$$D(G_n) = D^A(G_n) + D^{IA}(G_n) = \sum_{i \in V_n} d_i^A + \sum_{i \in V_n} d_i^{IA}.$$

Proposition 3.1. The total degree, the total number of nodes, edges and the total weights of edges of the weighted evolving network G_n are as follows.

$$D(G_n) = \frac{(3-2p)^n - 1}{(1-p)} D^A(G_0) + D(G_0), \tag{1}$$

$$|E_n| = \frac{(3-2p)^n - 1}{2(1-p)} D^A(G_0) + |E_0|, \tag{2}$$

$$|V_n| = \frac{(3-2p)^n - 1}{2(1-p)} D^A(G_0) + |V_0|, \tag{3}$$

$$W_n = r^n W_0 + D^A(G_0) \frac{(3-2p)^n - r^n}{3-2p-r}. \tag{4}$$

Proof. First, we introduce some notations.

1. Let $n_{v^A}(n)$ and $n_{v^{IA}}(n)$ denote the number of active and inactive nodes generated at nth iteration, respectively.
2. Let $n_{e^A}(n)$ and $n_{e^{IA}}(n)$ denote the number of active and inactive edges generated at nth iteration, respectively.

It is obvious that $n_{v^A}(n) = n_{e^A}(n)$ and $n_{v^{IA}}(n) = n_{e^{IA}}(n)$ at nth iteration. By the construction of network, we have $(n \geq 2)$

$$\begin{aligned} D^A(G_n) &= 2(1-p)D^A(G_{n-1}) + D^A(G_{n-1}) \\ &= (3-2p)D^A(G_{n-1}) \\ &= (3-2p)^n D^A(G_0). \end{aligned} \tag{5}$$

The two terms on the right side of the first line of $Eq.(5)$ can be explained as follows.

1. The first term represents the active degree generated at nth iteration.
2. The second term represents the total active degree of G_{n-1}.

Then,

$$\begin{aligned} D^{IA}(G_n) &= 2pD^A(G_{n-1}) + D^{IA}(G_{n-1}) \\ &= 2pD^A(G_{n-1}) + 2pD^A(G_{n-2}) + \cdots\cdots + 2pD^A(G_0) + D^{IA}(G_0) \\ &= 2p[(3-2p)^{n-1}D^A(G_0) + (3-2p)^{n-2}D^A(G_0) + \cdots\cdots \\ &\quad +(3-2p)^0 D^A(G_0)] + D^{IA}(G_0) \\ &= \frac{p(3-2p)^n - p}{(1-p)} D^A(G_0) + D^{IA}(G_0). \end{aligned} \tag{6}$$

Analogously, $Eq.(6)$ can be explained as follows.

1. The first term represents the inactive degree generated at nth iteration.
2. The second term represents the total inactive degree of G_{n-1}.

Then we can calculate the total degree of G_n.

$$\begin{aligned} D(G_n) &= D^A(G_n) + D^{IA}(G_n) \\ &= (3-2p)^n D^A(G_0) + 2p\frac{(3-2p)^n - 1}{2(1-p)} D^A(G_0) + D^{IA}(G_0) \\ &= \frac{(3-2p)^n - 1}{1-p} D^A(G_0) + D(G_0). \end{aligned}$$

By the definition of active degree and $Eq.(5)$, we can obtain the same result of $D(G_n)$ as follows.

$$\begin{aligned} D(G_n) &= 2G^A(G_{n-1}) + D(G_{n-1}) \\ &= 2[G^A(G_{n-1}) + G^A(G_{n-2}) + \cdots\cdots + G^A(G_0)] + D(G_0) \\ &= \frac{(3-2p)^n - 1}{1-p} D^A(G_0) + D(G_0). \end{aligned} \tag{7}$$

The two terms on the right side of the first line of $Eq.(7)$ can be explained as follows.

1. The first term represents the degree generated at nth iteration.
2. The second term represents the total degree of G_{n-1}.

By $Eq.(7)$, we can calculate the total number of edges as follows,

$$|E_n| = \frac{D(G_n)}{2} = \frac{(3-2p)^n - 1}{2(1-p)} D^A(G_0) + \frac{D(G_0)}{2}. \tag{8}$$

Then, by $Eq.(5)$, we obtain

$$\begin{aligned} |V_n| &= G^A(G_{n-1}) + |V_{n-1}| \\ &= G^A(G_{n-1}) + G^A(G_{n-2}) + \cdots\cdots + G^A(G_0) + |V_0| \\ &= \frac{(3-2p)^n - 1}{2(1-p)} D^A(G_0) + |V_0|. \end{aligned} \tag{9}$$

Next, we can calculate the number of active nodes and edges and the number of inactive nodes and edges, which generated at nth iteration.

By $Eq.(5)$, we know

$$n_{v^A}(n) = n_{e^A}(n) = (1-p)D^A(G_{n-1}) = (1-p)(3-2p)^{n-1}D^A(G_0), \tag{10}$$

and

$$n_{v^{IA}}(n) = n_{e^{IA}}(n) = pD^A(G_{n-1}) = p(3-2p)^{n-1}D^A(G_0). \tag{11}$$

Let $n_v(n)$ and $n_e(n)$ denote the number of nodes and edges generate at nth iteration, respectively. We can obtain

$$n_v(n) = n_e(n) = (3-2p)^{n-1} D^A(G_0). \tag{12}$$

Then, we calculate the total weight of all edges in G_n, which described as follows.

$$\begin{aligned} W_n &= D^A(G_{n-1}) + rW_{n-1} \\ &= r^n W_0 + r^{n-1} D^A(G_0) + r^{n-2} D^A(G_1) + \cdots\cdots + rD^A(G_{n-2}) + D^A(G_{n-1}) \\ &= r^n W_0 + \frac{(3-2p)^n - r^n}{3-2p-r} D^A(G_0).\ \Box \end{aligned} \tag{13}$$

Next, we will discuss the active and inactive degree of one active node in G_n. We know that each iteration will produce new nodes and new edges.

In G_n, all the same properties nodes, which simultaneously emerging, have identical degree and strength. First we introduce some notations.

1. Let $k_{v^A}^A(n_i, n)$ and $k_{v^A}^{IA}(n_i, n)$ denote the active and inactive degree of an active node in G_n, respectively, which was generated at $n_i th$ iteration, then $k_{v^A}^A(n_i, n_i) = 1$ and $k_{v^A}^{IA}(n_i, n_i) = 0$.

2. Let $k_{v^A}(n_i, n)$ and $k_{v^{IA}}(n_i, n)$ denote the degree of an active or inactive node in G_n, respectively, which was generated at $n_i th$ iteration, then $k_{v^{IA}}(n_i, n) = 1$ and $k_{v^A}(n_i, n) = k_{v^A}^A(n_i, n) + k_{v^A}^{IA}(n_i, n)$.

3. Let $s_{v^A}(n_i, n)$ denote the strength of an active node i in G_n, which was generated at the $n_i th$ iteration, then $s_{v^A}(n_i, n_i) = 1$.

4. Let $s_{v^{IA}}(n_i, n)$ denote the strength of an inactive node i in G_n which was generated at $n_i th$ iteration, then $s_{v^{IA}}(n_i, n) = 1$.

5. Let $w_{e^A}(n_i, n)$ denote the weight of an active edge in G_n, which was generated at the $n_i th$ iteration.

6. Let $w_{e^{IA}}(n_i, n)$ denote the weight of an inactive edge in G_n, which was generated at the $n_i th$ iteration. Then, we know $w_{e^A}(n_i, n) = w_{e^{IA}}(n_i, n) = 1$.

By construction of G_n,

$$\begin{aligned} k_{v^A}^A(n_i, n) &= k_{v^A}^A(n_i, n-1)(1-p) + k_{v^A}^A(n_i, n-1) \\ &= (2-p)^{n-n_i}. \end{aligned} \tag{14}$$

The two terms on the right sides of the first line of $Eq.(14)$ can be explained as follows.

1. The first term represents the active degree generation at n iteration.

2. The second term represents the active degree of active node i in G_{n-1}.

By $Eq.(14)$, we have

$$\begin{aligned} k_{v^A}^{IA}(n_i, n) &= k_{v^A}^A(n_{i,n-1})p + k_{v^A}^{IA}(n_i, n-1) \\ &= \frac{p(2-p)^{n-n_i} - p}{1-p}. \end{aligned} \tag{15}$$

Analogous, $Eq.(15)$ can be explained as $Eq.(14)$.

Then, we can obtain $k_{v^A}(n_i, n)$ by $Eq.(14)$ and $Eq.(15)$,

$$k_{v^A}(n_i, n) = \frac{(2-p)^{n-n_i} - p}{1-p}. \tag{16}$$

By construction of G_n, we have

$$\begin{aligned} s_{v^A}(n_i,n) &= rs_{v^A}(n_i,n-1)+k^A_{v^A}(n_i,n-1) \\ &= k^A_{v^A}(n_i,n-1)+rk^A_{v^A}(n_i,n-2)+\cdots+r^{n-n_i-1}k^A_{v^A}(n_i,n-1) \\ &\quad +r^{n-n_i}s_{v^A}(n_i,n_i) \\ &= (2-p)^{n-1-n_i}+r(2-p)^{n-2-n_i}+r^{n-n_i-1}+r^{n-n^i} \\ &= \frac{r^{n-n_i}-(2-p)^{n-n_i}}{r-(2-p)}+r^{n-n_i}, \end{aligned} \tag{17}$$

and

$$s_{v^{IA}}(n_i,n)=rs_{v^{IA}}(n_i,n-1)=r^{n-n_i}s_{v^{IA}}(n_i,n_i)=r^{n-n_i}. \tag{18}$$

we can obtain,

$$\begin{aligned} w_{e^A}(n_i,n) &= rw_{e^A}(n_i,n) \\ &= r^{n-n_i}. \end{aligned}$$

Then we obtain

$$w_{e^{IA}}(n_i,n)=r^{n-n_i}.$$

Let $w_e(n_i,n)$ denotes the weight of a edge in G_n, which was generated at n_ith iteration. Then we obtain

$$w_e(n_i,n)=r^{n-n_i}. \tag{19}$$

Next, we will calculate the degree distribution and the weight distribution of G_n.

4 Degree distribution and weight distribution

The degree distribution $P(k)$ of the weighted evolving network is the probability that a randomly chosen node has degree k. And we can use the cumulative degree distribution $P_{cum}(k)$ instead of degree distribution when a weighted network has a discrete sequence of node degree. This is the probability that a node has degree greater than or equal to k. This paper uses the following expression

$$P_{cum}(k)=\sum_{k'=k}^{\infty}P(k'). \tag{20}$$

For a network with a power-law degree distribution $P(k)\sim k^{-\gamma}$, their cumulative degree distribution is also power-law obeying $P_{cum}\sim k^{-(\gamma-1)}$.

Proposition 4.1. The degree distribution of the weighted evolving network G_n follows a power-law $P(k)\sim k^{-\gamma_k}$ with the exponent $\gamma_k=1+\frac{\ln(3-2p)}{\ln(2-p)}$.

Proof. We know the degrees of all inactive nodes are 1 because they doesn't create any new node in the next iteration. And by $Eq.(16)$, we know the degree of an active node i in G_n, which was generated at the n_ith iteration, is

$$k_{v^A}(n_i,n)=\frac{(2-p)^{n-n_i}-p}{1-p}.$$

From the above equation, we can obtain

$$n_i=n-\frac{\ln((1-p)k+p)}{\ln(2-p)}. \tag{21}$$

Thus, the cumulative degree distribution of G_n can be represented as

$$\begin{aligned} P_{cum}(k) &= \sum_{\mu\le n_i}\frac{n_{v^A}(\mu)}{N_n} \\ &= \frac{n_{v^A}(0)+D^A(G_0)(1-p)\frac{(3-2p)^{n_i}-1}{2-2p}}{N_0+D^A(G_0)\frac{(3-2p)^n-1}{2-2p}} \quad (k>1). \end{aligned} \tag{22}$$

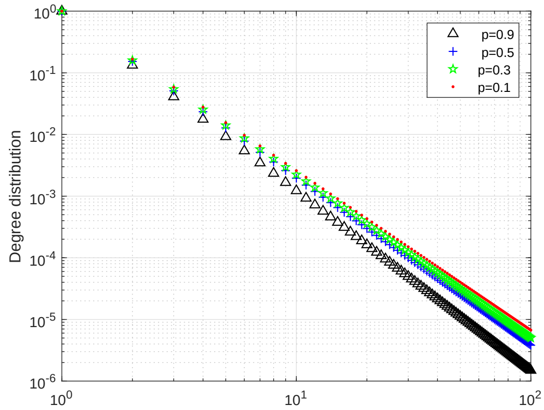

Figure 2: The degree distribution of G_n in log-log coordinate of $P(k) \sim k^{-1-\frac{\ln(3-2p)}{\ln(2-p)}}$ with probabilities $p = 0.9, 0.5, 0.3, 0.1$.

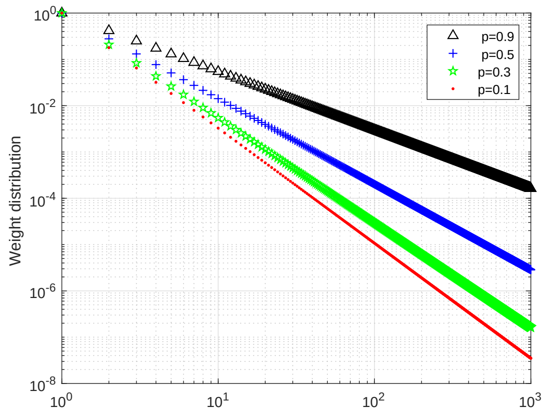

Figure 3: The weight distribution of G_n in log-log coordinate of $P(w) \sim w^{-1-\frac{\ln(3-2p)}{\ln r}}$ with probabilities $p = 0.9, 0.5, 0.3, 0.1$ and weight coefficient $r = 2$.

Substituting $Eq.(21)$ into $Eq.(22)$, we obtain

$$P_{cum}(k) = \frac{n_{v^A}(0) + D^A(G_0)(1-p)\frac{(3-2p)^n[(1-p)k+p]^{-\frac{\ln(3-2p)}{\ln(2-p)}}-1}{2-2p}}{N_0 + D^A(G_0)\frac{(3-2p)^n-1}{2-2p}} \quad (k > 1).$$

For $n \to \infty$, we have

$$P_{cum}(k) \sim (1-p)[(1-p)k+p]^{-\frac{\ln(3-2p)}{\ln(2-p)}}.$$

Therefore, the degree of nodes in the weighted evolving network G_n obeys a power-law form with exponent $\gamma_k = 1 + \frac{\ln(3-2p)}{\ln(2-p)}$. □

Proposition 4.2. The weight distribution of the graph G_n follows a power-law $P(w) \sim w^{-\gamma_w}$ with the exponent $\gamma_w = 1 + \frac{\ln(3-2p)}{\ln r}$.

Proof. We know that the weight of all simultaneously emerging edges is same both active edge and inactive edge. By $Eq.(19)$, we know

$$w_e(n_i, n) = r^{n-n_i}.$$

Form this equation, we obtain

$$n_i = n - \frac{\ln w_e(n_i, n)}{\ln r}. \tag{23}$$

Thus, the cumulative weight distribution of G_n can be represented as

$$\begin{aligned} P_{cum}(w) &= \sum_{\mu \le n_i} \frac{n_e(\mu)}{E_n} \\ &= \frac{D^A(G_0)[(3-2p)^{n_i} - 1] + (1-p)D(G_0)}{D^A(G_0)[(3-2p)^n - 1] + (1-p)D(G_0)}. \end{aligned} \tag{24}$$

Substituting $Eq.(23)$ into $Eq.(24)$, we obtain

$$P_{cum}(w) = \frac{D^A(G_0)[(3-2p)^n w^{-\frac{\ln(3-2p)}{\ln r}} - 1] + (1-p)D(G_0)}{D^A(G_0)[(3-2p)^n - 1] + (1-p)D(G_0)}.$$

Therefore, for large n, we have

$$P_{cum}(w) \sim w^{-\frac{\ln(3-2p)}{\ln r}}. \tag{25}$$

This implies that the weight distribution of G_n exhibits a power-law form $\gamma_w = 1 + \frac{\ln(3-2p)}{\ln r}$. $\square$

5 Conclusions

We consider the information dissemination network over the Internet. The more famous a website is, the more likely it is to spread information. But not everyone who receives information will pass it on. Inspired by the this idea, we constructed the weighted evolving network including activity and inactivity. The obtained results show that the degree distribution and the weight distribution of the weighted evolving network follow the power-law distribution. Thus, the model can well simulate the properties of real weighted network.

Acknowledgments

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