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A Class of Deviation Theorems for Weighted Sums of Continuous Random Variables

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Abstract: Consider a sequence of continuous random variables, we introduce the notion of log-likelyhood ratio of stochastic sequences, as a measure of dissimilarity between their joint distribution and the product of their marginals. We obtain some deviation theorems for the Jamison type weighted sums of continuous random variables by using the generating function method.

Keywords: Continuous random variables; generating function; weighted sums; deviation theorems

1 Introduction

In recent years, some results have been made in the field of deviation of random variables (see[1]-[4]) and their references). Liu (1997,1999) and Yang (2006) discussed the strong deviation theorems for discrete random variables by using generating function method. The present paper focuses on the study of the strong deviation theorems for weighted sums of continuous random variables.

Throughout this paper, let $\{X_n, n \geq 1\}$ be a continuous random variables defined on probability space $(\Omega, \mathcal{F}, P)$ taking valued in R^+.

Assume that the joint density functions of $\{X_n, n \geq 1\}$, are

$$p_n\left(x_1, \cdots, x_n\right), \ \ x_k \geq 0, \ \ 1 \leq k \leq n, \ \ n = 1, 2, ... \tag{1}$$

and $p_i\left(x_i\right), i = 1, 2, ...$are their marginal functions, let

$$q_n\left(x_1, \cdots, x_n\right) = \prod_{i=1}^{n} p_i\left(x_i\right), n \geq 1 \tag{2}$$

and let $F_k\left(x\right) = P\left(X_k \leq x\right)$, resp.

2 Definitions and Main Results

Definition 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with joint density functions* (1)*, q_n defined as* (2)*. Let A_n be a sequence of nonnegative real numbers, and $A_n \uparrow \infty$. The likelihood ratio and the limit log-likelihood ratio as follows:*

$$R_n(\omega) = \frac{q_n(X_1, X_2, \cdots, X_n)}{p_n(X_1, X_2, \cdots, X_n)} \tag{3}$$

and

$$\gamma(\omega) = -\liminf_{n} A_n^{-1} \log R_n(\omega). \tag{4}$$

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Obviously, $R_n(\omega) \equiv 1, a.e.$, if and only if $\{X_n, n \geq 1\}$ independent. And it will be shown that $\gamma(\omega) \geq 0, a.e.$ in any case.

In view of the above discussion of the limit log-likelihood ratio, it is natural to think of $\gamma(\omega)$ as a measure how far (the random deviation) of $\{X_n, n \geq 1\}$ is from being independent, how dependent they are. The smaller $\gamma(\omega)$ is, the smaller the deviation is.

Definition 2.2 *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables, and is said to be stochastically dominated by a nonnegative random variable X (we write $\{X_n, n \geq 1\} \prec X$) if there exists a constant C such that*

$$\sup_{n\geq 1} P\left\{X_n > x\right\} \leq CP\left\{X > x\right\}. \tag{5}$$

Definition 2.3 *Let $\{a_k, k \geq 1\}$ be a sequence of nonnegative real numbers, let $A_n = \sum_{k=1}^{n} a_k$,*

$$T_n(\omega) = A_n^{-1} \sum_{k=1}^{n} a_k [X_k - EX_k]. \tag{6}$$

The $T_n(\omega)$ is called Jamison type weighted sums.

Definition 2.4 *Let $\{X_n, n \geq 1\}$ be a continuous random variables defined on probability space $(\Omega, \mathcal{F}, P)$ taking valued in R^+. The density functions of $\{X_n, n \geq 1\}$ are $p_k(x), 1 \leq k \leq n, n = 1, 2, ...$ the moment transformation and tail probability moment transformation as follows:*

$$M_k(s) = \int_0^\infty s^{a_k x} p_k(x) dx, \tag{7}$$

and

$$W_k(s) = \int_0^\infty s^{a_k x} \int_x^\infty p_k(t) dt dx. \tag{8}$$

Next, some theorems will be presented prior to established the main results.

Theorem 2.1 *Let $\{X_n, n \geq 1\}$,X, $\{a_k, k \geq 1\}$, $\gamma(\omega)$, A_n, $M_k(s)$, and $W_k(s)$ be defined as before, $0 < a = \inf\{a_k\}$, $\sup\{a_k\} < b < +\infty$, and $\{X_n, n \geq 1\} \prec X$ and $EX < \infty$. Let*

$$\eta = \{\omega : \gamma(\omega) < +\infty\}, \tag{9}$$

then

$$\liminf_n T_n(\omega) \geq \alpha(\gamma(\omega)),\ a.e.,\ \omega \in \eta, \tag{10}$$

where

$$\alpha(x) = \sup\{\varphi(s, x), 0 < s < 1\},\ 0 \leq x < \infty, \tag{11}$$

$$\varphi(s, x) = \liminf_n A_n^{-1} \sum_{k=1}^{n} a_k [W_k(s) - EX_k] + \frac{x}{\ln s},\ \ 0 < s < 1,\ 0 \leq x < \infty. \tag{12}$$

and

$$\alpha(x) \leq 0,\ \lim_{x \to 0^+} \alpha(x) = \alpha(0) = 0. \tag{13}$$

Theorem 2.2 *Under the condition of Theorem 1, if there exist a $s_0 > 0$, $W_b(s_0) < +\infty$, then*

$$\limsup_n T_n(\omega) \le \beta(\gamma(\omega)), \ \ a.s., \ \omega \in H. \tag{14}$$

where

$$\begin{aligned} &\beta(x) = \inf\{\phi(s,x), 1 < s < s_0\}, \ 0 \le x < \infty, \\ &\phi(s,x) = \limsup_n {A_n}^{-1} \sum_{k=1}^{n} a_k [W_k(s) - EX_k] + \frac{x}{\ln s}, \ 1 < s < s_0, \ 0 \le x < \infty, \end{aligned} \tag{15}$$

and

$$\beta(x) \ge 0, \ \ \lim_{x \to 0^+} \beta(x) = \beta(0) = 0. \tag{16}$$

The proof of the above theorems will be given in Section 3.

3 The Proofs

Before providing the proofs of the main results in Section 2, we begin with some lemmas.

Lemma 3.1 *Let $g_n(x_1, \cdots, x_n)$ be a probability density function defined on probability space $(R^+)^n$, $p_n(x_1, \cdots, x_n)$ defined as before. Let σ_n be a sequence of nonnegative real numbers, and $\sigma_n \uparrow \infty$. Let*

$$Q_n(\omega) = \frac{g_n(X_1, X_2, \cdots, X_n)}{p_n(X_1, X_2, \cdots, X_n)} \tag{17}$$

then

$$\limsup_n \sigma_n \log Q_n(\omega) \le 0, \quad a.s. \tag{18}$$

Proof By Doob (1953),$\{Q_n, n \ge 1\}$ is a nonnegative supermartingale and $EQ_n \le 1$, by the Doobs martingale convergence theorem, there is a random variable $Q_\infty(\omega) < \infty$, such that $Q_n(\omega) \to Q_\infty(\omega)$ $a.e.$ $(n \to \infty)$. Hence, (18) follows.

Lemma 3.2 *Let $M_k(s)$ and $W_k(s)$ be defined by* (7) *and* (8) *resp., then*

$$\frac{M_k(s) - 1}{a_k \ln s} = W_k(s), \ \ W_k(1) = EX_k. \tag{19}$$

Proof

$$\begin{aligned} &W_k(s) \int_0^\infty s^{a_k x} \int_x^\infty p_k(t) dt dx \\ =& \int_0^\infty p_k(t) \int_0^t s^{a_k x} dx dt \\ =& \frac{1}{a_k \ln s} \left[\int_0^\infty p_k(t) s^{a_k t} dt - \int_0^\infty p_k(t) dt \right] \\ =& \frac{1}{a_k \ln s} \left[M_k(s) - 1 \right]. \end{aligned}$$

When $s = 1$, obviously $W_k(1) = EX_k$.

Lemma 3.3 *Let $\{X_n, n \geq 1\}$, X, $W_k(s)$ and a_k be defined as before,* $0 < a = \inf\{a_k\}$, $\sup\{a_k\} < b < \infty$. $\{X_n, n \geq 1\} \prec X$ *and* $EX < \infty$. *Let*

$$\varphi(s,x) = \liminf_n A_n^{-1} \sum_{k=1}^{n} a_k[W_k(s) - EX_k] + \frac{x}{\ln s},\ \ s > 0,\ 0 \leq x < \infty. \tag{20}$$

$$\phi(s,x) = \limsup_n {A_n}^{-1} \sum_{k=1}^{n} a_k[W_k(s) - EX_k] + \frac{x}{\ln s}, s > 0,\ \ 0 \leq x < \infty. \tag{21}$$

Then:
(i) $\varphi(s,x)(0 \leq x < \infty)$ *as a function of s, is continuous on* $(0,1)$.
(ii) Let $W_b(s) = \int_0^\infty s^{bx} \int_x^\infty p(t)dtdx$, *and there exists a* $s_0 > 1$, $W_b(s_0) < +\infty$. *Then* $\phi(s,x)(0 \leq x < \infty)$ *as a function of s, is continuous on* $(1, s_0)$.

Proof (i) Let $h(s) = \liminf_n A_n^{-1} \sum_{k=1}^{n} a_k[W_k(s) - EX_k]$. So to prove $\varphi(s,x)(0 \leq x < \infty)$ as a function of s, is continuous on $(0,1)$, it suffices to prove $h(s)$ is continuous on $(0,1)$.

If $0 < s < s+t < 1$, (we can let $t \leq s/2$), for $\forall \varepsilon > 0$, since $W_k(s) = \int_0^\infty \int_x^\infty p(t)dtdx = EX < \infty$, so $\exists M$, such that $\int_M^\infty \int_x^\infty p(t)dtdx < \frac{\varepsilon}{3}$. By $\{X_n, n \geq 1\} \prec X$, we have

$$W_k(s) \leq \int_0^M s^{a_k x} \int_x^\infty p_k(t)dtdx + \frac{\varepsilon}{3},$$

$$W_k(s+t) \leq \int_0^M (s+t)^{a_k x} \int_x^\infty p_k(t)dtdx + \frac{\varepsilon}{3},$$

and

$$\begin{aligned} &|W_k(s+t) - W_k(s)| \\ \leq &\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int_0^M ((s+t)^{a_k x} - s^{a_k x}) \int_x^\infty p_k(t)dtdx \\ \leq &\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int_0^M s^{a_k x}((1+\frac{t}{s})^{bx} - 1) \int_x^\infty p_k(t)dtdx. \end{aligned}$$

Let $f(x,y) = (1+y)^{bx}$, $f(x,y)$ is continuous on $[0,M] \times [0, \frac{1}{2}]$, hence $f(x,y)$ is uniformly continuous on $[0,M] \times [0, \frac{1}{2}]$. Then there exists $\delta > 0$, when $0 < t < \delta$,

$$\begin{aligned} &\left|(1+\frac{t}{s})^{bx} - 1\right| < \frac{\varepsilon}{3EX}, \\ &\int_0^M s^{ax}((1+\frac{t}{s})^{bx} - 1) \int_x^\infty p_k(t)dtdx < \frac{\varepsilon}{3}. \end{aligned}$$

According to the discussion above, we have

$$\begin{aligned} &|h(s+t) - h(s)| \\ =&\left|\liminf_n A_n^{-1} \sum_{k=1}^{n} a_k[W_k(s+t) - EX_k] - \liminf_n A_n^{-1} \sum_{k=1}^{n} a_k[W_k(s) - EX_k]\right| \\ =&\left|\liminf_n A_n^{-1} \sum_{k=1}^{n} a_k[W_k(s+t) - W_k(1)] + \limsup_n A_n^{-1} \sum_{k=1}^{n} a_k[W_k(1) - W_k(s)]\right| \\ \leq&\limsup_n A_n^{-1} \sum_{k=1}^{n} a_k \left|W_k(s+t) - W_k(s)\right| \\ <&\varepsilon. \end{aligned} \tag{22}$$

It means that $h(s)$ is continuous from the right.

If $0 < s+t < s < 1$, by $\{X_n, n \ge 1\} \prec X$, we have

$$\begin{aligned}
&|h(s) - h(s+t)| \\
=&\left|\liminf_n A_n^{-1}\sum_{k=1}^n a_k[W_k(s) - EX_k] - \liminf_n A_n^{-1}\sum_{k=1}^n a_k[W_k(s+t) - EX_k]\right| \\
=&\left|\liminf_n A_n^{-1}\sum_{k=1}^n a_k[W_k(s) - W_k(1)] + \limsup_n A_n^{-1}\sum_{k=1}^n a_k[W_k(1) - W_k(s+t)]\right| \\
\le&\limsup_n A_n^{-1}\sum_{k=1}^n a_k\,|W_k(s) - W_k(s+t)| \\
=&\limsup_n A_n^{-1}\sum_{k=1}^n a_k\int_0^{+\infty} s^{a_k x}(1-(1+\frac{t}{s})^{a_k x})\int_x^{+\infty} p_k(t)dtdx \\
\le&C\limsup_n A_n^{-1}\sum_{k=1}^n a_k\int_0^{+\infty} s^{ax}(1-(1+\frac{t}{s})^{bx})\int_x^{+\infty} p(t)dtdx \\
\le&C\int_0^{+\infty} s^{ax}(1-(1+\frac{t}{s})^{bx})\int_x^{+\infty} p(t)dtdx.
\end{aligned} \tag{23}$$

Since $1+\frac{t}{s} < 1$, we have $\int_0^{+\infty}(1+\frac{t}{s})dx$ is uniformly convergent. Then there exists $\delta > 0$, when $-\delta < t < 0$,$|h(s) - h(s+t)| < \varepsilon$. It means that $h(s)$ is continuous from left. So $h(s)$ is continuous on $(0,1)$, then (i) is proved.

(ii) Similarly, we can prove (ii).

Now, we present the proof of Theorem 2.1 as follows.

Proof of Theorem 2.1 Let s be an arbitrary positive real number, set

$$p_k(s; x_k) = \frac{s^{a_k x_k} p_k(x_k)}{M_k(s)}, \tag{24}$$

$$q_n(s; x_1, \cdots, x_n) = \prod_{k=1}^n p_k(s; x_k). \tag{25}$$

It is easy to see that $q_n(s; x_1, \cdots, x_n)$ is a product density function, let

$$R_n^*(\omega) = \frac{q_n(s; X_1, X_2, \cdots, X_n)}{p_n(X_1, X_2, \cdots, X_n)}. \tag{26}$$

By Lemma 1, there exists a set $A(s) \in F$, such that $P(A(s)) = 1$, and we have

$$\limsup_n A_n^{-1}\log R_n^*(\omega) \le 0, \quad \omega \in A(s). \tag{27}$$

That is,

$$\begin{aligned}
&\limsup_n A_n^{-1}[\log R_n(\omega) + \ln s\sum_{k=1}^n a_k x_k] \\
\le&\limsup_n A_n^{-1}\sum_{k=1}^n \ln M_k(s), \ \ \omega \in A(s).
\end{aligned} \tag{28}$$

By (4), (28), we have

$$\begin{aligned}
&\limsup_n A_n^{-1}\ln s\sum_{k=1}^n a_k x_k \\
\le&\limsup_n A_n^{-1}\sum_{k=1}^n \ln M_k(s) + \gamma(\omega), \ \ \omega \in A(s).
\end{aligned} \tag{29}$$

Set $0 < s < 1$, dividing two sides of (29) by $\ln s$, we have

$$\begin{aligned}&\limsup_n A_n^{-1}\sum_{k=1}^n a_k[X_k - EX_k]\\ \geq&\liminf_n A_n^{-1}\sum_{k=1}^n [\frac{\ln M_k(s)}{\ln s} - a_k EX_k] + \frac{\gamma(\omega)}{\ln s},\ \ \omega \in A\left(s\right).\end{aligned} \tag{30}$$

By (19) and the inequality $\ln x \leq x - 1$, we have

$$\begin{aligned}&\liminf_n A_n^{-1}\sum_{k=1}^n a_k[X_k - EX_k]\\ \geq&\liminf_n A_n^{-1}\sum_{k=1}^n [\frac{M_k(s)-1}{\ln s} - a_k EX_k] + \frac{\gamma(\omega)}{\ln s}\\ =&\liminf_n A_n^{-1}\sum_{k=1}^n a_k[W_k(s) - EX_k] + \frac{\gamma(\omega)}{\ln s}.\end{aligned} \tag{31}$$

Let Q be a rational numbers set in $(0, 1)$, and $A^* = \bigcap\limits_{s\in Q} A\left(s\right)$, then $P\left(A^*\right) = 1$. By (31) and we have for all $s \in Q$,

$$\begin{aligned}&\liminf_n A_n^{-1}\sum_{k=1}^n a_k[X_k - EX_k]\\ \geq&\liminf_n A_n^{-1}\sum_{k=1}^n a_k[W_k(s) - EX_k] + \frac{\gamma(\omega)}{\ln s}.\ \ \omega \in A^*.\end{aligned} \tag{32}$$

Let $s = 1$ in (28), we have

$$\gamma\left(\omega\right) \geq 0,\ \ \omega \in A\left(1\right). \tag{33}$$

By Lemma 3, since for any given $x \in (0, \infty)$, $\varphi(s, x)$ is a continuous function with respective to $s \in (0, 1)$, we have for each $s \in (0, 1)$, there exist a sequence $s_n \in Q$, such that

$$\lim_{n\to\infty} \varphi\left(s_n, \gamma(\omega)\right) = \alpha\left(\gamma(\omega)\right),\ \ \omega \in A^* \cap A\left(1\right) \cap H. \tag{34}$$

By (32) and (34), we have

$$\liminf_n \sum_{k=1}^n a_{n_k}[X_k - EX_k] \geq \alpha\left(\gamma(\omega)\right),\ \ \omega \in A^* \cap A\left(1\right) \cap H. \tag{35}$$

Since $P(A^* \cap A\left(1\right)) = 1$, (10) follows from (35).
Obviously, $\alpha(x) \leq 0$. Next, we will show that $\lim\limits_{x\to 0^+} \alpha(x) = \alpha(0) = 0$. Since $\{X_n, n \geq 1\} \prec X$, we have

$$\begin{aligned}&{A_n}^{-1}\sum_{k=1}^n a_k[W_k(s) - EX_k]\\ =&{A_n}^{-1}\sum_{k=1}^n a_k[\int_0^\infty (s^{a_k x}-1)\int_x^\infty p_k(t)dtdx], (s^{a_k x}-1<0)\\ \geq&C{A_n}^{-1}\sum_{k=1}^n a_k[\int_0^\infty (s^{a_k x}-1)\int_x^\infty p(t)dtdx]\\ \geq&C{A_n}^{-1}\sum_{k=1}^n a_k\left[W_b(s) - W_b(1)\right]\\ =&C[W_b(s) - W_b(1)].\end{aligned} \tag{36}$$

If $0 < x < 1$, by (36), we have

$$\begin{aligned}\alpha(x) &\geq \varphi(1-\sqrt{x}, x) \\ &\geq C[W_b(1-\sqrt{x}) - W_b(1)] + \frac{x}{\ln(1-\sqrt{x})}.\end{aligned} \tag{37}$$

If $x = 0$, we have

$$\alpha(0) \geq C[W_b(1-\frac{1}{n}) - W_b(1)], \ \ n \geq 1. \tag{38}$$

Noticing that $\alpha(x) \leq 0$, (13) follows from (37) and (38). Thus Theorem 2.1 is proved.□

Proof of Theorem 2.2 Set $1 < s < s_0$, dividing two sides of (29) by $\ln s$, we have

$$\begin{aligned}&\limsup_n A_n^{-1}\sum_{k=1}^{n} a_k[X_k - EX_k] \\ &\leq \limsup_n A_n^{-1}\sum_{k=1}^{n}[\frac{\ln M_k(s)}{\ln s} - a_k EX_k] + \frac{\gamma(\omega)}{\ln s}, \ \ \omega \in A\,(s)\,.\end{aligned} \tag{39}$$

The rest of the proof is similar to Theorem 2.1.

This completes the proof of Theorem 2.2. □

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