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A Property of the Solution Near the Travelling Wave of the Second-order Camassa-Holm Equation

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Abstract: This paper studies the property of the solution near the travelling wave $\mathcal{Q}$ of the second-order Camassa-Holm equation in the space H^2. The solution near the travelling wave is decomposed into $\lambda^{\frac{1}{2}}(t)u(x+x(t)) = \mathcal{Q}(x) + \varepsilon(t,x)$ by pseudo-conformal transformation. It is demonstrated that ε can be controlled by a fast-decaying exponential function when the initial value of ε is controlled by a fast-decaying exponential function. The solution of the second-order Camassa-Holm equation is equivalent to the travelling wave $\mathcal{Q}$ (up to scaling and translation) is proved when the solution exists globally.

Keywords: second-order Camassa-Holm equation; travelling wave; pseudo-conformal transformation; the property of the solution

1 Introduction

In [1–3], Merle and Martel studied the Cauchy problem of the critical generalized KdV equation in the space L^2:

$$\begin{cases} u_t + (u_{xx} + u^5)_x = 0, (t,x) \in [0,T) \times R, \\ u(0,x) = u_0(x), x \in R. \end{cases} \tag{1}$$

Let $\mathcal{Q}^*$ be the travelling wave of the KdV equation, λ_0 be the scaling invariant and x_0 be the translation invariant. Assuming that there exists a sequence u_n of solutions which satisfies H^1 bound, L^2 compact and $|u_n(0) - \mathcal{Q}^*|_{H^1} \to 0, n \to +\infty$, Martel and Merle proved that

$$u(t,x) = \lambda_0^{\frac{1}{2}} \mathcal{Q}^*(\lambda_0(x - x(x_0)) - \lambda_0^3 t).$$

On the other hand, in 2003, Constantin and Kolev [4] studied the infinite-dimensional Lie group of all smooth orientation-preserving diffeomorphisms of the circle with a Riemannian structure, they obtained a geodesic equation:

$$u_t = A_K^{-1} C_k(u) - uu_x, k \in N, \tag{2}$$

where

$$u = u(t,x), (t,x) \in R^+ \times R,$$

$$A_k(u) = \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u,$$

and

$$C_k(u) = -uA_k(u_x) + A_k(uu_x) - 2u_x A_k(u).$$

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We denote the convolution by *. The operator A_k^{-1} is given by the following convolution form:

$$A_k^{-1}(f)(x) = P_k * f = \int_R P_k(x-y) * f(y)dy, x \in R.$$

$\hat{P}_k$ is the Fourier transform of P_k:

$$\hat{P}_k = \frac{1}{1+\xi^2+\cdots+\xi^{2k}}, \forall \xi \in R.$$

When $k \geq 2$, equation (2) is the higher-order Camassa-Holm equation. When $k = 2$, equation (2) is the second-order Camassa-Holm equation:

$$u_t - u_{txx} + u_{txxxx} = 2u_x u_{xx} - 3uu_x + uu_{xxx} - 2u_x u_{xxxx} - uu_{xxxxx}, t > 0, x \in R. \tag{3}$$

Equivalently, equation (3) can be rewritten as the following form:

$$u_t = P_2 * (2u_x u_{xx} - 3uu_x + uu_{xxx} - 2u_x u_{xxxx} - uu_{xxxxx}), \tag{4}$$

where $P_2(x) = \frac{\sqrt{3}}{3} e^{-\frac{\sqrt{3}}{2}|x|} \sin(\frac{|x|}{2} + \frac{\pi}{6}), x \in R.$

In 2011, Tian, Zhang and Xia [5] proved that if $u_0 \in H^s(R), s > \frac{9}{2}$, the strong solution of equation (3) exists globally. They also obtained a conservation law of the second-order Camassa-Holm equation:

$$\int_R u^2(t,x) + u_x^2(t,x) + u_{xx}^2(t,x)dx = \int_R u_0^2(x) + {u_0}_x^2(x) + {u_0}_{xx}^2(x)dx. \tag{5}$$

In 2017, Ding [6] studied the travelling wave solutions of the higher-order Camassa-Holm equation. By the travelling wave transformation $u(t,x) = \mathcal{Q}(x-ct)$, the travelling wave solution of the second-order Camassa-Holm equation is as follows:

$$\mathcal{Q}_A(x-ct) = \begin{cases} Ae^{-\frac{\sqrt{3}}{2}(x-ct)} \left(\cos\frac{x-ct}{2} + \sqrt{3}\sin\frac{x-ct}{2} \right), x \geq ct, \\ Ae^{\frac{\sqrt{3}}{2}(x-ct)} \left(\cos\frac{x-ct}{2} - \sqrt{3}\sin\frac{x-ct}{2} \right), x < ct, \end{cases} \tag{6}$$

where $A > 0$ is the amplitude, $c > 0$ is the velocity and A is related to c.

There are also many papers about higher-order Camassa-Holm equation, such as [7–9].

In contrast to KdV equation, because of the loss of regularity of soliton, the methods of a series of works from Merle and Martel aren't available to the Camassa-Holm equation. In 2018, Molinet [10] proved a Liouville property for uniformly almost localized H^1-global solutions of the Camassa-Holm equation with a momentum density that is a non-negative finite measure. It should be noticed that the travelling wave of the second-order Camassa-Holm equation belongs to space H^3, but the soliton of the Camassa-Holm equation belongs to space H^1. Inspired by a series of works from Merle and Martel in [1–3] and Molinet's research in [10], we study the property of the solution of the second-order Camassa-Holm equation. The Cauchy problem of the second-order Camassa-Holm equation is as follows:

$$\begin{cases} u_t - u_{txx} + u_{txxxx} = 2u_x u_{xx} - 3uu_x + uu_{xxx} - 2u_x u_{xxxx} - uu_{xxxxx}, t > 0, x \in (-\infty, 0) \cup (0, +\infty), \\ u(0,x) = u_0(x). \end{cases} \tag{7}$$

By pseudo-conformal transformation, we decompose the solution of the second-order Camassa-Holm equation near the travelling wave $\mathcal{Q}$ into

$$\lambda^{\frac{1}{2}}(t)u(t, x + x(t)) = \mathcal{Q} + \varepsilon(t,x).$$

Let $a_2 = \sup_{t \geq 0} \|\varepsilon\|_{H^2}$ and let α be a positive constant. We define a neighborhood with $\mathcal{Q}$ as the center and α as the radius:

$$U_\alpha = \{u \in H^2(R); \inf_{r \in R} \|u(\cdot) - \mathcal{Q}(\cdot + r)\|_{H^2} \leq \alpha\}.$$

Let $\lambda(t)$ be the scaling invariant and $x(t)$ be the translation invariant, then the solution of equation (3) satisfies the following invariances:
(a) Translation invariance: if $u(t,x)$ is a solution of equation (3), then $u(t, x + x(t))$ is also a solution of equation (3);

(b) Scaling invariance: if $u(t,x)$ is a solution of equation (3), then $\lambda^{\frac{1}{2}}(t)u(\lambda^{\frac{1}{2}}(t)t,x)$ is also a solution of equation (3). From the proposition 3.1 in [11], there exist scaling invariant $\lambda'(t)$ and translation invariant $x'(t)$, such that

$$(\varepsilon(t,x),\mathcal{Q}_x)=(\varepsilon(t,x)),\mathcal{Q}_{xx})=0.$$

We define that the $\lambda(t)$ and $x(t)$ in the following paper are geometric parameters that satisfy the above orthogonality.

We have the following result:

Theorem 1 *Suppose*

$$u_0\in U_{\alpha_0}=\{u\in H^2(R);\inf_{r\in R}\|u(\cdot)-\mathcal{Q}(\cdot+r)\|_{H^2}\le\alpha_0\},$$

and assume that there exists a constant $C_1>0$, such that

$$\|u_0\|_{H^2}\ge C_1.$$

Let $\varepsilon_0=u_0-\mathcal{Q}$, which satisfies

$$|\varepsilon_0|\le e^{-\frac{\sqrt{3}}{2}|x|},x\ge 0.$$

If the solution of the Cauchy problem (7) exists globally in the space H^2, there exist scaling invariant $\lambda_0(t)\in C^1$ and translation invariant $x_0(t)\in C^1$, such that

$$u(t,x)=\lambda_0^{-\frac{1}{2}}(t)\mathcal{Q}\left(x-x_0(t)\right).$$

2 Properties of ε

For convenience, we take $A=1$ and denote $\zeta=x-ct$ in (6), so

$$\mathcal{Q}=\mathcal{Q}_1=2e^{-\frac{\sqrt{3}}{2}|\zeta|}\sin(\frac{|\zeta|}{2}+\frac{\pi}{6}),$$

and

$$\mathcal{Q}_{\zeta\zeta\zeta}=\begin{cases}4\sqrt{3}e^{-\frac{\sqrt{3}}{2}\zeta}\cos(\frac{\zeta}{2}+\frac{\pi}{6}),\zeta\ge 0\\ -4\sqrt{3}e^{\frac{\sqrt{3}}{2}\zeta}\cos(\frac{\zeta}{2}-\frac{\pi}{6}),\zeta<0\end{cases},$$

where the third order derivative of $\mathcal{Q}$ is not continuous at the point $\zeta=0$.

Property 2 *(O_1)Boundedness of λ: Suppose that there exists a constant $C_1>0$, such that*

$$\|u_0\|_{H^2}\ge C_1.$$

There exist λ_1, $\lambda_2>0$, such that

$$\forall t\ge 0,\lambda_1\le\lambda(t)\le\lambda_2.$$

(O_2)[11]Uniform boundedness of $\|\varepsilon\|_{H^2}$: If $\|u_0-\mathcal{Q}\|_{H^2}\le\alpha_0$, there exists a constant $C_3>0$, such that

$$\|\varepsilon\|_{H^2}\le C_3\alpha_0.$$

Proof. (O_1) Due to $C_1\le\|u_0\|_{H^2}$ and the conservation law:

$$\int_R u^2+u_y^2+u_{yy}^2dy=\int_R u_0^2+{u_0}_y^2+{u_0}_{yy}^2dy,$$

one gets

$$C_1\le\|u\|_{H^2}.$$

Since $\|u_0-\mathcal{Q}\|_{H^2}\le\alpha_0$, we have

$$\|u_0\|_{H^2}\le\|\mathcal{Q}\|_{H^2}+\alpha_0.$$

Therefore,

$$C_1 \leq \|u\|_{H^2} \leq C_2, \tag{8}$$

where $C_2 = \|\mathcal{Q}\|_{H^2} + \alpha_0 = \alpha + (2\sqrt{3})^{\frac{1}{2}}$.
Since $\lambda^{\frac{1}{2}}(t)u(\lambda^{\frac{1}{2}}(t)t, y)$ is a solution of equation (4), one gets

$$C_1 \leq \|\lambda^{\frac{1}{2}}(t)u(\lambda^{\frac{1}{2}}(t)t, y)\|_{H^2} \leq C_2. \tag{9}$$

So there exist $\lambda_1, \lambda_2 > 0$, such that

$$\forall t \geq 0, \lambda_1 \leq \lambda(t) \leq \lambda_2.$$

(O_2) It is detailed in Lemma 4.3 in [11]. ■

To derive the governing equation of ε:
Setting

$$v(t, y) = \lambda^{\frac{1}{2}}(t)u(t, y + x(t)), \tag{10}$$

one gets

$$\varepsilon(t, y) = v(t, y) - \mathcal{Q}(y) = \lambda^{\frac{1}{2}}(t)u(t, y + x(t)) - \mathcal{Q}(y). \tag{11}$$

We have

$$v_t = \frac{1}{2}\lambda^{-\frac{1}{2}}\lambda_t u + \lambda^{\frac{1}{2}}u_t + \lambda^{\frac{1}{2}}x_t u_y, \tag{12}$$

$$v_y = \lambda^{\frac{1}{2}}u_y, \tag{13}$$

and

$$v_{yy} = \lambda^{\frac{1}{2}}u_{yy}. \tag{14}$$

Applying (12)-(14) to equation (4), one has

$$\lambda^{\frac{1}{2}}v_t - \frac{1}{2}\lambda^{-\frac{1}{2}}\lambda_t v - \lambda^{\frac{1}{2}}x_t v_y = P_2 * (2v_y v_{yyy} - 3vv_y - 2v_y v_{yyyy} - vv_{yyyyy} + vv_{yyy}). \tag{15}$$

Setting

$$s = \int_0^t \frac{dt'}{\lambda^{\frac{1}{2}}(t')} \quad \text{or equivalently} \quad \frac{ds}{dt} = \frac{1}{\lambda^{\frac{1}{2}}(t)}, \tag{16}$$

one has

$$v_s = \frac{1}{2}\frac{\lambda_s}{\lambda}v + x_s v_y + P_2 * (2v_y v_{yyy} - 3vv_y - 2v_y v_{yyyy} - vv_{yyyyy} + vv_{yyy}). \tag{17}$$

Thus, we obtain

$$v_s = \frac{1}{2}\frac{\lambda_s}{\lambda}v + x_s v_y - P_2 * (3vv_y - v_y v_{yy}) + P_{2y} * (\frac{3}{2}v_{yy}^2 - vv_{yy}) - P_{2yy} * (v_y v_{yy}) - P_{2yyy} * (vv_{yy} - v_y v_{yy}). \tag{18}$$

Applying $v(s, y) = \mathcal{Q}(y) + \varepsilon(s, y)$ to (18), we obtain

$$\varepsilon_s - x_s \varepsilon_y = \frac{1}{2}\frac{\lambda_s}{\lambda}\varepsilon + \frac{1}{2}\frac{\lambda_s}{\lambda}\mathcal{Q} + x_s\mathcal{Q}_y + G(\varepsilon), \tag{19}$$

where

$$\begin{aligned} G(\varepsilon) = &\Big[-P_2 * (3\mathcal{Q}\mathcal{Q}_y - \mathcal{Q}_y\mathcal{Q}_{yy}) + P_{2y} * (\frac{3}{2}\mathcal{Q}_{yy}^2 - \mathcal{Q}\mathcal{Q}_{yy}) - P_{2yy} * (\mathcal{Q}_y\mathcal{Q}_{yy}) - P_{2yyy} * (\mathcal{Q}\mathcal{Q}_{yy} - \mathcal{Q}_y\mathcal{Q}_{yy})\Big] \\ &+ \Big[-P_2 * (3\mathcal{Q}\varepsilon_y - \mathcal{Q}_y\varepsilon_{yy}) + P_{2y} * (\frac{3}{2}\mathcal{Q}_{yy}\varepsilon_{yy} - \mathcal{Q}\varepsilon_{yy}) - P_{2yy} * (\mathcal{Q}_y\varepsilon_{yy}) - P_{2yyy} * (\mathcal{Q}\varepsilon_{yy} - \mathcal{Q}_y\varepsilon_{yy})\Big] \\ &+ \Big[-P_2 * (3\varepsilon\mathcal{Q}_y - \varepsilon_y\mathcal{Q}_{yy}) + P_{2y} * (\frac{3}{2}\varepsilon_{yy}\mathcal{Q}_{yy} - \varepsilon\mathcal{Q}_{yy}) - P_{2yy} * (\varepsilon_y\mathcal{Q}_{yy}) - P_{2yyy} * (\varepsilon\mathcal{Q}_{yy} - \varepsilon_y\mathcal{Q}_{yy})\Big] \\ &+ \Big[-P_2 * (3\varepsilon\varepsilon_y - \varepsilon_y\varepsilon_{yy}) + P_{2y} * (\frac{3}{2}\varepsilon_{yy}^2 - \varepsilon\varepsilon_{yy}) - P_{2yy} * (\varepsilon_y\varepsilon_{yy}) - P_{2yyy} * (\varepsilon\varepsilon_{yy} - \varepsilon_y\varepsilon_{yy})\Big]. \end{aligned} \tag{20}$$

Lemma 3 *There exists constants $C_5 > 0$ and $\tau_0 > 0$, such that $|G(\varepsilon)| \le C_5(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau_0|y|}, y \in R$.*

Proof. For convenience, we divide $G(\varepsilon)$ into two parts: $G_1(\varepsilon)$ and $G_2(\varepsilon)$, where

$$\begin{aligned} G_1(\varepsilon) = &\Big[-P_2*(3\mathcal{Q}\mathcal{Q}_y-\mathcal{Q}_y\mathcal{Q}_{yy})+P_{2y}*(\frac{3}{2}\mathcal{Q}_{yy}^2-\mathcal{Q}\mathcal{Q}_{yy})-P_{2yy}*(\mathcal{Q}_y\mathcal{Q}_{yy})-P_{2yyy}*(\mathcal{Q}\mathcal{Q}_{yy}-\mathcal{Q}_y\mathcal{Q}_{yy})\Big]\\ &+\Big[-P_2*(3\mathcal{Q}\varepsilon_y-\mathcal{Q}_y\varepsilon_{yy})+P_{2y}*(\frac{3}{2}\mathcal{Q}_{yy}\varepsilon_{yy}-\mathcal{Q}\varepsilon_{yy})-P_{2yy}*(\mathcal{Q}_y\varepsilon_{yy})-P_{2yyy}*(\mathcal{Q}\varepsilon_{yy}-\mathcal{Q}_y\varepsilon_{yy})\Big]\\ &+\Big[-P_2*(3\varepsilon\mathcal{Q}_y-\varepsilon_y\mathcal{Q}_{yy})+P_{2y}*(\frac{3}{2}\varepsilon_{yy}\mathcal{Q}_{yy}-\varepsilon\mathcal{Q}_{yy})-P_{2yy}*(\varepsilon_y\mathcal{Q}_{yy})-P_{2yyy}*(\varepsilon\mathcal{Q}_{yy}-\varepsilon_y\mathcal{Q}_{yy})\Big], \end{aligned}$$

and

$$G_2(\varepsilon) = -P_2*(3\varepsilon\varepsilon_y-\varepsilon_y\varepsilon_{yy})+P_{2y}*(\frac{3}{2}\varepsilon_{yy}^2-\varepsilon\varepsilon_{yy})-P_{2yy}*(\varepsilon_y\varepsilon_{yy})-P_{2yyy}*(\varepsilon\varepsilon_{yy}-\varepsilon_y\varepsilon_{yy}).$$

To estimate $G_1(\varepsilon)$:
Letting

$$\begin{aligned} F(y)+\Bigg|&\Big[-(3\mathcal{Q}\mathcal{Q}_y-\mathcal{Q}_y\mathcal{Q}_{yy})+(\frac{3}{2}\mathcal{Q}_{yy}^2-\mathcal{Q}\mathcal{Q}_{yy})-(\mathcal{Q}_y\mathcal{Q}_{yy})-(\mathcal{Q}\mathcal{Q}_{yy}-\mathcal{Q}_y\mathcal{Q}_{yy})\Big]\\ &+\Big[-(3\mathcal{Q}\varepsilon_y-\mathcal{Q}_y\varepsilon_{yy})+(\frac{3}{2}\mathcal{Q}_{yy}\varepsilon_{yy}-\mathcal{Q}\varepsilon_{yy})-(\mathcal{Q}_y\varepsilon_{yy})-(\mathcal{Q}\varepsilon_{yy}-\mathcal{Q}_y\varepsilon_{yy})\Big]\\ &+\Big[-(3\varepsilon\mathcal{Q}_y-\varepsilon_y\mathcal{Q}_{yy})+(\frac{3}{2}\varepsilon_{yy}\mathcal{Q}_{yy}-\varepsilon\mathcal{Q}_{yy})-(\varepsilon_y\mathcal{Q}_{yy})-(\varepsilon\mathcal{Q}_{yy}-\varepsilon_y\mathcal{Q}_{yy})\Big]\Bigg|, \end{aligned}$$

we have

$$|G_1(\varepsilon)| \le 2e^{-\frac{\sqrt{3}}{2}|y|}*F(y) = 2\int_{|y|\le|\tau|} e^{-\frac{\sqrt{3}}{2}|y-\tau|}*F(\tau)d\tau + 2\int_{|y|>|\tau|} e^{-\frac{\sqrt{3}}{2}|y-\tau|}*F(\tau)d\tau.$$

Case I: When $|y| \le |\tau|$, there exists a constant $\tau_1 > 0$, such that $|\tau| = (1+\tau_1)|y|$. we have $\tau = (1+\tau_1)y$ or $\tau = -(1+\tau_1)y$.
If $\tau = (1+\tau_1)y$, then

$$\int_{|y|\le|\tau|} e^{-\frac{\sqrt{3}}{2}|y-\tau|}\cdot F(\tau)d\tau = \int_{|y|\le|\tau|} e^{-\frac{\sqrt{3}}{2}|y-(1+\tau_1)y|}\cdot F(\tau)d\tau = e^{-\frac{\sqrt{3}}{2}\tau_1|y|}\int_{|y|\le|\tau|}F(\tau)d\tau.$$

If $\tau = -(1+\tau_1)y$,

$$\int_{|y|\le|\tau|} e^{-\frac{\sqrt{3}}{2}|y-\tau|}\cdot F(\tau)d\tau = \int_{|y|\le|\tau|} e^{-\frac{\sqrt{3}}{2}|y+(1+\tau_1)y|}\cdot F(\tau)d\tau = e^{-\frac{\sqrt{3}}{2}(2+\tau_1)|y|}\int_{|y|\le|\tau|}F(\tau)d\tau.$$

Therefore,

$$\int_{|y|\le|\tau|} e^{-\frac{\sqrt{3}}{2}|y-\tau|}\cdot F(\tau)d\tau \le e^{-\frac{\sqrt{3}}{2}\tau_1|y|}\int_{|y|\le|\tau|}F(\tau)d\tau.$$

Case II: When $|y| > |\tau|$, there exists a constant $\tau_2 > 0$, such that $|\tau| = (1-\tau_2)|y|$.
From triangle inequality, we know

$$\int_{|y|>|\tau|} e^{-\frac{\sqrt{3}}{2}|y-\tau|}\cdot F(\tau)d\tau \le \int_{|y|>|\tau|} e^{-\frac{\sqrt{3}}{2}(|y|-|\tau|)}\cdot F(\tau)d\tau = e^{-\frac{\sqrt{3}}{2}\tau_2|y|}\int_{|y|>|\tau|}F(\tau)d\tau. \tag{21}$$

Combining case I and case II, we obtain

$$\begin{aligned} |G_1(\varepsilon)| &\le 2e^{-\frac{\sqrt{3}}{2}\tau_1|y|}\int_{|y|\le|\tau|}F(\tau)d\tau + 2e^{-\frac{\sqrt{3}}{2}\tau_2|y|}\int_{|y|>|\tau|}F(\tau)d\tau\\ &\le 2(e^{-\frac{\sqrt{3}}{2}\tau_1|y|}+e^{-\frac{\sqrt{3}}{2}\tau_2|y|})\int_R F(\tau)d\tau\\ &\le 2[9\sqrt{3}+(2\sqrt{3})^{\frac{1}{2}}]e^{-\frac{\sqrt{3}}{2}\tau_0|y|}(5\|\varepsilon\|_{L^2}+4\|\varepsilon\|_{H^1}+5\|\varepsilon\|_{H^2}+1)\\ &\le 28[9\sqrt{3}+(2\sqrt{3})^{\frac{1}{2}}](a_2+1)e^{-\frac{\sqrt{3}}{2}\tau_0|y|}\\ &\le C_4(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau_0|y|}, \end{aligned} \tag{22}$$

where $\tau_0 = min\{\tau_1, \tau_2\}$ and $C_4 = 28[9\sqrt{3} + (2\sqrt{3})^{\frac{1}{2}}]$.
To estimate $G_2(\varepsilon)$:

$$|G_2(\varepsilon)| = \left|-P_2 * (3\varepsilon\varepsilon_y - \varepsilon_y\varepsilon_{yy}) + P_{2y} * (\frac{3}{2}\varepsilon_{yy}^2 - \varepsilon\varepsilon_{yy}) - P_{2yy} * (\varepsilon_y\varepsilon_{yy}) - P_{2yyy} * (\varepsilon\varepsilon_{yy} - \varepsilon_y\varepsilon_{yy})\right|$$
$$\le 2e^{-\frac{\sqrt{3}}{2}|y|} * \left|-(3\varepsilon\varepsilon_y - \varepsilon_y\varepsilon_{yy}) + (\frac{3}{2}\varepsilon_{yy}^2 - \varepsilon\varepsilon_{yy}) - (\varepsilon_y\varepsilon_{yy}) - (\varepsilon\varepsilon_{yy} - \varepsilon_y\varepsilon_{yy})\right|.$$

Letting $R(y) = \left|-(3\varepsilon\varepsilon_y - \varepsilon_y\varepsilon_{yy}) + (\frac{3}{2}\varepsilon_{yy}^2 - \varepsilon\varepsilon_{yy}) - (\varepsilon_y\varepsilon_{yy}) - (\varepsilon\varepsilon_{yy} - \varepsilon_y\varepsilon_{yy})\right|$, one obtains

$$|G_2(\varepsilon)| \le 2e^{-\frac{\sqrt{3}}{2}|y|} * R(y) = 2\int_{|y|\le|\tau|} e^{-\frac{\sqrt{3}}{2}|\overline{y}-\tau|} * R(\tau)d\tau + 2\int_{|y|>|\tau|} Ce^{-\frac{\sqrt{3}}{2}|y-\tau|} * R(\tau)d\tau.$$

Similar to $|G_1(\varepsilon)|$, it is clear that

$$\begin{aligned} |G_2(\varepsilon)| &\le 2(e^{-\frac{\sqrt{3}}{2}\tau_1|y|} + e^{-\frac{\sqrt{3}}{2}\tau_2|y|})\int R(\tau) \\ &\le 2e^{-\frac{\sqrt{3}}{2}\tau_0|y|}(\frac{3}{2}\|\varepsilon\|_{H^2}^2 + 3\|\varepsilon\|_{L^2}\|\varepsilon\|_{H^1} + 2\|\varepsilon\|_{L^2}\|\varepsilon\|_{H^2}) \\ &\le 13a_2e^{-\frac{\sqrt{3}}{2}\tau_0|y|}. \end{aligned} \tag{23}$$

In summary, $|G(\varepsilon)| \le C_5(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau_0|y|}$, where $C_5 = C_4 + 13$.
The proof is completed. ■

3 Estimate of ε

Lemma 4 *Let $\varepsilon_0(y) = \varepsilon(0, y)$. If $|\varepsilon_0(y)| < e^{-\frac{\sqrt{3}}{2}|y|}$, there exist $C_{10} > 0$ and $\tau' > 0$, such that*

$$|\varepsilon(s, y)| \le C_{10}(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau' y}(1 + e^{-\frac{\sqrt{3}}{4}s\tau'}), \forall y > 0.$$

Proof. Equation (19) can be rewritten as

$$\varepsilon_s - x_s\varepsilon_y = \frac{1}{2}\frac{\lambda_s}{\lambda}\varepsilon + f_1 + f_2, \tag{24}$$

where $f_1 = \frac{1}{2}\frac{\lambda_s}{\lambda}\mathcal{Q} + x_s\mathcal{Q}_y$ and $f_2 = G(\varepsilon)$.
To remove the term $\frac{1}{2}\frac{\lambda_s}{\lambda}\varepsilon$ in equation (24), we introduce the following transformation:

$$\eta(s, y) = \lambda^{-\frac{1}{2}}(s)\varepsilon(s, y), s \ge 0. \tag{25}$$

Therefore, equation (24) can be rewritten as

$$\eta_s - x_s\eta_y = g_1 + g_2, \tag{26}$$

where

$$g_1 = \frac{1}{2}\frac{\lambda_s}{\lambda}\mathcal{Q} + x_s\mathcal{Q}_y,$$

and

$$\begin{aligned} g_2 = &\left[-P_2 * (3\mathcal{Q}\mathcal{Q}_y - \mathcal{Q}_y\mathcal{Q}_{yy}) + P_{2y} * (\frac{3}{2}\mathcal{Q}_{yy}^2 - \mathcal{Q}\mathcal{Q}_{yy}) - P_{2yy} * (\mathcal{Q}_y\mathcal{Q}_{yy}) - P_{2yyy} * (\mathcal{Q}\mathcal{Q}_{yy} - \mathcal{Q}_y\mathcal{Q}_{yy})\right] \\ &+ \lambda^{\frac{1}{2}}\left[-P_2 * (3\mathcal{Q}\varepsilon_y - \mathcal{Q}_y\varepsilon_{yy}) + P_{2y} * (\frac{3}{2}\mathcal{Q}_{yy}\varepsilon_{yy} - \mathcal{Q}\varepsilon_{yy}) - P_{2yy} * (\mathcal{Q}_y\varepsilon_{yy}) - P_{2yyy} * (\mathcal{Q}\varepsilon_{yy} - \mathcal{Q}_y\varepsilon_{yy})\right] \\ &+ \lambda^{\frac{1}{2}}\left[-P_2 * (3\varepsilon\mathcal{Q}_y - \varepsilon_y\mathcal{Q}_{yy}) + P_{2y} * (\frac{3}{2}\varepsilon_{yy}\mathcal{Q}_{yy} - \varepsilon\mathcal{Q}_{yy}) - P_{2yy} * (\varepsilon_y\mathcal{Q}_{yy}) - P_{2yyy} * (\varepsilon\mathcal{Q}_{yy} - \varepsilon_y\mathcal{Q}_{yy})\right] \\ &+ \lambda\left[-P_2 * (3\varepsilon\varepsilon_y - \varepsilon_y\varepsilon_{yy}) + P_{2y} * (\frac{3}{2}\varepsilon_{yy}^2 - \varepsilon\varepsilon_{yy}) - P_{2yy} * (\varepsilon_y\varepsilon_{yy}) - P_{2yyy} * (\varepsilon\varepsilon_{yy} - \varepsilon_y\varepsilon_{yy})\right]. \end{aligned}$$

Let

$$\overline{\eta}(s,y)=\eta(s,y+x(s)),s\geq 0, \tag{27}$$

then equation (26) can be rewritten as

$$\overline{\eta}_s=\overline{g_1}+\overline{g_2}, \tag{28}$$

where

$$\overline{g_1}=\frac{1}{2}\frac{\lambda_s}{\lambda}\overline{\mathcal{Q}}+x_s\overline{\mathcal{Q}}_y,$$

and

$$\begin{aligned}\overline{g_2}=&\left[-\overline{P_2}*(3\overline{\mathcal{Q}}\overline{\mathcal{Q}}_y-\overline{\mathcal{Q}}_y\overline{\mathcal{Q}}_{yy})+\overline{P_2}_y*(\frac{3}{2}\overline{\mathcal{Q}}_{yy}^2-\overline{\mathcal{Q}}\overline{\mathcal{Q}}_{yy})-\overline{P_2}_{yy}*(\overline{\mathcal{Q}}_y\overline{\mathcal{Q}}_{yy})-\overline{P_2}_{yyy}*(\overline{\mathcal{Q}}\overline{\mathcal{Q}}_{yy}-\overline{\mathcal{Q}}_y\overline{\mathcal{Q}}_{yy})\right]\\&+\lambda^{\frac{1}{2}}\left[-\overline{P_2}*(3\overline{\mathcal{Q}}\overline{\eta}_y-\overline{\mathcal{Q}}_y\overline{\eta}_{yy})+\overline{P_2}_y*(\frac{3}{2}\overline{\mathcal{Q}}_{yy}\overline{\eta}_{yy}-\overline{\mathcal{Q}}\varepsilon_{yy})-\overline{P_2}_{yy}*(\overline{\mathcal{Q}}_y\overline{\eta}_{yy})-\overline{P_2}_{yyy}*(\overline{\mathcal{Q}}\overline{\eta}_{yy}-\overline{\mathcal{Q}}_y\overline{\eta}_{yy})\right]\\&+\lambda^{\frac{1}{2}}\left[-\overline{P_2}*(3\overline{\eta}\overline{\mathcal{Q}}_y-\overline{\eta}_y\overline{\mathcal{Q}}_{yy})+\overline{P_2}_y*(\frac{3}{2}\overline{\eta}_{yy}\overline{\mathcal{Q}}_{yy}-\overline{\eta}\overline{\mathcal{Q}}_{yy})-\overline{P_2}_{yy}*(\overline{\eta}_y\overline{\mathcal{Q}}_{yy})-\overline{P_2}_{yyy}*(\overline{\eta}\overline{\mathcal{Q}}_{yy}-\overline{\eta}_y\overline{\mathcal{Q}}_{yy})\right]\\&+\lambda\left[-\overline{P_2}*(3\overline{\eta}\overline{\eta}_y-\overline{\eta}_y\overline{\eta}_{yy})+\overline{P_2}_y*(\frac{3}{2}\overline{\eta}_{yy}^2-\overline{\eta}\overline{\eta}_{yy})-\overline{P_2}_{yy}*(\overline{\eta}_y\overline{\eta}_{yy})-\overline{P_2}_{yyy}*(\overline{\eta}\overline{\eta}_{yy}-\overline{\eta}_y\overline{\eta}_{yy})\right].\end{aligned}$$

Due to (5.6) in [11], one obtains $|x_s-1|\leq C_6\alpha$, more precisely

$$-C_6\alpha+1\leq x_s\leq C_6\alpha+1.$$

Let $\alpha\leq\frac{1}{2C_6}$, then $x_s\geq\frac{1}{2}$. By integration, we have $x(s)\geq\frac{1}{2}s, s\geq 0$.
Furthermore, one gets

$$\overline{\mathcal{Q}}=\mathcal{Q}(y+x(s))\leq e^{-\frac{\sqrt{3}}{2}|y+x(s)|}\leq e^{-\frac{\sqrt{3}}{2}(y+\frac{1}{2}s)},y>0. \tag{29}$$

From (5.6) in [11] and (29), one obtains

$$|\overline{g_1}|=|\frac{1}{2}\frac{\lambda_s}{\lambda}\overline{\mathcal{Q}}+x_s\overline{\mathcal{Q}}_y|\leq 2C_6(a_2+1)e^{-\frac{\sqrt{3}}{2}(y+\frac{1}{2}s)}. \tag{30}$$

For convenience, we divide $\overline{g_2}$ into two parts: $\overline{g_{21}}$ and $\overline{g_{22}}$, where

$$\begin{aligned}\overline{g_{21}}=&\left[-\overline{P_2}*(3\overline{\mathcal{Q}}\overline{\mathcal{Q}}_y-\overline{\mathcal{Q}}_y\overline{\mathcal{Q}}_{yy})+\overline{P_2}_y*(\frac{3}{2}\overline{\mathcal{Q}}_{yy}^2-\overline{\mathcal{Q}}\overline{\mathcal{Q}}_{yy})-\overline{P_2}_{yy}*(\overline{\mathcal{Q}}_y\overline{\mathcal{Q}}_{yy})-\overline{P_2}_{yyy}*(\overline{\mathcal{Q}}\overline{\mathcal{Q}}_{yy}-\overline{\mathcal{Q}}_y\overline{\mathcal{Q}}_{yy})\right]\\&+\lambda^{\frac{1}{2}}\left[-\overline{P_2}*(3\overline{\mathcal{Q}}\overline{\eta}_y-\overline{\mathcal{Q}}_y\overline{\eta}_{yy})+\overline{P_2}_y*(\frac{3}{2}\overline{\mathcal{Q}}_{yy}\overline{\eta}_{yy}-\overline{\mathcal{Q}}\overline{\eta}_{yy})-\overline{P_2}_{yy}*(\overline{\mathcal{Q}}_y\overline{\eta}_{yy})-\overline{P_2}_{yyy}*(\overline{\mathcal{Q}}\overline{\eta}_{yy}-\overline{\mathcal{Q}}_y\overline{\eta}_{yy})\right]\\&+\lambda^{\frac{1}{2}}\left[-\overline{P_2}*(3\overline{\eta}\overline{\mathcal{Q}}_y-\overline{\eta}_y\overline{\mathcal{Q}}_{yy})+\overline{P_2}_y*(\frac{3}{2}\overline{\eta}_{yy}\overline{\mathcal{Q}}_{yy}-\overline{\eta}\overline{\mathcal{Q}}_{yy})-\overline{P_2}_{yy}*(\overline{\eta}_y\overline{\mathcal{Q}}_{yy})-\overline{P_2}_{yyy}*(\overline{\eta}\overline{\mathcal{Q}}_{yy}-\overline{\eta}_y\overline{\mathcal{Q}}_{yy})\right],\end{aligned}$$

and

$$\overline{g_{22}}=\lambda\left[-\overline{P_2}*(3\overline{\eta}\overline{\eta}_y-\overline{\eta}_y\overline{\eta}_{yy})+\overline{P_2}_y*(\frac{3}{2}\overline{\eta}_{yy}^2-\overline{\eta}\overline{\eta}_{yy})-\overline{P_2}_{yy}*(\overline{\eta}_y\overline{\eta}_{yy})-\overline{P_2}_{yyy}*(\overline{\eta}\overline{\eta}_{yy}-\overline{\eta}_y\overline{\eta}_{yy})\right].$$

Similar to $|G_1(\varepsilon)|$, one obtains

$$\begin{aligned}|\overline{g_{21}}|&\leq 2[9\sqrt{3}\lambda_2^{\frac{1}{2}}+(2\sqrt{3})^{\frac{1}{2}}]e^{-\frac{\sqrt{3}}{2}\tau_0|\overline{y}|}(5\|\overline{\eta}\|_{L^2}+4\|\overline{\eta}\|_{H^1}+5\|\overline{\eta}\|_{H^2}+1)\\&\leq C_7(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau_0(y+\frac{1}{2}t)},\end{aligned} \tag{31}$$

and

$$\begin{aligned}|\overline{g_{22}}|&\leq 2\lambda_2e^{-\frac{\sqrt{3}}{2}\tau_0|\overline{y}|}(\frac{3}{2}\|\overline{\eta}\|_{H^2}^2+3\|\overline{\eta}\|_{L^2}\|\overline{\eta}\|_{H^1}+2\|\overline{\eta}\|_{L^2}\|\overline{\eta}\|_{H^2})\\&\leq C_8a_2e^{-\frac{\sqrt{3}}{2}\tau_0(y+\frac{1}{2}s)},\end{aligned} \tag{32}$$

where $\tau_0 = min\{\tau_1, \tau_2\}$, $C_7 = \max\{28[9\sqrt{3}\lambda_2^{\frac{1}{2}} + (2\sqrt{3})^{\frac{1}{2}}]\lambda_1^{-\frac{1}{2}}, 28[9\sqrt{3}\lambda_2^{\frac{1}{2}} + (2\sqrt{3})^{\frac{1}{2}}]\}$ and $C_8 = 13\lambda_2\lambda_1^{-1}$.
We obtain $|\overline{g_2}| \le |\overline{g_{21}}| + |\overline{g_{22}}| \le C_9(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau_0(y+\frac{1}{2}s)}$, where $C_9 = \max\{2C_7, 2C_8\}$.
Hence

$$|(\overline{\eta})_s| = |\overline{g_1}| + |\overline{g_2}| \le (C_9 + 2C_6)(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau'(y+\frac{1}{2}s)}, \tag{33}$$

where $\tau' = \min\{\tau_0, 1\}$.
By integration, we get

$$|\overline{\eta}(s,y)| \le (C_9 + 2C_6)\frac{2}{\sqrt{3}\tau'}(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau'(y+\frac{1}{2}s)} + |\overline{\eta}(0)|. \tag{34}$$

Due to (25) and (27), one has $\overline{\eta}(s,y) = \lambda^{-\frac{1}{2}}(s)\varepsilon(s, y - x(s))$.
Therefore,

$$|\varepsilon(s,y)| \le (C_9 + 2C_6)\frac{2}{\sqrt{3}\tau'}(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau'(y+\frac{1}{2}s)} + \lambda_2^{-\frac{1}{2}}|\varepsilon_0|.$$

Since $|\varepsilon_0| \le e^{-\frac{\sqrt{3}}{2}y}, y > 0$, we get

$$|\varepsilon(s,y)| \le C_{10}(a_2+1)e^{-\frac{\sqrt{3}}{2}\tau' y}(1 + e^{-\frac{\sqrt{3}}{4}s\tau'}),$$

where $C_{10} = (C_9 + 2C_6)\frac{2}{\sqrt{3}\tau'} + \lambda_2^{-\frac{1}{2}}$.
The proof is completed. ■

4 A property of the solution of the second-order Camassa-Holm equation

To prove theorem 1:
Proof. Assume that $\varepsilon(s,y) \not\equiv 0$. According to the continuity of $\varepsilon(s,y)$, there exists a sequence $\varepsilon_n(s,y) = \varepsilon(s_n, y_n)$ of solutions of equation (19), such that $|\varepsilon_n| > 0$.
Therefore, it is clear that

$$\varepsilon_{ns} - x_{ns}\varepsilon_{ny} = \frac{1}{2}\frac{\lambda_{ns}}{\lambda_n}\varepsilon_n + \frac{1}{2}\frac{\lambda_{ns}}{\lambda_n}\mathcal{Q} + x_{ns}\mathcal{Q}_y + G(\varepsilon_n). \tag{35}$$

Letting $y_n \to +\infty$ and combining lemma 3, lemma 4 and decay property of $\mathcal{Q}$ and $\mathcal{Q}_y$, one gets

$$\varepsilon_{ns} - x_{ns}\varepsilon_{ny} = 0. \tag{36}$$

Since

$$a_{2n} = \sup_{s\ge 0}\|\varepsilon_n\|_{H^2},$$

there exists $s_0 \ge 0$, such that $\|\varepsilon_n(s_0)\|_{H^2} \ge \frac{a_{2n}}{2}$.
Setting

$$\omega_n(s,y) = \frac{\varepsilon_n(s+s_0, y)}{a_{2n}},$$

One gets

$$\omega_{ns} - x_{ns}\omega_{ny} = 0. \tag{37}$$

Due to

$$\|\omega_n(0)\|_{H^2} = \|\frac{\varepsilon_n(s_0)}{a_{2n}}\|_{H^2} \ge \frac{1}{2}, \tag{38}$$

one has

$$\omega_n(s) \not\equiv 0.$$

On the other hand, since the characteristic line equation of equation (37) is

$$\begin{cases} \frac{ds}{d\mu} = 1, \frac{dy}{d\mu} = -x_{ns} \\ \frac{d\omega_{ns}}{d\mu} = 0, \frac{d\omega_{ny}}{d\mu} = 0 \\ \frac{d\omega_n}{d\mu} = 0 \end{cases},$$

one gets $\omega_n \equiv C$, where C is a constant.
Due to $(\varepsilon_n, \mathcal{Q}_y) = (\varepsilon_n, \mathcal{Q}_{yy}) = 0$, we obtain $(\omega_n, \mathcal{Q}_y) = (\omega_n, \mathcal{Q}_{yy}) = 0$. It follows that

$$C \int_R \mathcal{Q}_y dy = C \int_R \mathcal{Q}_{yy} dy = 0,$$

where

$$\mathcal{Q} = e^{-\frac{\sqrt{3}}{2}|y|}\sqrt{3}\sin\left(\frac{|y|}{2} + \frac{\pi}{6}\right).$$

Due to $\int_R \mathcal{Q}_{yy} dy \neq 0$, we have

$$\omega_n \equiv C \equiv 0. \tag{39}$$

However, (39) is contradicted to $\omega_n(s) \not\equiv 0$. So we have $\varepsilon \equiv 0$, and conclude that there exist $\lambda_0(t)$ and $x_0(t) \in C^1$, such that

$$u(t, y) = \lambda_0^{-\frac{1}{2}}(t)\mathcal{Q}\left(y + x_0(t)\right).$$

The proof is completed. ■

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