An Approximation Algorithm for the Solution of the Singularly Perturbed Volterra Integro-differential and Volterra Integral Equations

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Abstract: Recently, there has been an increasing interest in the study of singular and perturbed systems. In this paper, we propose a collocation method for solving singularly perturbed Volterra integro-differential and Volterra integral equations. The method is based upon radial basis functions, using zeros of the shifted Legendre polynomial as the collocation points. The results of numerical experiments are compared with the exact solution in illustrative examples to confirm the accuracy and efficiency of the presented scheme.

Keywords: singularly perturbed problems; Volterra integral equations; Volterra integro-differential equations; collocation method; radial basis functions; Legendre polynomials

1 Introduction

In the present work, we consider the singularly perturbed Volterra integro-differential equations (SVIDE)

\[
\varepsilon \frac{d}{dx}y(x) = u(x, \varepsilon, y(x)) + \int_0^x K(x, t, \varepsilon, y(t)) \, dt, \quad x \in I = [0, X],
\]

\[
y(0) = \alpha,
\]

where \(\alpha\) is a constant and \(\varepsilon\) is a known perturbation parameter which \(0 < \varepsilon \ll 1\). Smoothness assumptions on \(u\) and \(K\) imply existence of a unique solution of Eq. (1) for \(\varepsilon > 0\). It is assumed further that

\[\exists \lambda > 0 : \frac{d}{dy}u(x, \varepsilon, y(x)) \leq -\lambda < 0, \quad \forall x \in I.\]

By substituting \(\varepsilon = 0\) in Eq. (1), we obtain the reduced equation

\[
0 = u(x, y(x)) + \int_0^x K(x, t, y(t)) \, dt, \quad x \in I = [0, X],
\]

which is a Volterra integral equation (VIE) of the second kind. The singularly perturbed nature of Eq. (1) occurs when the properties of the solution with \(\varepsilon > 0\) are incompatible with those when \(\varepsilon = 0\). The interest here is in those problems which do imply such an incompatibility in the behavior of \(y(x)\) near \(x = 0\). This suggests the existence of a boundary layer near the origin where the solution undergoes a rapid transition. Singularly perturbed Volterra integro-differential equations arise in many physical and biological problems. Among these are diffusion-dissipation processes, epidemic dynamics, synchronous control systems, renewal processes and filament stretching. For a comprehensive review, see [1–3]. Finding the solutions of these problems has been widely studied by researchers in the last decade. Implicit Runge-Kutta methods were presented for singularly perturbed integro-differential-algebraic equations in [4] and for singularly perturbed integro-differential systems in [5]. In [6], Orsi applied a Petrov-Galerkin method to singularly perturbed integro-differential-algebraic equations. El-Gendi [7] applied spectral methods to obtain solution of singularly perturbed

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differential, integral and integro-differential equations. Hu [8] and Horvat et al. [9] solved the SVIDEs by using the spline collocation methods. Recently, in [10] a numerical procedure based on finite difference was presented for solving a class of SVIDEs. More recently, Ramos [11] applied Piecewise-quasilinearization techniques to obtain solution of SVIDEs. For more references about SVIDEs see [12–14].

In recent decades, the so-called meshless methods have been extensively used to find approximate solutions of various types of linear and nonlinear equations such as differential equations (DEs) and integral equations (IEs). Unlike the other methods which were used to mesh the domain of the problem, meshless method don’t require a structured grid and only make use of a scattered set of collocation points regardless of the connectivity information between the collocation points. For the last years, the radial basis functions (RBFs) method was known as a powerful tool for the scattered data interpolation problem. The main advantage of numerical methods which use radial basis functions is the meshless characteristic of these methods. The use of radial basis functions as a meshless method for the numerical solution of ordinary differential equations (ODEs) and partial differential equations (PDEs) is based on the collocation method. One of the domain-type meshless methods is given in [15] in 1990, which directly collocates radial basis functions, particularly the multiquadric (MQ), to find an approximate solution of linear and nonlinear DEs. Kansa’s method has recently received a great deal of attention from researchers [16–21].

Recently, Kansa’s method was extended to solve various ordinary and partial differential equations including the nonlinear Klein-Gordon equation [20], regularized long wave (RLW) equation [22], high order ordinary differential equations [23], the case of heat transfer equations [24], Hirota-Satsuma coupled KdV equations [25], second-order parabolic equation with nonlocal boundary conditions [26], Volterra’s Population model [27], steady flow of a third-grade fluid in a porous half space [28], Fokker-Planck equation [29], Second-order hyperbolic telegraph equation [30] and so on. All of the radial basis functions have global support, and in fact many of them, such as multiquadrics (MQ), do not even have isolated zeros [20, 22, 31]. The RBFs can be compactly and globally supported, infinitely differentiable, and contain a free parameter \( c \), called the shape parameter [22, 31, 32]. The interested reader is referred to the recent books and paper by Buhmann [31, 33] and Wendland [34] for more basic details about RBFs, compactly and globally supported and convergence rate of the radial basis functions.

There are two basic approaches for obtaining basis functions from RBFs, namely direct approach (DRBF) based on a differential process [35] and indirect approach (IRBF) based on an integration process [18, 23, 36]. Both approaches were tested on the solution of second order DEs and the indirect approach was found to be superior to the direct approach [18]. Some of the infinitely smooth RBFs choices are listed in Table 1. The RBFs can be of various types, for example: multiquadrics (MQ), inverse multiquadrics (IMQ), Gaussian forms (GA), hyperbolic secant (sech) form etc. In the cases of inverse quadratic, inverse multiquadric (IMQ), hyperbolic secant (sech) and Gaussian (GA), the coefficient matrix of RBFs interpolating is positive definite and, for multiquadric (MQ), it has one positive eigenvalue and the remaining ones are all negative [37].

In this paper, we use the multiquadrics direct radial basis function for finding the solution of SVIDEs. The MQ was first developed by Hardy [38] in 1971 as a multidimensional scattered interpolation method in modeling of the earth gravitational field. It was not recognized by most of the academic researchers until Franke [39] published a review paper on the evaluation of two-dimensional interpolation methods, whereas MQ was ranked as the best based on its accuracy, visual aspect, sensitivity to parameters, execution time, storage requirements, and ease of implementation.

For convenience the solution we use RBFs with collocation nodes \( \{ x_j \}_{j=1}^N \) which are the zeros of the shifted Legendre polynomial \( L_N(x) \), \( 0 \leq x \leq 1 \). The shifted Legendre polynomials \( L_i(x) \) are defined on the interval \([0, 1]\) and satisfy the following formulae [40]:

\[
L_0(x) = 1, \quad L_1(x) = 2x - 1, \\
L_{i+1}(x) = \frac{2i+1}{i+1} (2x - 1)L_i(x) - \frac{i}{i+1}L_{i-1}(x), \quad i = 1, 2, 3, \ldots
\]

This paper is arranged as follows: in Section 2, we describe the properties of radial basis functions. In Section 3, we introduce the Legendre-Gauss-Lobatto nodes and weights. In Section 4 we implement the problem with the proposed method.
method and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed methods. The conclusions are discussed in the final Section.

2 Radial basis functions

2.1 Definition of radial basis function

Let $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$ be the non-negative half-line and let $\phi: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $\phi(0) \geq 0$. A radial basis functions on $\mathbb{R}^d$ is a function of the form

$$\phi(\|X - X_i\|),$$

where $X, X_i \in \mathbb{R}^d$ and $\|\cdot\|$ denotes the Euclidean distance between $X$ and $X_i$s. If one chooses $N$ points $\{X_i\}_{i=1}^N$ in $\mathbb{R}^d$ then by custom

$$s(X) = \sum_{i=1}^N \lambda_i \phi(\|X - X_i\|), \quad \lambda_i \in \mathbb{R},$$

is called a radial basis functions as well [41]. The standard radial basis functions are categorized into two major classes [25]:

Class 1. Infinitely smooth RBFs [25, 42]:
These basis functions are infinitely differentiable and heavily depend on the shape parameter $c$ e.g. Hardy multiquadric (MQ), Gaussian(GA), inverse multiquadric (IMQ), and inverse quadric(IQ)(See Table 1).

Class 2. Infinitely smooth (except at centers) RBFs [25, 42]:
The basis functions of this category are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than the basis functions discussed in the Class 1. For example, thin plate spline, etc [25].

2.2 RBFs interpolation

The one dimensional function $y(x)$ to be interpolated or approximated can be represented by an RBFs as:

$$y(x) \approx y_N(x) = \sum_{i=1}^N \lambda_i \phi_i(x) = \Phi^T(x) \Lambda,$$

where

$$\phi_i(x) = \varphi(\|x - x_i\|),$$
$$\Phi^T(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_N(x)],$$
$$\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]^T,$$

$$j = 1, 2, \ldots, N.$$  

To brief discussion on coefficient matrix, we define:

$$\Lambda \Lambda = Y,$$
where

\[
Y = [y_1, y_2, ..., y_N]^T,
A = \Phi^T(x_1), \Phi^T(x_2), ..., \Phi^T(x_N)]^T,
\]

\[
= \begin{bmatrix}
\phi_1(x_1) & \phi_2(x_1) & \ldots & \phi_N(x_1) \\
\phi_1(x_2) & \phi_2(x_2) & \ldots & \phi_N(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(x_N) & \phi_2(x_N) & \ldots & \phi_N(x_N)
\end{bmatrix}, \tag{5}
\]

Note that \(\phi_i(x_j) = \varphi(||x_j - x_i||)\) therefore we have \(\phi_i(x_j) = \phi_j(x_i)\) consequently \(A = A^T\).

All the infinitely smooth RBFs choices are listed in Table 1 will give coefficient matrices \(A\) in Eq. (5) which are symmetric and nonsingular [37], i.e. there is a unique interpolant of the form Eq. (3) no matter how the distinct data points are scattered in any number of space dimensions. In the cases of inverse quadratic, inverse multiquadric (IMQ), hyperbolic secant (sech) and Gaussian (GA) the matrix \(A\) is positive definite and, for multiquadric (MQ), it has one positive eigenvalue and the remaining ones are all negative [37].

We have the following theorem about the convergence of RBFs interpolation:

**Theorem** [43, 44]: Assume \(x_i, (i = 1, 2, ..., N)\) are N nodes in \(\Omega\) which is convex, let

\[
h = \max_{x \in \Omega} \min_{1 \leq i \leq N} ||x - x_i||_2,
\]

when \(\hat{\phi}(\eta) < c(1 + |\eta|)^{-(2l+d)}\) for any \(y(x)\) satisfies \(\int (\hat{y}(\eta))^2 / \hat{\phi}(\eta) d\eta < \infty\) we have

\[
||y_N^{(\alpha)} - y^{(\alpha)}||_{\infty} \leq ch^{l-\alpha},
\]

where \(\phi(x)\) is RBFs and the constant \(c\) depends on the RBFs, \(d\) is space dimension, \(l\) and \(\alpha\) are nonnegative integer. It can be seen that not only RBFs itself but also its any order derivative has a good convergence.

**Proof:** A complete proof is given by Wu [43, 44]. □

The results of this section can be summarized in the following algorithm.

**Algorithm**

The algorithm works in the following manner:

1. Choose \(N\) center points \(\{x_j\}_{j=1}^N\) from the domain set \([a, b]\).
2. Approxime \(y(x)\) as \(y_N(x) = \Phi^T(x)A\).
3. Substitute \(y_N(x)\) into the main problem and creat residual function \(Res(x)\).
4. Substitute collocation points \(\{x_j\}_{j=1}^N\) into the \(Res(x)\) and create the \(N\) equations.
5. Solve the \(N\) equations with \(N\) unknown coefficients of members of \(A\) and find the numerical solution.

### 3 Legendre-Gauss-Lobatto nodes and weights

Let \(H_N[-1, 1]\) denote the space of algebraic polynomials of degree \(\leq N\)

\[
< P_i, P_j > = \frac{2}{2j + 1}\delta_{ij}.
\]

Here, \(<, \ldots, \rangle \) represent the usual \(L^2[-1, 1]\) inner product and \(\{P_i\}_{i \geq 0}\) are the well-known Legendre polynomials of order \(i\) which are orthogonal with respect to the weight function \(w(x) = 1\) on the interval \([-1, 1]\), and satisfy the following formulæ:

\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_{i+1}(x) = \frac{2i+1}{i+1}xP_i(x) - \frac{i}{i+1}P_{i-1}(x), \quad i = 1, 2, 3, ...
\]

Next, we let \(\{x_j\}_{j=0}^N\) as:

\[
(1 - x_j^2)\hat{P}(x_j) = 0; \\
-1 = x_0 < x_1 < x_2 < ... < x_N = 1,
\]

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where
behavior of the MQ-RBF method, we applied the following laws

1. The $L_2$ error norm of the solution which is defined by

$$L_2 = \| y^{\text{exact}}(x) - y_N^{\text{implicit}}(x) \|_2 = \left[ \sum_{j=1}^{N} (y^{\text{exact}}(x_j) - y_N^{\text{implicit}}(x_j))^2 \right]^{1/2},$$

where $\{x_j\}_{j=1}^{N}$ are interpolate nodes which are the zeros of shifted Legendre polynomial $L_N(x)$, $0 \leq x \leq 1$.

2. The $L_{\infty}$ error norm of the solution which is defined by

$$L_{\infty} = \| y^{\text{exact}}(x) - y_N^{\text{implicit}}(x) \|_{\infty} = \max_{0 \leq j \leq N} |y^{\text{exact}}(x_j) - y_N^{\text{implicit}}(x_j)|.$$

### 5.1 Problem 1

In this problem, we consider the following singularly perturbed Volterra integral equation [13, 14]

$$\varepsilon y(x) = \int_0^x (1 + t - y(t)) dt,$$  

which has the following exact solution:

$$y(x) = x + 1 - e^{-x/\varepsilon} - \varepsilon(1 - e^{-x/\varepsilon}).$$

We applied present method and solved Eq. (16) for different value of $N$. Table 2 shows the $L_2$-error, $L_{\infty}$-error norms, in some values of $\varepsilon$ obtained for $N = 16, 32, 64, 128$. Then in Figure 1 the exact and DRBF solutions for $N = 64$ and $\varepsilon = 2^0, 2^{-1}, 2^{-2}, ..., 2^{-10}$ is represented.

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**Table 2: Error norms of MQ RBF results with $\varepsilon = 0.1$ in problem 1.**

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<th>$\varepsilon$</th>
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<th>Note</th>
<th>$L_{\infty}$</th>
<th>$L_2$</th>
<th>Note</th>
<th>$L_{\infty}$</th>
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Figure 1: Comparison between DRBF approximations of $y(x)$ and exact solutions (points) for different values of $\varepsilon$ in problem 1.

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ε in some values of $y_0(x)$.

$$\text{We applied present method and solved Eq. (16) for different value of } \varepsilon.$$  

5.2 Problem 2

Now we consider the singularly perturbed Volterra integral equation [13, 14]

$$\varepsilon y(x) = \int_0^x (1 + x - t)(1 + t - y(t)) dt,$$  

which has the exact solution:

$$y(x) = x + 1 + \frac{1}{\gamma_1 - \gamma_2} \left( (\gamma_2 - 1 + 1/\varepsilon)e^{\gamma_1 x} - (\gamma_1 - 1 + 1/\varepsilon)e^{\gamma_2 x} \right),$$

where

$$\gamma_1 = 1/2\varepsilon(-1 + \sqrt{1 - 4\varepsilon}),$$  

$$\gamma_2 = 1/2\varepsilon(-1 - \sqrt{1 - 4\varepsilon}).$$

We applied present method and solved Eq. (16) for different value of $N$. Table 3 shows the $L_2$-error, $L_\infty$-error norms, in some values of $\varepsilon$ obtained for $N = 16, 32, 64, 128$. Then in Figure 2 the exact and DRBF solutions for $N = 64$ and

$\varepsilon = 2^{-1}, 2^{-1}, 2^{-2}, , 2^{-10}$ is represented.

5.3 Problem 3

Consider the linear singularly perturbed Volterra integral equation [9]

$$\varepsilon y(x) = \sin(x) - \int_0^x y(t) dt.$$  

Figure 2: Comparison between DRBF approximations of $y(x)$ and exact solutions (points) for different values of $\varepsilon$ in problem 2.

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Table 3: Error norms of MQ RBF results with $c = 0.1$ in problem 2.
Figure 3: Comparison between DRBF approximations of $y(x)$ and exact solutions (points) for different values of $\varepsilon$ in problem 3.

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Table 4: Error norms of MQ RBF results with $c = 0.1$ in problem 3.
Figure 4: Comparison between DRBF approximations of \( y(x) \) and exact solutions (points) for different values of \( \varepsilon \) in problem 4.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( L_2 )</th>
<th>( L_{\infty} )</th>
<th>( \varepsilon )</th>
<th>( L_2 )</th>
<th>( L_{\infty} )</th>
<th>( \varepsilon )</th>
<th>( L_2 )</th>
<th>( L_{\infty} )</th>
<th>( \varepsilon )</th>
<th>( L_2 )</th>
<th>( L_{\infty} )</th>
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<td>( 2^{-1} )</td>
<td>2.81e-004</td>
<td>8.51e-005</td>
<td>1.71e-007</td>
<td>3.65e-008</td>
<td>3.24e-015</td>
<td>4.94e-016</td>
<td>1.16e-035</td>
<td>4.17e-036</td>
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<tr>
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<td>2.08e-004</td>
<td>8.15e-005</td>
<td>1.33e-007</td>
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<td>2.35e-034</td>
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<tr>
<td>( 2^{-3} )</td>
<td>1.55e-004</td>
<td>7.93e-005</td>
<td>9.55e-008</td>
<td>3.35e-008</td>
<td>1.98e-015</td>
<td>4.97e-016</td>
<td>9.16e-035</td>
<td>3.34e-035</td>
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<tr>
<td>( 2^{-4} )</td>
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<td>4.73e-005</td>
<td>6.44e-008</td>
<td>3.01e-008</td>
<td>1.50e-015</td>
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<td>3.30e-035</td>
<td>1.20e-035</td>
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<td>( 2^{-5} )</td>
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<td>( 2^{-6} )</td>
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<td>4.64e-016</td>
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</tr>
</tbody>
</table>

The exact solution is

\[
y(x) = \frac{1}{1 + \varepsilon^2} (\cos(x) + \varepsilon \sin(x) - e^{-x/\varepsilon}).
\]

We applied present method and solved Eq. (16) for different value of \( N \). Table 4 shows the \( L_2 \)-error, \( L_{\infty} \)-error norms, in some values of \( \varepsilon \) obtained for \( N = 16, 32, 64, 128 \). Then in Figure 3 the exact and DRBF solutions for \( N = 64 \) and \( \varepsilon = 2^{-1}, 2^{-2}, ..., 2^{-10} \) is represented.

### 5.4 Problem 4

Opposed to previous problems, in this problem we consider the singularly perturbed Volterra integro-differential equation [9–11]

\[
\varepsilon \frac{d}{dx} y(x) = (1 + \varepsilon) e^{-x} - \varepsilon - y(x) + \int_0^x (1 + \varepsilon) y(t) dt, \tag{19}
\]

with the initial condition \( y(0) = 1 + e^{-1} \). The solution is

\[
y(x) = e^{x-1} + e^{-x/\varepsilon(1+\varepsilon)}.
\]

We applied present method and solved Eq. (16) for different value of \( N \). Table 5 shows the \( L_2 \)-error, \( L_{\infty} \)-error norms, in some values of \( \varepsilon \) obtained for \( N = 16, 32, 64, 128 \). Then in Figure 4 the exact and DRBF solutions for \( N = 64 \) and \( \varepsilon = 2^{-1}, 2^{-2}, ..., 2^{-10} \) is represented.

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In this paper, we discussed some integral equations which have the singularly perturbed properties. We proposed

\[ y(x) = \int_0^x g(t)\,dt + f(x) \]

with the initial condition \( y(0) = 0 \). The exact solution is

\[ y(x) = 10 - (10 + x)e^{-x} + 10e^{-x}/\varepsilon. \]

We applied present method and solved Eq. (16) for different values of \( \varepsilon \). Table 6 shows the \( L_2 \)-error, \( L_\infty \)-error norms, in some values of \( \varepsilon \) obtained for \( N = 16, 32, 64, 128 \). Then in Figure 5 the exact and DRBF solutions for \( N = 64 \) and

\[ \varepsilon = 2^0, 2^{-1}, 2^{-2}, \ldots, 2^{-10} \]

is represented.

## 6 Conclusion

In this paper, we discussed the some of integral equations which have the singularly and perturbed properties. We proposed a numerical scheme to solve this equations using collocation points and approximating the solution using the multiquadric (MQ) radial basis function. For convenience the solutions we used RBFs with collocation nodes \( \{x_j\}_{j=1}^N \) which are the zeros of the shifted Legendre polynomial \( L_N(x) \), \( 0 \leq x \leq 1 \). Additionally, through the comparison with exact solutions.
we show that the RBFs methods have good accuracy and efficiency and results obtained using the RBFs method are with low error.

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References


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