Ritz-Galerkin Method for Solving a Class of Inverse Problems in the Parabolic Equations

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Abstract: In this paper a type of inverse problem belongs to the class of parabolic equations is considered. The approximation of the problem is based on the Ritz-Galerkin method. This approximation provides greater flexibility in which to impose initial and boundary conditions. The results of numerical experiments are compared with analytical solutions and perspicuous examples are included to confirm the accuracy and efficiency of the presented scheme.

Keywords: inverse problem; parabolic equations; heat equation; Ritz-Galerkin method

1 Introduction

Consider the following parabolic equation

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( a(x) \frac{\partial u}{\partial x} \right) + h(x, t), \quad 0 < x < 1, \quad 0 \leq t \leq T,
\]

with the initial condition

\[
u(x, 0) = u_0(x), \quad 0 < x < 1,
\]

and boundary conditions:

\[
u(1, t) = g(t), \quad u_x(0, t) = 0, \quad 0 \leq t \leq T,
\]

and extra specification

\[
u(x, T) = u_1(x), \quad 0 < x < 1.
\]

This model describes the heat conduction procedure in a given homogenous medium with some input source \( h(x, t) \). The coefficient \( a(x) \) represents a heat conduction property whose physical name is heat capacity. The literature on the numerical solutions of parabolic partial differential equation is growing rapidly and much attention has been devoted to the investigation of parabolic inverse problems, see e.g. [1-17]. The uniqueness result was obtained in [18]. In addition, we refer the interested reader for more theoretical discussion and more background on this problem to the references [19-20]. Apparently the well known approach for solving the present inverse problem is the least squares method, but it has some drawbacks. For example, it is usually not evident that the solution to the optimization problem solves the original inverse problem and the error functional may be based on data which do not uniquely determine the unknown function. Also some authors considered the problem under an optimization control framework [21-24].

In the present article a new technique based on the Ritz-Galerkin method is used to solve the inverse problem introduced by equations (1-4). Compared with other published methods its fascinating merit is the high accuracy in computations. This article arranged as follows:

In section 2, we describe the property of Ritz-Galerkin method required for our subsequent development. Section 3 is devoted to the solution of problem (1-4) by using the Ritz-Galerkin method. In section 4, we report our numerical findings and demonstrate the accuracy of the proposed scheme by considering numerical examples. Section 5 consists of a brief summary.

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2 The Ritz-Galerkin method

Consider the differential equation
\[ L[y(x)] + f(x) = 0, \tag{5} \]
over the interval \( a \leq x \leq b \). Multiplying (5) by any arbitrary weight function \( w(x) \) and integrating over the interval \([a,b]\) one obtains
\[ \int_a^b w(x)(L[y(x)] + f(x))dx = 0, \tag{6} \]
for any arbitrary \( w(x) \). Equations (5) and (6) are equivalent, because \( w(x) \) is any arbitrary function.

We introduce a trial solution \( u(x) \) to (5) of the form
\[ u(x) = \varphi_0(x) + \sum_{j=1}^n c_j \varphi_j(x). \tag{7} \]
and replace \( y(x) \) with \( u(x) \) on the left side of (5). The residual is defined as follows
\[ r(x) = L[u(x)] + f(x). \]
The goal is to construct \( u(x) \) so that the integral of the residual will be zero for some choices of weight functions. That is, \( u(x) \) will satisfy (6) in the sense that
\[ \int_a^b w(x)(L[u(x)] + f(x))dx = 0, \]
for some choices of \( w(x) \). One of the most important weighted residual methods was introduced by the Russian mathematician, Boris Grigoryevich Galerkin (February 20, 1871 - July 12, 1945). Galerkin’s method selects the weight functions in a special way: they are chosen from the basis functions that used for the trial solution, so \( w(x) \in \{ \varphi_i(x) \}_{i=1}^n \). It is required that the following \( n \) equations hold true
\[ \int_a^b \varphi_i(x)(L[u(x)] + f(x))dx = 0, \quad i = 1, 2, ..., n. \]
To apply the method, we solve these \( n \) equations for the coefficients \( \{c_j\}_{j=1}^\infty \).

Suppose we wish to solve a boundary value problem over the interval \([a,b]\) with the above method, we select \( \varphi_i(x), \quad i = 1, 2, ..., m \) so that satisfy the homogeneous form of the specified essential boundary conditions and \( \varphi_0 \) must satisfy the specified essential boundary conditions.

3 Solution of inverse problem via Ritz-Galerkin method

Consider the inverse problem (1-4). Our strategy is based upon converting the inverse problem to the direct one.

By using Eqs.(1,3,4) we have
\[ a(x) = \frac{\int_0^x (u_x(s, T) - h(s, T))ds}{(u_1)_x(x)}. \]
Now we only need to find an approximation for \( u(x, t) \).

Let us set
\[ w(x, t) = u(x, t) - x^2 g(t), \tag{8} \]
we have
\[ w(1, t) = 0, \quad w_x(0, t) = 0. \]
Now equations (1-4) are equivalent to
\[ \frac{\partial w}{\partial t} - \frac{\partial}{\partial x} (a(x) \frac{\partial w}{\partial x}) = h(x, t) - x^2 g'(t) + 2a(x) g(t) + 2xa'(x)g(t), \tag{9} \]

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with the homogenous boundary conditions
\[ w(1, t) = w_x(0, t) = 0, \]
and initial conditions
\[ w(x, 0) = u_0(x) - x^2 g(0), \quad w(x, T) = u_1(x) - x^2 g(T). \] (11)

Let
\[ F(w) = \frac{\partial w}{\partial t} - \frac{\partial}{\partial x} (a(x) \frac{\partial w}{\partial x}) - [h(x, t) - x^2 g'(t) + 2a(x)g(t) + 2x a'(x)g(t)] = 0. \] (12)

Now a Ritz-Galerkin approximation to (12) is constructed as follows. The approximation \( \hat{w} \) is sought in the form of the truncated series
\[ \hat{w}(x, t) = \sum_{i=0}^{M} \sum_{j=0}^{N} x^2(x - 1)i(t - T)c_{ij}\varphi_{ij}(x, t) + s(x, t), \] (13)
where \( \varphi_{ij}(x, t) \) are basis functions and \( s(x, t) \) is an interpolating function as
\[ s(x, t) = u_0(x) - x^2 g(0) + \frac{t}{T}(u_1(x) - u_0(x) + x^2 g(0) - g(T)). \]

This approximation satisfies the boundary and initial conditions (10,11).

The expansion coefficients \( c_{ij} \) are determined by the Galerkin equations:
\[ < F(\hat{w}), \varphi_{ij}(x, t) > = 0, \quad i = 0, 1, ..., M, \quad j = 0, 1, ..., N, \] (14)
where \(< . >\) denotes the inner product defined by
\[ < F(\hat{w}), \varphi_{ij}(x, t) > = \int_{0}^{1} \int_{0}^{T} F(\hat{w})(x, t)\varphi_{ij}(x, t) dt dx. \]

Galerkin equations (14) give a system of non-linear equations which can be solved for the elements of \( c_{ij}, i = 0, 1, ..., M, \quad j = 0, 1, ..., N \), using Newton’s iterative method. The initial values required to start Newton’s iterative method can be chosen by using the physical behavior of the given equation.

We applied the method presented in this paper and solved two examples. Also we use Legendre polynomials as basis.

4 Illustrative examples:

4.1 Example 1

Consider the equation:
\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} (a(x) \frac{\partial u}{\partial x}) \quad 0 < x < 1, 0 \leq t \leq 1. \]

With the initial condition \( u(x, 0) = x^3, 0 < x < 1 \), boundary condition:
\[ u(1, t) = e^t, \quad u_x(0, t) = 0, \quad 0 \leq t \leq 1. \]

And extra specification \( u(x, 1) = x^3 e^{-t} \). The exact solution is \( u(x, t) = x^3 e^t \) and \( u(x) = \frac{e^x}{12} \).

By equation (13) we have
\[ \hat{w}(x, t) = \sum_{i=0}^{M} \sum_{j=0}^{N} x^2(x - 1)i(t - 1)c_{ij}\varphi_{ij}(x, t) + s(x, t), \]
where
\[ \varphi_{ij}(x, t) = P_i(x)P_j(t), \]
and
\[ s(x, t) = x^3 - x^2 + t(x^3 e - x^3 + x^2 (1 - e)). \]

Here, \( P_i(t) \) are the well-known shifted Legendre polynomials of order \( i \) which are orthogonal with respect to the weight function \( W(t) = 1 \) on the interval \([0, 1]\).

By employing the strategy presented in this paper to solve the problem with \( M = 1, N = 3 \) we have approximate solution:
\[ \hat{a}(x) = -3.58516 \times 10^{-8} x + 0.0833334x^2 - 4.38519 \times 10^{-8} x^3, \quad 0 < x < 1. \]

In Figure (1) the exact and approximate solutions of \( u(x, t) \) are plotted. Figures 2 and 3 shows error for \( a(x) \) and \( u(x, t) \).
4.2 Example 2

Consider (1) with the initial condition
\[ u(x,0) = x^2e^x, \quad 0 < x < 1, \]
boundary conditions:
\[ u(1,t) = e^{t+1}, \quad u_x(0,t) = 0, \quad 0 \leq t \leq 1 \]
and extra specification \( u(x,1) = x^2e^{x+1} \). The exact solution is
\[ u(x,t) = x^2e^{x+t} \]
and
\[ a(x) = \frac{e^{t}(x^2-2x+2)-2}{x(2+x)e^t}. \]
Again by using the presented strategy to solve the problem with \( M = 1, N = 3 \) we have an approximate solution. In figure(4) the exact and approximate solution for \( u(x,t) \) are plotted. The absolute error between \( a(x), u(x,t) \) and their approximations are plotted in figures(5,6).

\[ \text{Figure 1: Exact and approximate solution of } u(x,t) \text{ for example 1.} \]

\[ \text{Figure 2: The absolute error of } a(x) \text{ for example 1.} \]

\[ \text{Figure 3: The absolute error of } u(x,t) \text{ for example 1.} \]

\[ \text{Figure 4: Exact and approximate solution of } u(x,t) \text{ for example 2.} \]

\[ \text{Figure 5: The absolute error of } u(x,t) \text{ for example 2.} \]

\[ \text{Figure 6: The absolute error of } a(x) \text{ for example 2.} \]
5 Conclusions

The Ritz-Galerkin method is used to reduce the solution of the parabolic inverse problem to the solution of algebraic equations. The choice of approximation provides greater flexibility in which to impose initial and boundary conditions. Moreover, only a small number of basis are needed to obtain a satisfactory result. Illustrative examples are included to demonstrate the validity and applicability of the technique.

References