Some Results on the Optimal Correction of Infeasible Second Order Conic Linear Inequalities

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Abstract: It is known that the optimal correction of infeasible linear inequalities in the nonnegative orthant can be achieved by solving a lower dimensional convex problem. In this paper our goal is to investigate the problem of optimal correction of infeasible second order conic linear inequalities. We prove that under certain conditions analogous result to the classic linear case holds. Finally, the efficiency of the new approach using several randomly generated numerical examples is shown.

Keywords: linear inequalities; conic linear inequalities; infeasible systems; second order cone optimization; interior point methods

1 Introduction

The resulting mathematical model of many real world applications might be infeasible. The infeasibility might be due to the error in problem data and many other reasons. A specific case of these models are infeasible linear systems. The optimal correction of such systems is a well studied problem and appears in many disciplines [1–3]. The aim of this paper is to investigate the optimal correction of infeasibility in the second order conic linear setting. This is due to variety of applications where optimization over second order cones appear [4, 5]. Thus let us first define a second order cone and introduce the linear optimization problem over these cones.

Definition 1.1 A second order cone in \( \mathbb{R}^n \) is defined as

\[
Q_n = \{ x \in \mathbb{R}^n \mid ||x|| \leq x_1 \}, \text{ where } x = (x_2, \ldots, x_n)^T.
\]

Throughout this paper by \( x \succeq Q_n 0_n \) we mean \( x \in Q_n \).

Analogous to the nonnegative orthant as a cone, it is a closed, convex, self-dual, pointed cone with nonempty interior that enables one to extend interior point algorithms from linear optimization (LO) to second order cone optimization (SOCO) [4, 6]. However there are differences between duality results between LO and SOCO. To have strong duality for SOCO one require stronger conditions, namely strict feasibility of primal and dual SOCO than just feasibility of primal and dual problems as in LO. Further details on differences between SOCO and LO can be found in [4].

Similar to LO, a SOCO in the standard primal form is given by

\[
\begin{align*}
\min & \quad c^T x \\
Ax & = b \\
x & \succeq Q_n 0_n,
\end{align*}
\]

(1)

where \( A = [A_1, A_2, \ldots, A_k] \in R^{m \times n}, b \in R^m, c \in R^n, 0_n \) is all zero vector of dimension \( n \), \( x^T = (x_1^T, \ldots, x_k^T) \). Moreover, \( x \succeq Q_n 0_n \Leftrightarrow x^i \succeq Q_{ni_1} 0_{ni_1}, \quad i = 1, \ldots, k, n_1 + \cdots + n_k = n, Q_n = Q_{n_1} \times Q_{n_2} \times \cdots \times Q_{n_k}, \) and its dual is

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given by

\[
\max \quad b^T y \\
\quad c - A^T y \succeq Q_n 0_n,
\]
(2)
or in the slack form

\[
\max \quad b^T y \\
\quad A^T y + s = c \\
\quad s \succeq Q_n 0_n.
\]
(3)

Now suppose we have the following infeasible second order conic linear inequalities (SOCLIQs)

\[
c_i - A_i^T y \succeq Q_{n_i} 0_{n_i}, \quad y \in \mathbb{R}^{m}, \quad i = 1, \cdots, k.
\]
(4)

Correcting this infeasible system to a feasible one by minimal changes in the vector \(c = (c_1^T, c_2^T, \cdots, c_k^T)^T\) is equivalent to solve

\[
\min_{y,r} \quad \sum_{i=1}^k ||r_i|| \\
\quad r_i + c_i - A_i^T y \succeq Q_{n_i} 0_{n_i}, \quad i = 1, \cdots, k.
\]
(5)

Now to solve this problem we can present it as the dual form (2) and use interior point based software packages like Mosek and SeDuMi [7, 8]:

\[
\max_{y,r} \quad -\sum_{i=1}^k t_i \\
\quad r_i + c_i - A_i^T y \succeq Q_{n_i} 0_{n_i}, \quad i = 1, \cdots, k, \\
\quad \begin{pmatrix} t_i \\ r_i \end{pmatrix} \succeq Q_{n_i+1} 0_{n_i+1}.
\]
(6)

In this paper, our goal is to see whether, instead of solving (6), it would be possible to find the optimal correction vector \(r^T = (r_1^T, \cdots, r_k^T)^T\) by solving a lower dimensional convex problem, analogous to the case in nonnegative orthant? We show that in certain cases this reduction is possible, however in general the lower dimensional problem is not convex. Finally, several numerical examples are presented to show the efficiency of the new approach in the convex case.

2 Optimal Correction

In this section instead of (6), we present a lower dimensional problem which should be solved in order to find the optimal correction. Moreover, under certain conditions this lower dimensional problem is convex.

We may write problem (5) as

\[
\min_{y} \min_{r} \quad \sum_{i=1}^k ||r_i|| \\
\quad r_i + c_i - A_i^T y \succeq Q_{n_i} 0_{n_i}, \quad i = 1, \cdots, k.
\]
(7)

Now for a given \(y\), let us consider the inner minimization problem. It can be written as

\[
\max \quad -\sum_{i=1}^k t_i \\
\quad ||r_i|| \leq t_i, \quad i = 1, \cdots, k, \\
\quad r_i + c_i - A_i^T y \succeq Q_{n_i} 0_{n_i}, \quad i = 1, \cdots, k
\]
(8)
and its dual is

\[
\min \sum_{i=1}^{k} (c_i - A_i^T y)^T z_i \\
u_i = 1, \ i = 1, \ldots, k, \\
z_i + v_i = 0, \ i = 1, \ldots, k, \\
(u_i, v_i^T) \in Q_{n+1}, \ i = 1, \ldots, k, \\
z_i \in Q_{n}, \ i = 1, \ldots, k.
\]  

(9)

Let $\tilde{A}_i = A_i(2 : m, ;)$ and $\tilde{c}_i = c_i(2 : m)$. Then it is easy to check that if $|c_i(1) - A_i^T (1,:) y| < ||\tilde{c}_i - \tilde{A}_i^T y||$, then the optimal $r_i$ in (8) is given by

\[
r_i = \left(-\frac{|c_i - A_i^T y| - (c_i(1) - A_i^T (1,:) y)}{2||c_i - A_i^T y||} \right)
\]

(10)

and in the dual problem (9) $z_i = \frac{1}{\sqrt{2}} (1, -\frac{c_i - A_i^T y}{||c_i - A_i^T y||})^T u_i = 1, v_i = -z_i$. However if $|c_i(1) - A_i^T (1,:) y| \geq ||\tilde{c}_i - \tilde{A}_i^T y||$, then the optimal $r_i$ is given by

\[
r_i = -c_i + A_i^T y.
\]

(11)

and $z_i = -\frac{c_i - A_i^T y}{||c_i - A_i^T y||}, u_i = 1, v_i = -z_i$. Otherwise $r_i = 0$ in the optimal solution.

However, since for each $i = 1, \ldots, k$, we might have $|c_i(1) - A_i^T (1,:) y| < ||\tilde{c}_i - \tilde{A}_i^T y||$ or $|c_i(1) - A_i^T (1,:) y| \geq ||\tilde{c}_i - \tilde{A}_i^T y||$, then one can not determine the optimal $r_i$ value in prior. In the sequel we discuss some specific cases which one can find the optimal $r_i$ values by solving a lower dimensional convex problem.

First let us introduce the following index sets:

\[I = \{1, 2, \ldots, k\},\]

\[I_1 = \{i \in I \mid |c_i(1) - A_i^T (1,:) y| < ||\tilde{c}_i - \tilde{A}_i^T y||, \ \forall y \in R^m\},\]

\[I_2 = \{i \in I \mid |c_i(1) - A_i^T (1,:) y| \geq ||\tilde{c}_i - \tilde{A}_i^T y||, \ \forall y \in R^m\}.\]

**Lemma 2.1** If $I_1$ and $I_2$ are known in prior, then the optimal $r$ value is given by

\[
r_i = \left(-\frac{|c_i - A_i^T y| - (c_i(1) - A_i^T (1,:) y)}{2||c_i - A_i^T y||} \right) \quad i \in I_1
\]

and

\[
r_i = -c_i + A_i^T y^* \quad i \in I_2,
\]

where $y^*$ is the optimal solution of

\[
\min \frac{1}{\sqrt{2}} \sum_{i \in I_2} (||\tilde{c}_i - \tilde{A}_i^T y|| - (c_i(1) - A_i^T (1,:) y)) + \sum_{i \in I_2} ||c_i - A_i^T y||
\]

(14)

which is equivalent to the following SOCO:

\[
\min \frac{1}{\sqrt{2}} \sum_{i \in I_1} t_i + \sum_{i \in I_2} w_i \\
||\tilde{c}_i - \tilde{A}_i^T y|| \leq (c_i(1) - A_i^T (1,:) y) + t_i, \quad i \in I_1,
\]

\[
||c_i - A_i^T y|| \leq w_i, \quad i \in I_2.
\]

(15)

**Proof.** It follows from the previous discussion.
Lemma 2.2 If \( c_i(1) - A_i^T(1,:)y \geq 0 \) \( \forall y, \forall i \in I \), then \( I_1 = 1 \) and the optimal \( r \) value is given by (12) where \( y^* \) is an optimal solution of

\[
\min \frac{1}{\sqrt{2}} \sum_{i \in I} (||\tilde{c}_i - \tilde{A}_i^T y|| - (c_i(1) - A_i^T(1,:)y)).
\]  

(16)

This itself is equivalent to the following SOCO:

\[
\min \frac{1}{\sqrt{2}} \sum_{i \in I} t_i
\]

\[
||\tilde{c}_i - \tilde{A}_i^T y|| \leq (c_i(1) - A_i^T(1,:)y) + t_i
\]  

(17)

Proof. It follows from the previous discussion.

Remark 2.3 It is easy to see that (15) has \( k + m \) variables and constraints belong to \( Q_{n_1+1} \times \cdots Q_{n_k+1} \). However, problem (6) has \( 2k + m \) variables and constraints belong to \( Q_{n_1+1} \times \cdots Q_{n_k+1} \times Q_{n_1} \times \cdots Q_{n_k} \). Therefore if the index sets \( I_1 \) and \( I_2 \) are known, then doing the minimal correction by solving (15) should be much faster than (6), as it is verified by our numerical experiments.

Remark 2.4 When \( k = 1 \) in (1), then we can do the optimal correction by solving three lower dimensional convex problems without any specific condition like the general case. However, by increasing \( k \), this number increases exponentially. For \( k = 1 \), for each \( y \), either \( c_1(1) - A_1^T y \) is positive or negative. Therefore we solve the following three lower dimensional SOCO problems:

\[
\min \frac{1}{\sqrt{2}} (||\tilde{c}_1 - \tilde{A}_1^T y|| - (c_1(1) - A_1^T(1,:)y))
\]

\[
c_1(1) - A_1^T(1,:)y \geq 0,
\]  

(18)

\[
\min \frac{1}{\sqrt{2}} (||\tilde{c}_1 - \tilde{A}_1^T y|| - (c_1(1) - A_1^T(1,:)y))
\]

\[
c_1(1) - A_1^T(1,:)y \leq 0,
\]  

(19)

\[
\min ||c_1 - A_1^T y||
\]

\[
c_1(1) - A_1^T(1,:)y \leq 0.
\]  

(20)

Then we put the optimal solutions of these problem in (12) and (13). The smallest one would be the the optimal \( r \) value.

3 Numerical Results

In this section we present several examples showing the efficiency of the reduced problem compared to (6). To generate infeasible problems in the form (4), we have used the conic Farkas Lemma as follows:

Lemma 3.1 The set of linear inequalities

\[
c - A^T y \succeq_{Q_n} 0_m
\]  

(21)

has no solution if there exists a vector \( x \) such that

\[
x \succeq_{Q_n} 0_n, \ Ax = 0, \ c^T x < 0.
\]  

(22)

- Example: Let \( x = \text{rand}(n-1,1) \) and \( x(1) = \text{norm}(x(2 : end)) + 0.1 \). Obviously \( x \succeq_{Q_n} 0_n \) and further let \( A = (\text{null}(x^T))^T \) and \( c(1) = 0.01, c(2 : end) = -\text{rand}(n-1,1) - 0.01 \). By the conic Farkas lemma, for these data (22) is feasible, thus (21) is infeasible. We further modify this system to satisfy condition in Lemma 2.2. If \([m,n] = \text{size}(A)\), we let \( A = [0_{m \times 1} A], c = [0.01; c], x = [\text{norm}(x) + 0.1; x] \). For this modified system obviously (22) is feasible, thus

\[
c - A^T y \succeq_{Q_n} 0_n
\]

is infeasible. This procedure can be repeated to have several systems of different dimensions that are infeasible. In the following table we have summarized results for a few systems with various dimensions.

As we see, the lower dimensional problem is much faster than the original one specially in large dimensions. Thus under certain conditions, it is an efficient choice for the optimal corrections of infeasible conic linear inequalities.
Table 1: Comparison of problems (6) and (17)

| $m, n$          | (6) (time(sec), $||r||$)     | (17) (time(sec), $||r||$)     |
|-----------------|-----------------------------|-------------------------------|
| $k = 1, n_1 = 101, m = 99$ | (0.4, 5.7781)               | (0.06, 5.7781)                |
| $k = 1, n_1 = 501, m = 499$ | (14.1, 13.6389)             | (2.9, 13.6389)               |
| $k = 1, n_1 = 1001, m = 999$ | (103.5, 20.0493)            | (20.8, 20.0493)              |
| $k = 2, n_1 = 201, n_2 = 301, m = 199$ | (2.6, 22.9193)             | (0.85, 22.9193)              |
| $k = 2, n_1 = 501, n_2 = 701, m = 499$ | (30.9, 35.1854)            | (12.6, 35.1854)              |
| $k = 2, n_1 = 801, n_2 = 1001, m = 799$ | (123, 41.7916)             | (41.9, 41.7916)              |

4 Conclusions

In this paper, we have studied correcting a set of SOCLIQs by minimal changes in the vector $b$. We have shown that under certain conditions, we can achieve this goal by solving lower dimensional SOCO problems analogous to the case of linear inequalities in the nonnegative orthant. Finally, several illustrative examples are presented to show the efficiency of the new approach.

References