Dynamic Analysis of a Fractional Order Rössler System

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Abstract: Based on the qualitative theory, the existence and uniqueness of solution for the fractional order Rössler system is investigated theoretically in this paper. The stability of the corresponding equilibria is also argued similarly to the integer order counterpart. Finally, Numerical solutions, together with simulations are given to verify the correctness of our analysis.

Keywords: fractional order Rössler system; qualitative theory; numerical simulation

1 Introduction

The notion of fractional order derivative dates back three centuries [1], but its applications to physics and engineering have been attracted lots of attention recently. It has been found that in interdisciplinary fields, many systems can be described by fractional differential equations. For instance, viscoelastic systems [2], dielectric polarization [3], electrode-electrolyte polarization [4], electromagnetic waves [5], quantitative finance [6] and quantum evolution of complex systems [7].

It is known that some fractional order differential systems behave chaotically, e.g., the fractional order Chua circuit [8], the fractional order Duffing system [9], the fractional order jerk model [10], the fractional order Chen system [11], the fractional order Lü system [12], the fractional order Rössler system [13], the fractional order Arneodo system [14], and the fractional order Newton-Leipnik system [15].

Due to essential differences between ordinary differential equations and fractional order differential equations, most of characteristics or conclusions of the ordinary differential equations systems cannot be directly extended to the case of the fractional order differential equations systems. Therefore, the fractional order systems have been paid more attention. Recently, many efforts have been devoted to the study of chaotic dynamics of fractional order differential systems [16-18]. Many current results about fractional order chaotic systems, however, are attained only by numerical simulations. Because of low accuracy associated with some of the numerical methods or limitations of them to detect chaos [19,20], wrong results have been reported in special cases. Due to this deficiency, a logical need is observed to develop analytical methods in order to investigate chaos in fractional order systems. One of these methods which is constructed based on the stability analysis in fractional order systems [21]. In Ref.[22], some interesting results for stability analysis of fractional order time variant ordinary differential systems (ODEs) are presented by using the Hermit eigenvalues analysis. These results are directly extended based on the related results of commensurate integer order ODEs. Based on theoretical results, one can design practical control scheme for some fractional dynamical system. There are some differences between Lorenz system and Rössler system. The Lorenz system describes the main properties of dissipative systems, while Rössler system is an emanative system. It is important to study the solution of different systems. Furthermore, Rössler system is easy realized in experiments and practical, however, the output amplitudes of variables in the group of Lorenz model are much large, it is realized by scale narrowed in circuit, otherwise electronic components cannot work safely in circuit. Theoretically the existence of solutions of Rössler system is studied in this paper, so we can design and choose circuit element parameters in Rössler circuit to ensure the safety of working circuit.

In this paper, our aim is to investigate the existence and uniqueness about solution of the fractional order Rössler system, similar to the results presented in Ref. [23]. At the same time, the stability of equilibria of the system will also be analyzed. The paper is organized as follows. In Section 2, the definition of fractional order derivative and some of its basic properties are introduced. In Section 3, the stability of equilibria for the fractional order Rössler system are
the function $f$ (for both definitions) is

\[ \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau, \]  

where $\Gamma(.)$ is the gamma function and $n - 1 \leq \alpha < n$. The Caputo definition of the fractional derivative, which sometimes is called smooth fractional derivative, is described as

\[ \frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - \tau)^{m-\alpha-1} f(\tau) d\tau, & m - 1 < \alpha < m, \\ \frac{d^m f(t)}{dt^m}, & \alpha = m, \end{cases} \]  

where $m$ is the first integer larger than $\alpha$. For zero initial conditions, the Laplace transform of the fractional derivative (for both definitions) is

\[ L\{\frac{d^\alpha f(t)}{dt^\alpha}\} = s^\alpha L\{f(t)\}. \]  

In the rest of this paper, the notation $\frac{d^\alpha}{dt^\alpha}$ represents the Caputo fractional derivative of order $\alpha$.

A fractional order nonlinear system can be defined by the following model:

\[ \begin{cases} \frac{d^\alpha x}{dt^\alpha} = f(t, x), \\ x(0) = x_0, \end{cases} \]  

where $0 < \alpha < 1$.

**Theorem 1** (D. Matignon, [21]) Assume that $D = [0, T^*] \times [x_0 - \delta, x_0 + \delta]$ with some $T^* > 0$ and some $\delta > 0$, and let the function $f : D \to \mathbb{R}$ be continuous. Furthermore, define $T = \min\{T^*, (\frac{d\alpha+1}{\|f\|_\infty})^\frac{1}{\alpha}\}$, then there exists a function $x : [0, T] \to \mathbb{R}$ solving the initial value problem (4). Notice that $\|f\|_\infty$ is the norm of function $f$.

**Theorem 2** (D. Matignon, [21]) Assume that $D = [0, T^*] \times [x_0 - \delta, x_0 + \delta]$ with some $T^* > 0$ and some $\delta > 0$, and let the function $f : D \to \mathbb{R}$ be bounded on $D$ and fulfill a Lipschitz condition with respect to the second variable, i.e.,

\[ |f(t, x) - f(t, y)| \leq L|x - y| \]  

with some constant number $L > 0$ independent of $t, x, y$. Then denoting $T$ as Theorem 1, there exists at most one function $x : [0, T] \to \mathbb{R}$ solving the initial value problem (4).

**Remark 1** The results of Theorems 1 and 2 can be easily generalized to the initial value problem of the n-dimension fractional order differential equations

\[ \begin{cases} \frac{d^\alpha x(t)}{dt^\alpha} = f(t, X(t)), \\ X(0) = X_0, \end{cases} \]  

where $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{R}^n, 0 < \alpha_i < 1, i = 1, 2, \ldots, n$. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, system (6) is called a commensurate order system; otherwise, system (6) indicates an incommensurate order system.

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3 Stability analysis of fractional order Rössler system

3.1 Existence and uniqueness of solution

Rössler designed the Rössler attractor in 1976, and the originally theoretical equations were later found to be useful in modeling equilibrium in chemical reactions. The defining equations are:

\[
\begin{align*}
\frac{dx}{dt} &= -y - z, \\
\frac{dy}{dt} &= x + ay, \\
\frac{dz}{dt} &= b + z(x - c),
\end{align*}
\]

(7)

where \(x, y, z\) are the state variables, and \(a, b, c\) are three system parameters. Two types chaotic attractors were found in integer Rössler system: one is a single lobe chaotic attractor (spiral-type), the other is a more complicate chaotic attractor (screw-type).

Here, we consider the fractional order system denoted as system (8) below. The standard derivative is replaced by a fractional derivative as the followings:

\[
\begin{align*}
\frac{d^{\alpha_1}x}{dt^{\alpha_1}} &= -y - z, \\
\frac{d^{\alpha_2}y}{dt^{\alpha_2}} &= x + ay, \\
\frac{d^{\alpha_3}z}{dt^{\alpha_3}} &= b + z(x - c),
\end{align*}
\]

(8)

where \(0 < \alpha_i \leq 1\) \((i = 1, 2, 3)\) is derivative order.

**Theorem 3** The initial value problem of the fractional order Rössler system (8) can be represented in the following form:

\[
\begin{align*}
D^{\alpha}X(t) + x(t)BX(t) + E, t \in (0, T], \\
X(0) = X_0,
\end{align*}
\]

(9)

where \(\alpha = (\alpha_1, \alpha_2, \alpha_3)^T, 0 < \alpha_i < 1\) \((i = 1, 2, 3)\), \(X(t) = (x(t), y(t), z(t))\),

\[
A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix},
\]

some constant \(T > 0\), then it has a unique solution.

**Proof** Let \(F(X(t)) = AX(t) + x(t)BX(t) + E\), which is obviously continuous and bounded on the interval \([x_0 - \delta, x_0 + \delta]\) for any \(\delta > 0\). Furthermore, one has

\[
|F(X(t)) - F(X_1(t))| = |A(X(t) - X_1(t)) + x(t)BX(t) - x_1(t)BX_1(t)|
\]

\[
\leq \|A\| + \|B\|\|X(t) - X_1(t)\| \leq L\|X(t) - X_1(t)\|,
\]

where \(L = \|A\| + \|B\|\|\|X_0\| + \delta\| > 0\), \(X(t), Y(t) \in R^3\), \(\|\cdot\|\) and \(\|\cdot\|\) denote matrix norm and vector norm, respectively. The above inequality manifests that \(F(x(t))\) satisfies a Lipschitz condition. Based on the results of Theorems 1 and 2, we can conclude that the initial value problem of fractional order Rössler system has a unique solution.

3.2 The analysis of equilibria

Consider the following commensurate fractional order system:

\[
\frac{d^{\alpha}x}{dt^{\alpha}} = f(x),
\]

(10)

where \(0 < \alpha < 1\) and \(x \in \mathbb{R}^n\). The equilibrium points of system(10) are calculated by solving the following equation:

\[
f(x) = 0.
\]

(11)
The eigenvalues of Eq. (14) are at least one positive root when the corresponding characteristic polynomial is

\[ |\arg(\lambda)| > \frac{\alpha \pi}{2} \]  

(12)

Besides, consider the Cauchy problem of the nonlinear autonomous fractional order system described by:

\[
\frac{d^\alpha X(t)}{dt^\alpha} = F(X(t)),
\]

(13)

with the initial value \( X(0) = X_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T \), where

\[ F(X(t)) = \begin{pmatrix}
    f_1(x_1(t), x_2(t), \ldots, x_n(t)) \\
    f_2(x_1(t), x_2(t), \ldots, x_n(t)) \\
    \vdots \\
    f_n(x_1(t), x_2(t), \ldots, x_n(t))
\end{pmatrix}. \]

The equilibrium points of system (8) are asymptotically stable if all eigenvalues \( \lambda \) of the Jacobian matrix \( J = \frac{\partial F}{\partial X} \) evaluated at the equilibrium points satisfy:

\[
d^\alpha X(t) = F(X(t)),
\]

(13)

Theorem 4 (D. Matignon, [25]) The equilibrium points of system (10) are locally asymptotically stable if all eigenvalues \( \lambda \) of the Jacobian matrix \( J = \frac{\partial F}{\partial X} \) evaluated at the equilibrium points satisfy:

\[
|\arg(\lambda)| > \frac{\alpha \pi}{2}.
\]

(12)

Theorem 5 (D. Matignon, [28]) Let \( \dot{X} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \) is the equilibrium of system (13), i.e., \( \frac{d^\alpha X}{dt^\alpha} = F(\dot{X}) = 0 \), and \( \Lambda = \frac{\partial F}{\partial X} \) is the Jacobian matrix at the point \( \dot{X} \), then the point \( \dot{X} \) is asymptotically stable when \( |\arg(eig(\Lambda))| > \frac{\alpha \pi}{2} \), where \( \alpha_m = \max_{1 \leq i \leq n} \{\alpha_i\} \).

Proposition 1 If \( c^2 = 4ab \), the system (8) only one equilibrium \( S_0 = (\frac{e}{c}, \frac{f}{c}, \frac{e}{c}) \); if \( c^2 > 4ab \), it has two equilibria \( S_1 = (\frac{e + \sqrt{c^2 - 4ab}}{2a}, \frac{c + \sqrt{c^2 - 4ab}}{2a}, \frac{c + \sqrt{c^2 - 4ab}}{2a}) \) and \( S_2 = (\frac{e - \sqrt{c^2 - 4ab}}{2a}, \frac{c - \sqrt{c^2 - 4ab}}{2a}, \frac{-c - \sqrt{c^2 - 4ab}}{2a}) \).

Remark 2 We let the right of system (8) is equal to zero, we can conclude the proposition 1.

Proposition 2 With respect to the system (8), we have

1) The equilibrium \( S_0 \) is unstable when \( Q_0 < 0 \), where \( Q_0 = 1 + \frac{c}{2a} - \frac{ac}{2} \).

2) The equilibrium \( S_1 \) asymptotically stable if \( P_1 > 0, Q_1 > 0, \) \( P_1Q_1 + M > 0 \), where \( P_1 = \frac{c - M}{2a} - a, Q_1 = 1 + \frac{M + c - M}{2}, M = \sqrt{c^2 - 4ab} \).

3) The equilibrium \( S_2 \) asymptotically stable if \( P_2 > 0, Q_2 > 0, \) \( P_2Q_2 - M > 0 \), where \( P_2 = \frac{c + M}{2a} - a, Q_2 = 1 + \frac{M - c - M}{2a}, M = \sqrt{c^2 - 4ab} \).

Proof 1) The Jacobian matrix of system (8) at the point \( S_0 \) is

\[
\begin{pmatrix}
    0 & -1 & -1 \\
    1 & a & 0 \\
    \frac{c}{2a} & 0 & -\frac{c}{2a}
\end{pmatrix},
\]

whose corresponding characteristic polynomial is

\[ f(\lambda) = \lambda^3 - (a - \frac{c}{2})\lambda^2 + (1 + \frac{c}{2a} - \frac{ac}{2})\lambda \]

(14)

The eigenvalues of Eq.(14) are at least one positive root when \( 1 + \frac{c}{2a} - \frac{ac}{2} < 0 \). Therefore the equilibrium \( S_0 \) is unstable.

2) \( f(\lambda) = \lambda^3 - \left( \frac{c - \sqrt{c^2 - 4ab}}{2} - a \right)\lambda^2 + \left( 1 + \frac{c + \sqrt{c^2 - 4ab}}{2a} + \frac{\sqrt{c^2 - 4ab} - c}{2} \right)\lambda - \sqrt{c^2 - 4ab} \)

(15)
3) \[
\begin{pmatrix}
0 & -1 & -1 \\
1 & a & 0 \\
c - \frac{\sqrt{c^2 - 4ab}}{2a} & 0 & - \frac{\sqrt{c^2 - 4ab - c}}{2a}
\end{pmatrix}
\]
is the Jacobian matrix of system (8) at the point \(S_2\), whose corresponding characteristic polynomial is
\[
f(\lambda) = \lambda^3 - \left(\frac{c + \sqrt{c^2 - 4ab}}{2} - a\right)\lambda^2 + \left(1 + \frac{c - \sqrt{c^2 - 4ab}}{2a}\right)\lambda + \sqrt{c^2 - 4ab}.
\]
Let \(P_2 = \frac{c + M}{2} - a, Q_2 = 1 + \frac{c - M}{2} - a, M = \sqrt{c^2 - 4ab}\), from Routh-Hurwitz criteria, if \(P_2 > 0, Q_2 > 0, P_2Q_2 - M > 0\), the eigenvalues of Eq.(16) are all negative, so the equilibrium \(S_2\) is asymptotically stable.

4 Numerical methods and simulations

By exploiting the Adams-Bashforth-Moulton scheme, the fractional order system (8) can be discretized as followings:

\[
x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1 + 2)}(-y_{n+1} + z_{n+1}) + \frac{h^{q_2}}{\Gamma(q_2 + 2)}(x_{n+1} + ay_{n+1}) + \frac{h^{q_3}}{\Gamma(q_3 + 2)}(b + z_{n+1}(x_{n+1} - c)),
\]
\[
y_{n+1} = y_0 + \frac{h^{q_1}}{\Gamma(q_1 + 2)}(x_{n+1} + ay_{n+1}) + \frac{h^{q_2}}{\Gamma(q_2 + 2)}(x_{n+1} + ay_{n+1}) + \frac{h^{q_3}}{\Gamma(q_3 + 2)}(b + z_{n+1}(x_{n+1} - c)),
\]
\[
z_{n+1} = z_0 + \frac{h^{q_1}}{\Gamma(q_1 + 2)}(-y_{n+1} + z_{n+1}) + \frac{h^{q_2}}{\Gamma(q_2 + 2)}(x_{n+1} + ay_{n+1}) + \frac{h^{q_3}}{\Gamma(q_3 + 2)}(b + z_{n+1}(x_{n+1} - c)),
\]
\[
\begin{align*}
\beta_{i,j,n+1} &= \begin{cases} 
(n^{q_{i+1}} - (n - q_i)(n + 1)^{q_i} & j = 0, \\
(n - j + 2)^{q_i+1} - (n - j)^{q_i+1} - 2(n - j + 1)^{q_i+1} & 1 \leq j \leq n, \\
1 & j = n + 1,
\end{cases} \\
\gamma_{i,j,n+1} &= \frac{h^q}{q_i}((n - j + 1)^{q_i} - (n - j)^{q_i}), 0 \leq j \leq n, i = 1, 2, 3.
\end{align*}
\]

To verify the effectiveness of the obtained results, some numerical simulations for the fractional order Rössler system have been conducted. All the differential equations are solved by using the method mentioned above. In the following simulations, let \(q_1 = 0.91, q_2 = 0.95, q_3 = 0.98\) and \(h = 0.01\).

We set \(a = b = 1, c = 2\), i.e., \(c^2 = 4ab\), \(Q_0 < 0\). The equilibrium \(S_0\) of system (8) is unstable. Figure 1 shows that the state \(x(t)\) of system (8) is dramatically decreasing towards infinite, while the state \(y(t)\) of system (8) is rapidly increasing towards infinite, where the corresponding initial state are set as \(x_0 = 0.99, y_0 = -0.99, z_0 = 0.99\).

Figure 2 demonstrates that the equilibrium \(S_1\) of system (8) is stable, where \(a = 0.2, b = 5, c = 4\), i.e., \(P_1 > 0, Q_1 > 0, P_1Q_1 + M > 0\). Figure 3 shows that the equilibrium \(S_1\) of system (8) is unstable, where \(a = 1, b = 2, c = 4\), i.e., \(P_1 < 0\).

Figure 4 demonstrates that the equilibrium \(S_2\) of system (8) is stable, where \(a = 0.4, b = 5, c = 4\), i.e., \(P_2 > 0, Q_2 > 0, P_2Q_2 - M > 0\). Figure 5 shows that the equilibrium \(S_2\) of system (8) is unstable, where \(a = 1, b = 2, c = 4\), i.e., \(Q_2 < 0\).
Figure 1: The equilibrium $S_0$ of system (8) is unstable

Figure 2: The equilibrium $S_1$ of system (8) is stable
Figure 3: The equilibrium $S_1$ of system (8) is unstable.

Figure 4: The equilibrium $S_2$ of system (8) is stable.
5 Conclusions

The dynamics for the fractional order Rössler system has been extensively investigated in this paper. A rigorous proof of existence and uniqueness of solution for the fractional order Rössler system has been provided. It has been shown that the fractional order Rössler system has the similar equilibria and stability with the integer order counterpart. Finally, the numerical solutions and simulations are given to verify the feasibility of the results.

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