Approximate Symmetry Reduction Approach: Infinite Series Reduction to the Perturbed Burgers Equations

Suqing Qian¹ *, Li Wei²
¹, ² Faculty of Science, Jiangsu University
Zhenjiang, Jiangsu, 212013, P.R. China
¹ School of Mathematics and Statistic, Changshu Institute of Technology
Suzhou, Jiangsu, 215500, P.R. China
(Received 14 November 2010, accepted 13 March 2011)

Abstract: From the approximate symmetry point of view, the perturbed burgers equation is investigated. The symmetry of a system of the corresponding PDEs which approximates the perturbed burgers equation is constructed and the corresponding general approximates symmetry reduction is derived, which enables infinite series solutions and general formulae. Study shows that the similarity solutions of zero-order are in the forms of Kummer and hyperbolic tangent functions.

Keywords: perturbed burgers equation; approximate symmetry reduction; series reduction solution

1 Introduction

Nonlinear evaluation arise in many areas of science and technology. Many researchers have applied various methods to solution of nonlinear equations, such as CK’s direct method [1]. In many physical systems, the asymptotic analysis often leads to a nonlinear perturbed differential equation depending on a small parameter. In order to better understand such phenomena as well as further apply them in practical scientific research, it is important to study the properties of the perturbed partial differential equations(PDEs). The approximate symmetry perturbation approach [2] is a powerful method for such substitutions and has been applied to many classical PDEs [3]-[10]. Maybe in theory it is not difficult, however the concrete study shows that it indeed is not a trivial work to construct the different order perturbation solutions and even give the general formulas.

In this paper, we take the perturbed Burgers equation as an example ,which reads

\[ u_t = 2u u_x + u_{xx} + \varepsilon (3\alpha_1 u^2 u_x + 3\alpha_2 uu_{xx} + 3\alpha_3 u_x^2 + \alpha_4 u_{xxx}), \quad \varepsilon \ll 1, \]  

(1)

where \( \alpha_i \) are constants, \( \varepsilon \) is a perturbative parameter, and subscripts denote partial differentiation [11]. It appears in the long-wave, small amplitude limit of expended systems dominated by dissipation, but where dispersion is also present at a higher order. When, the equation is reduced to the sum of Burgers with the first higher-order equation of the Burgers hierarchy [12]. In this paper, we will take account of the special case of the perturbed Burgers equation

\[ u_t = 2u u_x + u_{xx} + \varepsilon \lambda (u^2 u_x + uu_{xx} + u_x^2 + u_{xxx}), \quad \varepsilon \ll 1, \]  

(2)

where \( \lambda \) is a constant.

2 Approximate symmetry reduction approach to Eq. (2)

According to the perturbation theory, in the actual case,solutions of perturbed partial differential equations can be expressed as the form containing finite portion of the series up to some order of a small parameter. Specifically, for Eq. (2),

* Corresponding author. E-mail address: qsp@cslg.edu.cn

Copyright © World Academic Press, World Academic Union
IUNS.2011.04.15/456
the solution can be supposed to be of the general form
\[ u = \sum_{j=0}^{\infty} \epsilon^j u_j, \]  
(3)

with \( u_j \) being functions of \( x \) and \( t \). Substituting Eq. (3) into Eq. (2) and vanishing the coefficients of all different powers of \( \epsilon \), we obtain a system of partial differential equations:

\[
\begin{align*}
O(\epsilon^0) : & \quad u_{0t} - u_{0xx} - 2u_0u_{0x} = 0, \\
O(\epsilon^1) : & \quad u_{1t} - \lambda u_{1xx} - 2(u_1u_{0x} + u_0u_{1x}) - 3\lambda(u_0^2 + u_0u_{0x} + u_0u_{0xx}) = 0, \\
O(\epsilon^2) : & \quad u_{2t} - \lambda u_{2xx} - 2(u_2u_{0x} + u_2u_{0x} + u_1u_{1x}) - 3\lambda(u_0u_{1x} + u_0u_{0x} + u_1u_{0x}) \nonumber \\
& \quad - 6\lambda(u_{0x}u_{1x} + u_0u_{10x}) = 0 \\
& \quad \cdots \cdots, \end{align*}
\]

(4a)

(4b)

(4c)

(4d)

Substituting Eq. (5) into Eq. (7) and eliminating \( u_j \) \((j = 0, 1, \cdots)\) in terms of Eq. (4) leads to the determining equations by vanishing all coefficients of the different partial derivatives of \( u_j \) for the unknown functions \( X, T \) and \( U_j \) \((j = 0, 1, \cdots)\), which are over-determined and have the solution

\[
\begin{align*}
X &= \frac{cx}{2} + x_0, \quad T = ct + t_0, \quad U_0 = -\frac{cu_0}{2}, \quad U_1 = -cu_1, \quad U_2 = -\frac{3cu_2}{2}, \\
U_3 &= -2cu_3, \quad U_4 = -\frac{5cu_4}{2}, \cdots, \quad U_j = -\frac{(j+1)u_j}{2}, \cdots, 
\end{align*}
\]

(8)
where \( c, x_0 \) and \( t_0 \) are arbitrary constants. Subsequently, solving the characteristic equations

\[
\frac{dx}{X} = \frac{dt}{T} = \frac{du_0}{U_0} = \frac{du_1}{U_1} = \frac{du_2}{U_2} = \cdots = \frac{du_j}{U_j} = \cdots
\]  

(9)

lead to the similarity solutions to Eq. (4) which can be distinguished in the following three subcases.

### 2.1 Symmetry perturbations of the Kummer function solutions

When \( c \neq 0, x_0 \neq 0, t_0 \neq 0 \) the similarity solutions are Eq. (9), we can choose the similarity variable as

\[
\xi = \frac{cx + 2x_0}{\sqrt{ct + t_0c}}
\]

(10)

then the similarity solutions for fields \( u_j \) are

\[
\begin{align*}
u_0 &= (ct + t_0)^{-1/2} V_0(\xi), \quad u_1 = (ct + t_0)^{-1} V_1(\xi), \quad u_2 = (ct + t_0)^{-3/2} V_2(\xi), \\
u_3 &= (ct + t_0)^{-2} V_3(\xi), \quad u_4 = (ct + t_0)^{-5/2} V_4(\xi), \quad \cdots, \quad u_j = (ct + t_0)^{-(j+1)/2} V_j(\xi), \quad \cdots
\end{align*}
\]

(11)

Accordingly, the perturbation series solution of Eq. (2) is of the form

\[
u = \sum_{j=0}^{\infty} \varepsilon^j (ct + t_0)^{-(j+1)} V_j(\xi)
\]

(12)

and the similarity reduction equation related to the similarity solutions

\[
\begin{align*}
O(\varepsilon^0) : \quad V_{0\xi} &= -\frac{c \xi}{2} V_{0\xi} - 2V_0 V_{0\xi} - \frac{c}{2} V_0, \\
O(\varepsilon^1) : \quad V_{1\xi} &= -\frac{c \xi}{2} V_{1\xi} - 2(V_0 V_{1\xi} + V_1 V_{0\xi}) - 3\lambda(V_0 V_{0\xi} + V_{0\xi}^2 + V_0^2 V_{0\xi}) - cV_1 \\
&\quad - \lambda V_{0,\xi,\xi,\xi}, \\
O(\varepsilon^2) : \quad V_{2\xi} &= -\frac{c \xi}{2} V_{2\xi} - 3\lambda(V_1 V_{0\xi} + V_{0\xi}^2 V_{1\xi} + V_0 V_{1\xi\xi}) - 2(V_1 V_{1\xi} + V_0 V_{2\xi} + V_2 V_{0\xi}) \\
&\quad - 6\lambda(V_0 V_{1\xi} + V_0 V_{1\xi\xi} + \frac{3c}{2} V_{2\xi} - \lambda V_{1,\xi,\xi,\xi} \\
O(\varepsilon^3) : \quad V_{3\xi} &= -\frac{c \xi}{2} V_{3\xi} - 3\lambda(V_2 V_{0\xi} + V_{1\xi}^2 + V_1 V_{2\xi} + V_{0\xi}^2 V_{2\xi} + V_0 V_{2\xi\xi} + V_2^2 V_{0\xi}) \\
&\quad - 2(V_2 V_{1\xi} + V_1 V_{2\xi} + V_0 V_{3\xi} + V_0 V_{3\xi}) - 6\lambda(V_0 V_2 V_{1\xi} + V_0 V_{2\xi} + V_0 V_2 V_{0\xi}) \\
&\quad - 2cV_3 - \lambda V_{2,\xi,\xi,\xi} \\
&\quad \cdots, \quad \cdots, \\
O(\varepsilon^j) : \quad V_{j\xi} &= -\frac{c \xi}{2} V_{j\xi} - \frac{j+1}{2} V_j - \lambda V_{j-1,\xi,\xi,\xi} - 2 \sum_{i=0}^{j-1} V_i V_{j-i,\xi} \\
&\quad - 3\lambda \sum_{i=0}^{j-1} V_i V_{j-1-i,\xi} + 2 \sum_{i=0}^{j-3} V_i V_{j-1-i,\xi} + V_j \sum_{i=0}^{j-1} \sum_{s=0}^{j-1-i} V_s V_{j-1-i-s,\xi} \\
O(\varepsilon^{j+1}) : \quad V_{j+1,\xi} &= -\frac{c \xi}{2} V_{j+1,\xi} - \frac{j+1}{2} V_j - \lambda V_{j-1,\xi,\xi,\xi} - 2 \sum_{i=0}^{j-1} V_i V_{j-i,\xi} \\
&\quad - 3\lambda \sum_{i=0}^{j-1} V_i V_{j-1-i,\xi} + 2 \sum_{i=0}^{j-3} V_i V_{j-1-i,\xi} + V_j \sum_{i=0}^{j-1} \sum_{s=0}^{j-1-i} V_s V_{j-1-i-s,\xi} \\
&\quad + V_{(j+1)/2,\xi} \\
&\quad j = 0, 2, 4, \cdots, 2n, \cdots \\
&\quad \cdots, \quad \cdots,
\end{align*}
\]

IJNS homepage: http://www.nonlinearscience.org.uk/
with $V_{-1,\xi} = 0, V_{-2,\xi} = 0, V_{-1,\xi,\xi,\xi} = 0, V_{-1,\xi,\xi} = 0, V_{-1} = 0$.

Eq. (13a) has the Kummer function solution

$$V_0 = \frac{2C_1\left(\frac{\varepsilon}{2} + C_1C_2K_1\left(\frac{3c+2C_1}{2c} - \frac{c_0^2}{4}\right) - 2c\left(-\frac{\varepsilon}{2} - C_1\right)K_2\left(\frac{3c+2C_1}{2c} - \frac{c_0^2}{4}\right)\right)}{c_0^2(K_1\left(\frac{3c+2C_1}{2c} - \frac{c_0^2}{4}\right) + K_2\left(\frac{3c+2C_1}{2c} - \frac{c_0^2}{4}\right))}$$

+ \frac{2}{c_0^2} \left(K_1\left(\frac{3c+2C_1}{2c} - \frac{c_0^2}{4}\right) + K_2\left(\frac{3c+2C_1}{2c} - \frac{c_0^2}{4}\right)\right),

(14)

where $C_1$ and $C_2$ are arbitrary constants, and the two types of Kummer functions $K_1(\mu, \nu, z)$ and $K_2(\mu, \nu, z)$ solve the equation

$$zy''(z) + (\nu - z)y'(z) - \mu y(z) = 0.$$

Remark: Infinite series Eq. (12) dominates superior convergence, since general term becomes infinitesimal for sufficiently large time $t$.

### 2.2 Symmetry perturbation of tanh function solution

When $c = 0, x_0 \neq 0$ and $t_0 \neq 0$, the similarity solutions are

$$u_0 = V_0(\xi), \quad u_1 = V_1(\xi), \quad u_2 = V_2(\xi), \quad u_3 = V_3(\xi), \quad u_4 = V_4(\xi), \ldots, \quad u_j = V_j(\xi), \ldots,$$

(16)

with the similarity variable $\xi = -\frac{t_{01} + t_0 x}{t_0}$, and the perturbation solution to Eq. (2) is

$$u = \sum_{j=0}^{\infty} \varepsilon^j V_j(\xi).$$

The similarity reduction equations related to similarity solutions

$$O(\varepsilon^0): \quad V_{0\xi\xi} = -2V_0 V_{0\xi} - \frac{x_0}{t_0} V_{0\xi\xi}$$

(18a)

$$O(\varepsilon^1): \quad V_{1\xi\xi} = -\frac{x_0}{t_0} V_{1\xi} - 2\left(V_0 V_{1\xi} + V_1 V_{0\xi}\right) - 3\lambda(V_{0\xi}^2 + V_0 V_{0\xi\xi} + V_{0\xi}^2 V_{0\xi}) - \lambda V_{0\xi\xi\xi}\xi$$

(18b)

$$O(\varepsilon^2): \quad V_{2\xi\xi} = -\frac{x_0}{t_0} V_{2\xi} - \lambda V_{1\xi\xi\xi}\xi - 2(V_2 V_{1\xi} + V_1 V_{1\xi\xi} + V_0 V_{2\xi}) - 3\lambda(V_1 V_{0\xi\xi} + V_0^2 V_{1\xi\xi} + V_0 V_{1\xi\xi\xi} - 6\lambda(\lambda V_{0\xi\xi\xi\xi\xi}\xi + V_0 V_{1\xi\xi\xi})$$

(18c)

$$O(\varepsilon^3): \quad V_{3\xi\xi} = -\frac{x_0}{t_0} V_{3\xi} - \lambda V_{2\xi\xi\xi}\xi - 2(V_3 V_{2\xi} + V_2 V_{1\xi\xi} + V_1 V_{2\xi\xi} + V_0 V_{3\xi}) - 6\lambda(V_0 V_{1\xi\xi\xi}\xi + V_0 V_{2\xi\xi\xi\xi} + V_0 V_{1\xi\xi\xi\xi\xi}\xi + V_1 V_{2\xi\xi\xi})$$

(18d)

$$\ldots \ldots \ldots$$

$$O(\varepsilon^j): \quad V_{j\xi\xi} = -\frac{x_0}{t_0} V_{j\xi} - \lambda V_{j-1,\xi\xi\xi}\xi - 2\sum_{i=0}^{j-1} V_i V_{j-1-i,\xi\xi} - 3\lambda\sum_{i=0}^{j-1} V_i V_{j-1-i,\xi\xi\xi} + 2\sum_{i=0}^{j-3} V_i V_{j-1-i,\xi\xi\xi}\xi$$

(18e)

$$+ \sum_{i=0}^{j-1} V_i \sum_{s=0}^{j-i-1} V_s V_{j-1-i-s,\xi\xi}\xi\xi$$

$$O(\varepsilon^{j+1}): \quad V_{j+1\xi\xi} = -\frac{x_0}{t_0} V_{j+1\xi} - \lambda V_{j,\xi\xi\xi}\xi - 2\sum_{i=0}^{j-1} V_i V_{j-i,\xi\xi}\xi - 3\lambda\sum_{i=0}^{j-1} V_i V_{j-i,\xi\xi\xi} + 2\sum_{i=0}^{j-3} V_i V_{j-i,\xi\xi\xi}\xi$$

(18f)

+ \sum_{i=0}^{j-1} V_i \sum_{s=0}^{j-i-1} V_s V_{j-i-s,\xi\xi}\xi\xi + V_{j+1,\xi\xi\xi}\xi\xi + 2V_{j+1,\xi\xi\xi\xi}\xi\xi\xi$$

$$\ldots \ldots \ldots$$
with $V_{-1, \xi} = 0, V_{-2, \xi} = 0, V_{-1, \xi, \xi} = 0, V_{-1, \xi, \xi, \xi} = 0, V_{-1} = 0$. Eq. (18a) has the tanh function solution

$$V_0 = \frac{1 - C_1 x_0 - 2 \tanh \left( \frac{\xi + C_2}{C_1} \right) t_0}{2}$$

(19)

where $C_1, C_2$ are arbitrary constants.

### 2.3 Symmetry perturbation of tanh function solution

When $c = 0$, $x_0 = 0$ and $t_0 \neq 0$, the similarity solution are

$$u_0 = V_0(\xi), u_1 = V_1(\xi), u_2 = V_2(\xi), u_3 = V_3(\xi), \ldots, u_j = V_j(\xi), \ldots,$$

(20)

with the similarity variable $\xi = x$. From Eq. (3), the perturbation series solution to Eq. (2) is

$$u = \sum_{j=0}^{\infty} \varepsilon^j V_j(\xi)$$

(21)

and the similarity reduction equation related are

$$O(\varepsilon^0) : \quad V_0 \xi \xi = -2V_0 V_0 \xi,$$

(22a)

$$O(\varepsilon^1) : \quad V_1 \xi \xi = -\lambda V_0 \xi \xi \xi - 2(V_0 V_1 \xi + V_1 V_0 \xi) - 3\lambda(V_0^2 V_1 \xi + V_0 V_1 \xi \xi + V_1 V_0 \xi \xi),$$

(22b)

$$O(\varepsilon^2) : \quad V_2 \xi \xi = -\lambda V_1 \xi \xi \xi - 2(V_0 V_2 \xi + V_1 V_1 \xi + V_2 V_0 \xi) - 3\lambda(V_0^2 V_1 \xi \xi + V_0 V_1 \xi \xi \xi + V_1 V_0 \xi \xi \xi + V_0 V_1 \xi \xi \xi + V_1 V_0 \xi \xi \xi)$$

$$-6\lambda(V_0 V_1 \xi + V_0 V_1 \xi \xi + V_0 V_1 \xi \xi \xi),$$

(22c)

$$O(\varepsilon^3) : \quad V_3 \xi \xi = -\lambda V_2 \xi \xi \xi - 2(V_0 V_2 \xi + V_1 V_1 \xi + V_2 V_0 \xi) - 3\lambda(V_0^2 V_1 \xi \xi \xi + V_0 V_1 \xi \xi \xi \xi + V_1 V_0 \xi \xi \xi \xi + V_0 V_1 \xi \xi \xi \xi + V_1 V_0 \xi \xi \xi \xi)$$

$$+6\lambda(V_0 V_1 \xi \xi + V_0 V_1 \xi \xi \xi + V_0 V_1 \xi \xi \xi \xi),$$

(22d)

$$\ldots \ldots \ldots$$

$$O(\varepsilon^j) : \quad V_j \xi \xi = -\lambda V_{j-1} \xi \xi \xi - 2 \sum_{i=0}^{j-1} V_i V_{j-i, \xi} - 3\lambda \sum_{i=0}^{j-1} V_i V_{j-i, \xi \xi} + 2 \sum_{i=0}^{j-3} V_{i, \xi} V_{j-i, \xi \xi \xi \xi} + \sum_{i=0}^{j-1} V_i \sum_{s=0}^{j-1-i} V_s V_{j-i-s, \xi}$$

(22e)

$$O(\varepsilon^{j+1}) : \quad V_{j+1} \xi \xi = -\lambda V_{j+1} \xi \xi \xi - 2 \sum_{i=0}^{j-1} V_i V_{j-i, \xi} - 3\lambda \sum_{i=0}^{j-1} V_i V_{j-i, \xi \xi} + 2 \sum_{i=0}^{j-3} V_{i, \xi} V_{j-i, \xi \xi \xi \xi} + \sum_{i=0}^{j-1} V_i \sum_{s=0}^{j-1-i} V_s V_{j-i-s, \xi} + V_{(j+1)/2, \xi}$$

(22f)

$$j = 0, 2, 4, \ldots, 2n, \ldots$$

$$\ldots \ldots \ldots$$

with $V_{-1, \xi} = 0, V_{-2, \xi} = 0, V_{-1, \xi, \xi} = 0, V_{-1, \xi, \xi, \xi} = 0, V_{-1} = 0$. Eq. (20a) has the tanh function solution

$$V_0 = \frac{\tanh \left( \frac{\xi + C_2}{C_1} \right)}{C_1}$$

(23)

where $C_1, C_2$ are arbitrary constants.
3 Conclusion and discussion

In summary, the perturbed Burgers equation is studied by applying the approximate symmetry approach. The symmetry of the corresponding PDEs which approximates the perturbed Burgers equation is constructed. The general approximate symmetry reductions and the general form of the infinity approximate series solutions are obtained respectively. Similarity equations of the zero-order are in the form of Kummer function and tanh function.

Nonetheless, we haven not mentioned the convergence of infinite series solutions because of its difficulty. Moreover, the approximate symmetry reduction approach can be used to search for similar results of other perturbed nonlinear differential equations with small parameters.

References


