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New Oscillation Criteria for Second Order Nonlinear Delay Differential Equations

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Abstract: New oscillation criteria for second order nonlinear differential equation

$$\left[a(t)\ |z'(t)|^{m-1}\ z'(t)\right]' + \sum_{i=1}^{n} q_i(t)\ f_i\left(y(\tau_i(t))\right) = 0$$

where $z(t) = y(t) + p(t)\ y(\sigma(t))$, for $t \geq t_0$, by using the so called a class of new function $\Phi(t,s,l)$ and H-method defined in the sequel. Furthermore, the results of the paper contain several results obtained previously as special cases.

Keywords: oscillation; second order; nonlinear differential equations; neutral type.

1 Introduction

In this paper, we are concerned with the oscillation behavior of solutions of the second order neutral differential equation of the form:

$$\left[a(t)\ |z'(t)|^{m-1}\ z'(t)\right]' + \sum_{i=1}^{n} q_i(t)\ f_i\left(y(\tau_i(t))\right) = 0 \tag{1}$$

where $z(t) = y(t) + p(t)\ y(\sigma(t))$, $m > 0$, for $t \geq t_0$.

We assume that:

(I_1) $a(t) \in C([t_0,\infty), (0,\infty))$; $a'(t) \geq 0$, $R(t) = \int_0^\infty a^{\frac{-1}{m}}(s)\ ds = \infty$,
(I_2) $p \in C([t_0,\infty), R)$; *and* $-1 < p(t) \leq p_0 \leq 1$, p_0 constant.
(I_3) $\sigma(t) \in C([t_0,\infty), R)$; *and* $\sigma(t) \leq t$ *for* $t \geq t_0$ *and* $\lim_{t\to\infty} \sigma(t) = \infty$,
(I_4) $\tau_i(t) \in C([t_0,\infty), R)$; *and* $\tau_i(t) \leq t$ *for* $\tau_i'(t) > 0$ *for* $t \geq t_0$
and $\lim_{t\to\infty} \tau_i(t) = \infty$, $i = 1, 2, .., n$
(I_5) $q_i(t) \in C([t_0,\infty), (0,\infty))$; *and* $i = 1, 2, \ldots, n$

The aim of this paper is to improve and extend some of the main results of [1, 2]. We will use the function class Υ to study the oscillatory behavior of Eq. (1). We say that a function $\Phi = \Phi(t,\ s, l)$ belongs to the function class Υ, denoted by $\Phi \in \Upsilon$ if $\Phi \in C(E, R)$ where $E = \{(t,s,l); t_0 \leq l \leq s \leq t < \infty\}$.
which satisfies $\Phi(t,t,l) = 0$, $\Phi(t,s,l) \neq 0$ for $l < s < t$, and has the partial derivative $\frac{\partial\Phi}{\partial s}$ on E such that $\frac{\partial\Phi}{\partial s}$ is locally integrable with respect to s in E.

Define the operator $A[.\ ; l, t]$ by

$$A[g; l, t] = \int_l^t \Phi^2(t,s,l)\ g(s)\ ds \tag{2}$$

For $t \geq s \geq l \geq t_0$ and $g(s) \in C[t_0,\infty)$, the function $\varphi = \varphi(t,s,l)$ is defined by

$$\frac{\partial\Phi}{\partial s} = \varphi(t,s,l)\ \Phi(t,s,l) \tag{3}$$

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It is easy to verify that $A\,[.\,;\,l\,,t\,]$ is linear operator and satisfies

$$A\,[g'\,;l\,,t\,]\;=-\,2\,A\,[g\,\phi\,;l\,,t\,]\;\;for\;g\in C'\,[t_{\,0}\,,\infty\,). \tag{4}$$

Define Also the function class Γ which will be needed extensively. Let, $D_0\;=\{\,(t\,,s\,)\,\in\,R^2\,;\;t\,>\,s\,>\,t_0\,\}$ and $D_0\,=\{\,(t\,,s\,)\in R^2\,;\;t>s>t_0\,\}$. We say that the function $H\in(D,R)$ belong to the class Γ, denoted by $H\in\Gamma$. If: $H(t,t)=0$ for $t\geq t_0\,,\;H\,(\,t\,,s)>0\;on\;(t\,,s)\in D_0$.
Thus it has a continuous and non positive partial derivative on D_0 with respect to the second variable, such that:

$$\frac{\partial}{\partial\,s}H(t\,,s\,)=-\,h\,(t\,,s)\;H\,(t\,,s)\qquad for\;\;(t\,,s)\in D_0\,. \tag{5}$$

where $h\in C(D,R)$. For a given functions $h\in C(D,R)$, $\rho\in C'\,(\,[t_0\,,\infty)\,,\;R^+\,)$ and $\eta\in C'\,(\,[t_0\,,\infty\,),\;R\,)$, we set

$$\lambda\,(t\,,s)=h\,(t\,,s)-\;\frac{\rho'\,(s)}{\rho\,(s)} \tag{6}$$

$$\theta_i\,(t,\,s)=Q_i(s)\;-\eta'\,(s)-\lambda\,(t\,,s)\;\eta\,(s) \tag{7}$$

where

$$Q_i(t)=\sum_{i=1}^{n}\mu_i\,q_i(t)\;\;[1-p\,(\tau_j(t))]^{\,m} \tag{8}$$

and

$$\phi_i\;(\,t\,,s\,)=Q^*_i(s)\;\;-\;\eta'\,(s\,)-\;\lambda\;(\,t\,,s\,)\;\,\eta\,(s) \tag{9}$$

where $Q^*_i(t)=\sum_{i=1}^{n}\mu_i\,q_i(t)$ for $i=1,2,\ldots,n$

2 Main results

In order to prove our main theorems we shall need the following lemma:

Lemma 2.1 *Suppose that $y(t)$ is an eventually positive of* (1). *Let*

$$z\,(t)=y\,(t)+p\,(t\,)\;y\,(\sigma\,(t\,)) \tag{10}$$

For (i) $0\leq p\,(\,t\,)\leq 1$ *, and (ii)* $-1<p_0<p\,(t\,)\leq 0$.
Then there exist a number $t_1\geq t_0$ *such that*
$z\,(t\,)>0\;,\;\;z'\,(\,t\,)>\;\;0$ *and* $z''\,(\,t\,)\leq 0\;,t\;\;\geq\;\;t_1$

The proof of the above lemma is similar to the proof of lemma 1 ([7]).

Theorem 2.2 *Let the conditions* (I_1-I_5) *be hold for* $0\leq p(t)\leq 1$. *Assume that for each* $l\geq t_0$ *there exists a function* $\Phi\in\Upsilon$ *such that*

$$\lim_{t\,\to\,\infty}\;\sup A\;[\sum_{i=1}^{n}\mu_i\,q_i(s)\;[1-p\,(\,\tau_i\,(s))]^{\,m}\;\;-(\,\frac{2\,\phi}{m+1})^{\,m+1}\;\frac{a\,(\tau_i(\,s))}{(\tau_i^{'}\,(s\,))^{\,m}}\;\;;\;l\,,t\,]>0\,, \tag{11}$$

where the operator A is defined by (2) *and $\phi=\phi\,(t\,,s\,,l\,)$ is defined by* (3), *then Eq.* (1) *is oscillatory.*

Proof. Suppose the contrary that there exists a solution $y(t)$ of Eq. (1) such that $y\,(t\,)\;>\;0\;,\;\;y\,(\sigma\,(t\,))\;>\;0$ and $y\,(\tau_j\,(t\,))\;\;>0$ for $t\geq t_1\geq t_0$. Then by lemma 2.1, we have
$z\,(t)>0\,,\;\;z'\,(t\,)>0$. But since $a\,(t\,)\;\;|z'\,(t\,)|^{\,m-1}\;\;z'\,(t\,)\;\;=\;\;a\;(t)\;\;(z'\,(t\,))^{\,m}$ is decreasing function then
$a\,(t)\;\;(z'\,(t\,))^{\,m}\;\;\leq\;\;a\;(\tau_i(t))\;\;(z'\,(\tau_i(t)\,))^{\,m}$ for $t\geq t_1$
Thus it follows that

$$\frac{z'\,(\tau_i\,(t))}{z'\,(t\,)}\geq\left(\frac{a\,(t\,)}{a\,(\tau_i(t))}\right)^{\frac{1}{m}} \tag{12}$$

Now from (1) and (I_5), we have

$\left[a(t)\ \ (z'(t))^{m}\right]' + \sum_{i=1}^{n} \mu_i\, q_i(t)\ y^{m}(\tau_i(t)) \quad \leq 0\ ,\quad t \geq t_1$, that is

$$\begin{aligned} 0 \ &\geq \left[a(t)\ \ (z'(t))^{m}\right]' + \sum_{i=1}^{n} \mu_i\, q_i(t)\ y^{m}(\tau_i(t)) \\ &= \left[a(t)\ \ (z'(t))^{m}\right]' + \sum_{i=1}^{n} \mu_i\, q_i(t)\ \left[z(\tau_i(t)) - p(\tau_i(t))\ y(\sigma(\tau_i(t)))\right]^{m} \\ &\geq \left[a(t)\ \ (z'(t))^{m}\right]' + \sum_{i=1}^{n} \mu_i\, q_i(t)\, z^{m}(\tau_i(t))\ \left[1 - p(\tau_i(t))\right]^{m} \end{aligned}$$

So we have

$$\frac{\left[a(t)\,(z'((t))^{m}\right]'}{(z(\tau_i(t)))^{m}} \leq -\sum_{i=1}^{n} \mu_i\, q_i(t)\ \left[1 - p(\tau_i(t))\right]^{m} \tag{13}$$

Define

$$w(t) = \frac{a(t)\ (z'(t))^{m}}{(z(\tau_i(t))^{m}}\ ,\qquad t \geq t_1 \tag{14}$$

Then from (12)-(14) we have

$$w'(t) \leq -\sum_{i=1}^{n} \mu_i\, q_i(t)\ \left[1 - p(\tau_i(t))\right]^{m} - m\ \tau_i'(t)\ \frac{w^{\frac{m+1}{m}}(t)}{a^{\frac{1}{m}}(\tau_i(t))} \tag{15}$$

Applying the operator $A\ [.\ ,l\ ,t]\ ,\ t \geq l \geq t_0$, to (15) we have

$$A[w'(t), l, t] \leq A[-\sum_{i=1}^{n} \mu_i\, q_i(s)\ \left[1 - p(\tau_i(s))\right]^{m} - m\ \tau_i'(s)\ a^{-\frac{1}{m}}(\tau_i(s))\ w^{\frac{m+1}{m}}(s)\ ; l, t] \tag{16}$$

Thus using (4) with the inequality (16), we have

$$A[\sum_{i=1}^{n} \mu_i\, q_i(s)\ \left[1 - p(\tau_i(s))\right]^{m}\ ;\ l, t] \leq 2A[w(s)\ \phi(s); l, t] - A\ [m\,\tau_i'(s)\, a^{-\frac{1}{m}}(\tau_i(s))\ w^{\frac{m+1}{m}}(s)\ ; l, t] \tag{17}$$

where $t \geq l \geq t_1$.
Now setting $F(w)\ = 2\phi\ w\ - m\,\tau_i'(s)\ a^{-\frac{1}{m}}(\tau_i(s))\ w^{\frac{m+1}{m}}\ , w > 0$, then by simple calculation, we obtain

$$w = \left(\frac{2\,\phi\, a^{\frac{1}{m}}(\tau_i(s))}{(m+1)\,\tau_i'(s)}\right)^{m},$$

and F(w) have the maximum

$$\left(\frac{2\,\phi}{(m+1)}\right)^{m+1} \frac{a(\tau_i(s))}{(\tau_i'(s))^{m}}\ ,$$

i.e.

$$F(w) \leq F_{\max} = \left(\frac{2\,\phi}{(m+1)}\right)^{m+1} \frac{a(\tau_i(s))}{(\tau_i'(s))^{m}}\quad , \tag{18}$$

Thus from (17) and (18), we get

$$A[\sum_{i=1}^{n} \mu_i\, q_i(s)\ \left[1 - p(\tau_i(s))\right]^{m} - \left(\frac{2\,\phi}{(m+1)}\right)^{m+1} \frac{a(\tau_i(s))}{(\tau_i'(s))^{m}}\ ;\ l, t] \leq 0\ , t \geq l \geq t_1\ .$$

Thus

$$\limsup_{t\to\infty} A[\sum_{i=1}^{n} \mu_i\, q_i(s)\ \left[1 - p(\tau_i(s))\right]^{m} - \left(\frac{2\,\phi}{(m+1)}\right)^{m+1} \frac{a(\tau_i(s))}{(\tau_i'(s))^{m}}\ ;\ l, t] \leq 0\ ,$$

This contradicts the assumption (11). This completes the proof. ■

Remark 2.1 *If $n = 1$ then Eq.* (1) *include the case discussed by [2] and Theorem 2.2 cover Theorem 2.1 of [2].*

Now choosing $\Phi\,(t\,,s\,,l\,)=\rho\,(s)\;\;(t-s)^{\,\alpha}\,(s\,-l)^{\,\beta}$ for $\alpha\;,\;\beta>\frac{1}{2}$ and $\rho\,(s)\;\in C'\;(\,[t_{\,0},\infty)\,,(\,0\,,\infty\,)\,)$, we have

$$\phi\,(t\,,s\,,l\,)=\frac{\rho'\,(s)}{\rho\,(s)}+\frac{\beta\,t-(\alpha\,+\beta\,)\,s+\alpha\,l}{(t-s)\;(s-l)}.$$

Thus by Theorem 2.2, we can easily prove the following oscillation result:

Theorem 2.3 *Let the hypotheses of Theorem 2.2 be satisfied. Assume that for each* $l\geq t_{\,0}$ *there exists a function* $\rho\,(s)\,\in C'\,(\,[t_{\,0}\,,\infty\,)\,,\;(0\,,\infty\,)\,)$ *and two constants* $\alpha,\beta>\frac{1}{2}$ *such that*

$$\lim_{t\rightarrow\infty}\,\sup\int_{l}^{t}[\,\rho^{\,2}\,(s)\,(t-s)^{\,2\,\alpha}\,(s-l)^{\,2\beta}\left[\begin{array}{l}\sum\limits_{i=1}^{n}\mu_{\,i}\,q_{i}(s)\;\;[1-p\,(\tau_{\,i}(s))]^{\,m}\,-\\ (\frac{2}{m+1})^{\,m+1}\,\frac{a\,(\tau_{\,i}(s))}{(\tau_{\,i}^{\prime}(s))^{\,m}}\,\left[\frac{\beta\,t-(\alpha+\beta)\,s+\alpha\,l}{(t-s)\;(s-l)}\right]^{\,m+1}\end{array}\right]\,ds>0\,. \tag{19}$$

Then Eq. (1) *is oscillatory.*

Note that Now if we choose $m=1,a(t)=1,\tau_{i}(t)=\lambda_{\,i}\,t\;\;(0<\lambda_{\,i}\leq1\,)$, and choose $\beta=1$ and $\rho\,(s)=1$ in Theorem 2.3, then we have the following result.

Corollary 2.1 *Let* $m=1,a(t)=1,\;\tau_{i}(t)=\lambda_{\,i}\,t\;\;(0<\lambda_{\,i}\leq1\,)$*, and for each* $l\geq t_{\,0}$ *there exists a constant* $\alpha>\frac{1}{2}$ *such that*

$$\lim_{t\rightarrow\infty}\sup\frac{1}{t^{\,2\,\alpha+1}}\int_{l}^{t}[\,(t-s)^{\,2\,\alpha}\,(s-l)^{\,2}\sum_{i=1}^{n}\lambda_{\,i}\,\mu_{\,i}q_{i}\,(s)\;[1-p\,(\tau_{i}(s))]\;ds\;>\;\frac{\alpha}{(2\alpha-1)\;(2\alpha+1)} \tag{20}$$

Then Eq. (1) is oscillatory. Our corollary includes corollary 2.1 in [2].

Corollary 2.2 *Let* $m=1,a(t)=1,\;\tau_{i}(t)=\lambda_{\,i}\,t\;\;(0<\lambda_{i}\leq1\,)$*, and for each* $l\geq t_{0}$ *there exists a constant* $\beta>\frac{1}{2}$ *such that*

$$\lim_{t\rightarrow\infty}\sup\frac{1}{t^{2\,\beta+1}}\int_{l}^{t}(t-s)^{\,2}\,(s-l)^{\,2\beta}\,\sum_{i=1}^{n}\lambda_{\,i}\,\mu_{\,i}\,q_{\,i}\,(s)\;\;[1-p\,(\tau_{i}(s))]\;ds\;>\;\frac{\beta}{(2\beta-1)\;(2\beta+1)} \tag{21}$$

Then Eq. (1) *is oscillatory.*

The proof is similar to proof of corollary 2.1 of [2].

Theorem 2.4 *Let the conditions* $(I_{1}-I_{5})$ *hold and suppose that* $H\,\in\,\Upsilon\,,\;\rho\,\in\,C'\,(\,[t_{\,0}\,,\infty\,)\;,R^{+}\,)$*and* $\eta\,\in\,C'\,(\,[\,t_{\,0}\,,\,\infty\,)\,,\,R\,)$*, then Eq.* (1) *is oscillatory provided that one of the following conditions hold*
(c_{1}) $0\leq p(t)\leq1$ *and*

$$\lim_{t\rightarrow\infty}\;\sup\frac{1}{H\,(t\,,t_{\,0})}\;\int_{l}^{t}H(t\,,\;s)\;\rho(s)\;\left[\theta_{\,i}(t\,,s\,)\;\;-\frac{k\,\tau_{i}^{'}(s)}{a^{\frac{1}{m}}\,(s)}\;\;|\lambda\,(t\,,\;s)|^{\,m+1}\right]\;ds=\infty \tag{22}$$

(c_{2}) $-1<p_{0}\leq p(t)\leq0$*, and*

$$\lim_{t\rightarrow\infty}\;\sup\frac{1}{H\,(\,t\,,t_{\,0})}\int_{t_{0}}^{t}H(t\,,\;s)\;\rho\,(s)\;\left[\phi_{\,i}(t\,,s\,)\;\;-\frac{k\,\tau_{i}^{'}(s)}{a^{\frac{1}{m}}\,(s)}\;\;|\lambda\,(t\,,s\,)|^{m+1}\right]\;ds=\infty \tag{23}$$

where $k=(\,m+1\,)^{\,-\,(\,m+1\,)}$

Proof. Consider the case (c_{1}), let y(t) be a nonoscillatory solution of (1) w.l.g, we may assume that *y(t)>0* for $t\,\geq\,t_{0}$. The case of *y(t)<0* is similar by the substitution $u=-y$. Now going through as in Theorem 2.1 in [1] we have $y\,(\,\tau_{i}(t))\geq[\,1-p\,(\tau_{i}(t))]\,z(\tau_{i}(t))$, thus from (1) and (I_{5}) we have

$$\begin{array}{rl}0&\geq\left[a\,(t)\;\;|\,z'\;(t\,)|^{\,m-1}\,z'\,(\,t\,)\right]^{\,\prime}+\sum_{i=1}^{n}\,\mu_{\,i}\,q_{i}(t)\;y^{\,m}(\tau_{i}(t))\\ &\geq\;\left[a\,(t)\;\;|\,(z'\,(t))|^{\,m-1}\;z'\,(t)\right]^{\,\prime}+\sum_{i=1}^{n}\,\mu_{\,i}\,q_{i}(t\,)\;\;[1-p\,(\tau_{i}(t))]^{\,m}\;z^{\,m}\,(\tau_{i}(t))\end{array} \tag{24}$$

Now define

$$v\,(t) = \rho\,(t)\ \left[\frac{a\,(t)\ \ |z'\,(t)|^{\,m\,-1}\,z'\,(t)}{z^{\,m}\,(\tau_i(t))} -\ \eta\,(t\,)\right]\ \ ,\ \ t \geq t_0 \tag{25}$$

Differentiating (25) and using (24) we get

$$v'(t)\ \leq \frac{\rho'\,(t)}{\rho\,(t)}\ v\,(t)\ -\rho\,(t)\,\eta'\,(t)\ -\rho\,(t)\ \sum_{i=1}^{n}\mu_i\,q_i(t)\ \left[1-p\,(\tau_i(t))\right]^{m} - \frac{m\,\rho\,(t)\,\tau_i(t)}{a^{\frac{1}{m}}\,(t)}\ \left[\frac{v\,(t)}{\rho\,(t)} - \eta\,(t)\right]^{\frac{m+1}{m}} \tag{26}$$

Thus it follows from (8) that

$$v'\,(t)\ \leq\ -\,\rho\,(t)\ \left[Q\,(t) - \eta'\,(t)\right] + \frac{\rho'\,(t)}{\rho\,(t)}\ v\,(t)\ \ - \frac{m\,\rho\,(t)\,\tau_i'\,(t)}{a^{\frac{1}{m}}\ (t)}\ \left[\frac{v\,(t)}{\rho\,(t)} - \eta\,(t)\right]^{\frac{m+1}{m}} \tag{27}$$

Thus by (7), we get

$$\begin{aligned}&\textstyle\int_T^t H\,(t\,,\,s)\ \rho(s)\ \left[\theta_i\,(t\,,s) + \lambda\,(t\,,s)\ \eta\,(s)\right]\ ds \leq\ H\,(t\,,T)\ v\ (T) - \int_T^t H\ (\,t\,,s\,)\ \lambda\ (\,t\,,s\,)\ v\ (s)\ ds\\ &-m\ \textstyle\int_T^t\ \frac{H\,(t\,,s)\,\rho\,(s)\ \tau_i'(s)}{a^{\frac{1}{m}}\,(s)}\ \ \left|\frac{v\,(s)}{\rho\,(s)} - \eta\ (s)\right|^{\frac{m+1}{m}}\ ds\end{aligned}$$

Hence

$$\begin{aligned}&\textstyle\int_T^t H\ (t\,,s\,)\ \rho\ (s)\ \theta_i\ (t\,,s)\ ds \leq H\ (\,t\,,T\,)\ v\ (T) + \int_T^t H\ (t\,,s\,)\ \rho(s)\ |\,\lambda\ (t\,,s)\,|\ \left|\frac{v\,(s)}{\rho\,(s)} - \eta\ (s)\right|\ ds\\ &-m\ \textstyle\int_T^t\ \frac{H\,(t\,,s\,)\,\rho\,(s)\,\tau_i'(s)}{a^{\frac{1}{m}}\,(s)}\ \ \left|\frac{v\,(s)}{\rho\,(s)} - \eta\ (s)\right|^{\frac{m+1}{m}}\ ds\end{aligned} \tag{28}$$

Following [1], we have

$$\int_T^t H\ (t\,,s\,)\ \rho\ (s\,)\ \theta_i\,(t\,,s\,)\ ds \leq H\ (t\,,T)\ v\ (T) + k\int_T^t \frac{H\ (t\,,s)\ \rho\ (s)\ \tau_i'\,(s)}{a^{\frac{1}{m}}\ (s)}\ \ |\,\lambda\ (t\,,s)\,|^{m+1}\ ds \tag{29}$$

Setting $T = T_0$, so $\int_T^t\ H\ (t\,,s\,)\ \rho\ (s\,)\ \ \left[\theta_i\ (t\,,\,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}(s)}\ \ |\,\lambda\ (t\,,s)|^{m+1}\right]\ ds \leq H\ (t\,,T_0)\ \ v\ (T_0)$

Thus by (6), we obtain

$$\begin{aligned}&\textstyle\int_{t_0}^t\ H\ (t\,,s)\ \rho\ (s)\ \left[\theta_i\ (t\,,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}\,(s)}\ |\,\lambda\ (t\,,s\,)\,|^{m+1}\right]\ ds\\ &= \left(\textstyle\int_{t_0}^{T_0} + \int_{T_0}^t\right)\ H\ (t,\,s)\ \rho\ (s)\ \left[\theta_i\,(t\,,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}\,(s)}\ \ |\lambda\ (t\,,s)|^{m+1}\right] ds\\ &\leq H\ (t\,,s)\ \left(\ \textstyle\int_{t_0}^{T_0}\rho(s)\ \left[\theta_i\,(t\,,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}\,(s)}\ \ |\,\lambda\ (t\,,s)|^{m+1}\right]\ ds + \ |\,v\ (T_0)\,|\right)\end{aligned} \tag{30}$$

Dividing by) $H\ (\,t\,,t_0)$ and taking the limits as $t \to \infty$, we get a contradiction to the condition (22). This completes the proof of case (c_1) . ■

In the case (c_2), we follow Theorem 2.1 of [1] to get

$$v'\,(t\,)\ \leq\ -\rho\,(t)\ [Q_i^*(t) -\ \eta'\,(t)]\ +\ \frac{\rho'\,(t)}{\rho\,(t)}\ v\,(t) - \frac{m\,\rho\,(t)\tau_i'\,(t)}{a^{\frac{1}{m}}\ (t)}\ \ \left[\frac{v\,(t)}{\rho\,(t)}\ -\ \eta\,(t\,)\right]^{\frac{m+1}{m}}$$

The rest of the proof is similar to that of case (c_1).

Remark 2.2 *If* $\sum\limits_{i=1}^{n} q_i\ (t)\ f_i\ (y\ (\tau_i\ (t))\,) = q_i\ (t)\ \ |y\ (t-\tau)\,|^{\,\alpha-1}\ y\ (t-\tau) + q_i\,(t)\ \ |y\ (t-\tau)\,|^{\,\beta-1}\ y\ (t-\beta\,)$
Then Theorem 2.3 includes Theorem 2.1 of [1].

Theorem 2.5 *Let the hypotheses of Theorem 2.3.be satisfied assume that*

$$0 < \inf_{s \geq t_0} \left\{ \lim_{t \to \infty} \inf \frac{H(t,s)}{H(t,t_0)} \right\} < \infty. \tag{31}$$

and

$$\lim_{t \to \infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{H(t,s)\,\rho(s)\,a(s)}{\left(\tau_i'(s)\right)^m} \left|\lambda(t,s)\right|^{m+1} ds < \infty \tag{32}$$

Then Eq(1) *is oscillatory provided that one of the following conditions hold*
(c_1) $0 \leq p(t) \leq 1$. *and there exists* $\psi \in ([t_0, \infty), R)$ *such that*

$$\int_{t}^{\infty} \frac{\rho(s)\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left(\frac{\psi(s)}{\rho(s)} - \eta(s)\right)^{\frac{m+1}{m}} ds = \infty\,. \tag{33}$$

and for any $T \geq t_0$.

$$\lim_{t \to \infty} \sup \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\,\rho(s) \left[\theta_i(t,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left|\lambda(t,s)\right|^{m+1}\right] ds \geq \psi(T) \tag{34}$$

where $\psi(s) = \{\psi(s), 0\}$.
(c_2) $-1 < p_0 \leq p(t) \leq 0$. *and there exists* $\psi \in ([t_0, \infty), R)$ *such that* (33) *holds and for all* $T \geq t_0$.

$$\lim_{t \to \infty} \sup \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\,\rho(s) \left[\varphi_i(t,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left|\lambda(t,s)\right|^{m+1}\right] ds \geq \psi(T) \tag{35}$$

Proof. Proceeding as in the proof of case (c_1) of theorem 2.3. We get (28) and (29) from (29), we get

$$\lim_{t \to \infty} \sup \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\,\rho(s) \left[\theta_i(t,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left|\lambda(t,s)\right|^{m+1}\right] ds \leq v(T)$$

For all $t \geq T \geq T_0$
Also, by (33), we have

$$\psi(T) \leq v(T),\ T \geq T_0 \tag{36}$$

Define

$$\mathrm{P}(t) = \frac{1}{H(t,T_0)} \int_{T_o}^{t} H(t,s)\,\rho(s)\,\left|\lambda(t,s)\right| \left|\frac{v(s)}{\rho(s)} - \eta(s)\right| ds\,.$$

and

$$\mathrm{O}(t) = \frac{m}{H(t,T_0)} \int_{T_0}^{t} \frac{H(t,s)\,\rho(s)\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left|-\frac{v(s)}{\rho(s)} - \eta(s)\right|^{\frac{m+1}{m}} ds$$

Then by (28) and (34), we see that

$$\left[\mathrm{O}(t) - \mathrm{P}(t) \leq v(T_0) - \int_{T}^{t} H(t,s)\,\rho(s)\,\theta_i(t,s)\,ds\right]$$

Thus

$$\lim_{t \to \infty} \inf\left[\mathrm{O}(t) - \mathrm{P}(t)\right] \leq v(T_0) - \psi(T_0) < \infty \tag{37}$$

Now claim that

$$\int_{T_0}^{\infty} \frac{\rho(s)\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left|-\frac{v(s)}{\rho(s)} - \eta(s)\right|^{\frac{m+1}{m}} ds < \infty \tag{38}$$

If this is false, then by (37), there exists a sequence $\{T_n\}_{n=1}^{\infty}$ in $[t_0, \infty)$ with $\lim_{t\to\infty} T_n = \infty$ satisfying

$$\lim_{t \to \infty} \left[\mathrm{O}(T_n) - \mathrm{P}(T_n)\right] = \lim_{t \to \infty} \inf\left[\mathrm{O}(t) - \mathrm{P}(t)\right] < \infty$$

Following the same lines as [1], we have

$$\lim_{n\to\infty} \frac{\mathrm{P}^{m+1}(T_n)}{\mathrm{O}^{m}(T_n)} = \infty \tag{39}$$

On the other hand, by Holder's inequality with the definition of P, we get

$$P(T_n) \le \left(\frac{m}{H(T_n,T_o)} \int_{T_o}^{T} \frac{H(T_n,s)\,\rho(s)\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left| \frac{v(s)}{\rho(s)} - \eta(s) \right|^{\frac{m+1}{m}} \right)^{\frac{m}{m+1}} \times \left(\frac{1}{m^m H(T_n,T_0)} \int_{T_o}^{T_n} \frac{H(T_n,s)\,a(s)\,\rho(s)}{(\tau_i'(s))^m} \,|\lambda(t,s)|^{m+1}\, ds \right)^{\frac{1}{m+1}} .$$

and, accordingly, we obtain,

$$\frac{\mathrm{P}^{m+1}(T_n)}{\mathrm{O}^{m}(T_n)} \le \frac{1}{m^m H(T_n,T_o)} \int_{T_o}^{T_n} \frac{H(T_n,s)\,a(s)\,\rho(s)}{(\tau_i'(s))^m} \,|\lambda(t,s)|^{m+1}\, ds$$

So, because of (39), we have

$$\lim_{t\to\infty} \frac{1}{H(T_n,T_o)} \int_{T_o}^{T_n} \frac{H(T_n,s)\,a(s)\,\rho(s)}{(\tau_i'(s))^m} \,|\lambda(t,s)|^{m+1}\, ds = \infty$$

This leads to

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T_o)} \int_{T_o}^{t} \frac{H(T_n,s)\,a(s)\,\rho(s)}{(\tau_i'(s))^m} \,|\lambda(t,s)|^{m+1} ds = \infty$$

Contradicting (32). Therefore (38) holds. Now, in view of (36), from (38), we have

$$\int_{T_o}^{\infty} \frac{\rho(s)\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left(\frac{\psi(s)}{\rho(s)} - \eta(s) \right)^{\frac{m+1}{m}} ds \le \int_{T_o}^{\infty} \frac{\rho(s)\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \left| \frac{v(s)}{\rho(s)} - \eta(s) \right|^{\frac{m+1}{m}} ds < \infty$$

this contradicts (33), and the proof is completed. ■

Theorem 2.6 *Let the hypotheses of Theorem 2.3 be satisfied. Suppose that* (31) *holds, and*

$$\lim_{t\to\infty} \inf \frac{1}{H(t,t_o)} \int_{t_o}^{t} \frac{H(t,s)\,a(s)\,\rho(s)}{(\tau_i'(s))^m} \,|\lambda(t,s)|^{m+1}\, ds < \infty . \tag{40}$$

Then Eq (1) *is oscillatory provided that one of the following conditions hold*
(c_1), and there exists $\psi \in ([t_0,\infty), R)$ such that (33) *holds. and for any $T \ge t_0$.*

$$\lim_{t\to\infty} \inf \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\,\rho(s) \left[\theta_i(t,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \,|\lambda(t,s)|^{m+1} \right] ds \ge \psi(T) \tag{41}$$

(c_2), and there exists $\psi \in ([t_0,\infty), R)$ such that (33) *holds. and for any $T \ge t_0$.*

$$\lim_{t\to\infty} \inf \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\,\rho(s) \left[\varphi_i(t,s) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}(s)} \,|\lambda(t,s)|^{m+1} \right] ds \ge \psi(T) \tag{42}$$

Theorem 2.7 *Let the conditions of Theorem 2.3 be satisfied. Suppose that* (31) *holds, then Eq.* (1) *is oscillatory provided that one of the following conditions hold*
(c_1), and for any $T \ge t_0$.

$$\lim_{t\to\infty} \inf \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\,\rho(s)\,\theta_i(t,s)\, ds \le \infty \tag{43}$$

and there exists $\psi \in ([t_0,\infty), R)$ such that (33) *holds, and* (41) *hold.*
(c_2), and for any $T \ge t_0$.

$$\lim_{t\to\infty} \inf \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\,\rho(s)\,\varphi_i(t,s)\, ds \le \infty \tag{44}$$

and there exists such that (33) *holds, and* (42) *hold.*

3 Example

Example 1 *Consider the following equation*

$$\left[\, | \, z' \, (t) \, |^{\, m-1} \, z' \, (t) \right]' + \sum_{i=1}^{2} \frac{a_i}{t^2} \, e^{\lambda_i \mu_i t} \, y \, (\lambda_i t) = 0 \tag{45}$$

where $z\,(t) = \, y\,(t) + p_0\, y\,(t\,-1)\,, a(t) = 1, -1 < p_0 < 1, m > 0, \mu_i > 0, 0 < \lambda_i < 1, q_1 = \frac{a_1}{t^2}, q_2 = \frac{a_2}{t^2}, a_i > 0, \tau_i(t) = \lambda_i t, t \geq 1, f_i\,(y\,(\tau_i\,(t))) = e^{\lambda_i \mu_i t}\, y\,(\lambda_i t)\,, i = 1, 2, \sigma\,(t) = t - 1, \lim_{t\to\infty} \tau_i\,(t) = \infty, q_1, q_2 > 0$

For Theorem 2.5, let $H\,(t\,, s) = (\,t - s\,)^{\,m+1}\,,\ \rho\,(t) = t^{-\,(m+1)}\,,\ \eta\,(t) = 0$ *. then* $h\,(t\,,\,s) = \frac{m+1}{t-s}\,,\ \lambda\,(t\,,\,s) = \frac{(m+1)\,t}{(t-s)\,s}$ *and Eq.* (32)

$\lim_{t\to\infty} \sup \frac{1}{H\,(t\,,t_o)} \int_{t_o}^{t} \frac{H\,(t\,,s)\;a(s)\;\rho(s)}{\left(\tau_i'(s)\right)^m} \; |\lambda\,(t\,,s)\,|^{\,m+1}\; ds \;=\; \lim_{t\to\infty} \sup \frac{(\,m+1)^{\,m+1}\; t^{\,m+1}}{\lambda_i^m\;(t-t_0)^{\,m+1}} \int_{t_0}^{t} s^{-\,2\,(m+1)}\, ds \;<\; \infty$

Now, we consider the following two cases:

Case 1: $0 \leq p_o \leq 1$.

$$Q_i\,(t) = \sum_{i=1}^{2} \mu_i\, q_i(\,t\,)\;\left[1 - p\,(\tau_i\,(t))\right]^m = \sum_{i=1}^{2} \mu_i\,(1 - p_o)^m\; \frac{q_i}{t^2}$$

and

$$\theta_i\,(t\,,\,s) = Q_i\,(\,s\,) - \eta'\,(\,s\,) - \lambda\,(t\,,s\,)\;\eta'\,(\,s) = Q_i\,(\,s\,) = \sum_{i=1}^{2} \frac{\mu_i\,(1-p_o)^m\; a_i}{t^2}$$

Then the Eq. (34) *takes the form*

$$\begin{aligned}
&\lim_{t\to\infty} \sup \frac{1}{H\,(t\,,T)} \int_T^t H\,(\,t\,,s\,)\;\rho(s)\;\left[\theta_i\,(\,t\,,s\,) - \frac{k\,\tau_i'(s)}{a^{\frac{1}{m}}(s)}\;|\,\lambda\,(\,t\,,s\,)\,|^{\,m+1}\right]\; ds \\
&= \lim_{t\to\infty} \sup \frac{1}{(t-T\,)^{\,m+1}} \int_T^\infty (\,t\,-s\,)^{\,m+1}\; s^{-\,(m+1)} \left[\frac{\sum_{i=1}^{2}\mu_i\,a_i\,(1-p_o)^m}{s^2} - \frac{\lambda_i\,w\,(m{+}1)^{m+1} t^{\,m+1}}{(t-s)^{\,m+1} s^{\,m+1}}\right] ds \\
&= \frac{\sum_{i=1}^{2}\mu_i\,a_i\,(1-p_o)^m}{T^{\,m+1}} - \frac{\lambda_i}{(2m{+}1)}\;\frac{1}{T^{\,2\,m+1}} \geq \frac{(1-p_o)^m\;\sum_{i=1}^{2}\mu_i\,a_i}{T}
\end{aligned}$$

For $t > T \in (-\infty, 0)$

Let $\psi\,(\,T\,) \;=\; \frac{(1-p_o)^m\;\sum_{i=1}^{2}\mu_i\,a_i}{T}$*, from* (33) *we have*

$$\int_T^\infty \frac{\rho\,(\,s\,)\;\tau_i'\,(s)}{a^{\frac{1}{m}}\,(s)}\;\left(\frac{\psi\,(\,s\,)}{\rho\,(s)} - \eta\,(\,s)\right)^{\frac{m+1}{m}} ds = \lambda_i\;(1 - p_0)^{\,m+1}\left(\sum_{i=1}^{2}\mu_i\,a_i\right)^{\frac{m+1}{m}}\;\int_T^\infty ds = \infty$$

Hence, it follow from Theorem 2.5 that Eq. (45) *is oscillatory.*

Case 2: $-1 < p_0 \leq 0$*. Note that*

$$\phi_i\,(t\,,s\,) = Q_i^*(\,t\,) = \frac{\sum_{i=1}^{2}\mu_i\,a_i}{t^2}$$

The rest of the proof is similar to that of case 1. Hence by Theorem 2.5 Eq. (45) *is oscillatory.*

In the above example, if $m = 1$, $\sum_{i=1}^{2} a_i\,\mu_i\,\lambda_i \geq 2$, $i = 1, 2$ *and* $p_0 = \frac{1}{2}$*, then from* (22) *in corollary 2.2, we have the left hand side is in the from*

$$\begin{aligned}
&\lim_{t\to\infty} \sup \frac{1}{t^{\,2\,\beta+1}} \int_l^t (\,t\,-s\,)^2\,(s-l)^{\,2\,\beta} \sum_{i=1}^{n} \lambda_i\,\mu_i\,q_i(s)\;\left[1 - p(\tau_i(s)\,)\right]\; ds = \\
&\lim_{t\to\infty} \sup \frac{1}{2\,t^3} \int_l^t .\;\frac{(t-s)^2\,(s-l)^2}{s^2} \sum_{i=1}^{2} \lambda_i\,\mu_i\,a_i\, ds = \frac{1}{6} \sum_{i=1}^{2} \lambda_i\,\mu_i\,a_i \geq \frac{1}{3}
\end{aligned} \tag{46}$$

where $\beta = 1$*, and the right hand side.*

$$\frac{\beta}{(2\beta\;-1)\;(2\,\beta+1)} \;=\frac{1}{3} \tag{47}$$

From (46) *and* (47)*, the inequality* (22) *holds then Eq.* (45) *is oscillatory.*

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