ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.12(2011) No.1,pp.42-47

An Analysis of Two Iterative Techniques for Solution of Cahn-Hilliard Equation

S. Hussain, A. Shah *
Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.
(Received 9 January 2011, accepted 19 May 2011)

Abstract: In this article, we provide an approximate analytical solution to Cahn-Hilliard (CH) equation by using Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM). These methods are simple in calculations and produce a good and satisfactory results. Both the methods have the ability to solve linear and nonlinear differential equations with acceptable accuracy. The approximate analytical solution of the problem is calculated in form of a converging series with easily calculated factors.

Keywords: Cahn-Hilliard Equation; Homotopy Perturbation Method; Variational Iteration Method

1 Introduction

The solution of nonlinear equations is the most important field of applied mathematics. Since, nonlinear equations comes from almost all areas of science, engineering and social sciences. Also, most of the physical processes in nature are fundamentally nonlinear and are delineated by nonlinear equations. In general, it is very difficult to solve such equations exactly due to their complicated nature. In the recent years, there has been great advancement in iterative techniques to solve such types of problems. Some of them are Homotopy Perturbation Method (HPM)[1–3], Adomian Decomposition Method (ADM) [4–6], Homotopy Analysis Method (HAM) [7, 8] and Variational Iteration Method (VIM) [9–11]. Nonlinear equations can easily be solved by using HPM and VIM without any transformation, linearization or use of small parameters like in classical methods. Both methods provide magnificent solutions with high accuracy. The computations become more simpler due to availability of many symbolic computation tools (like Mathematica and Maple). In this work, we applied the HPM and VIM for finding an approximate analytical solution to the Cahn Hilliard equation.

Cahn Hilliard equation plays an important role in phase separation models and describes number of phenomenon in binary systems, like interfacial dynamics, phase transition effects and spinodal decomposition.

The rest of the paper is organized as follows; In Section 2, the basic introduction and implementation of the homotopy perturbation method is given. In Section 3, basic idea and application of the variational iteration method is given. Comparison of solutions obtained by HPM and VIM with exact solution are given in subsections 2.1 and 3.1. Section 4 conclude this paper.

2 Homotopy perturbation method

In order to elaborate this method, we suppose the function given as;

$$P(u) - \varphi(\tau) = 0, \qquad \tau \in \Omega \tag{2.1}$$

subject to the boundary conditions:

$$\beta(u, \frac{\partial u}{\partial n}) = 0. \tag{2.2}$$

where P is a differential operator, β is boundary operator, $\varphi(\tau)$ is known analytic function and Γ is boundary of the domain Ω. Now P can be further separated into two parts, $L(u)$ and $N(u)$, where $L(u)$ is linear operator and $N(u)$ is nonlinear operator. Therefore, Eq.(2.1) can be written as:

$$L(u) + N(u) - \varphi(\tau) = 0. \tag{2.3}$$

* **Corresponding author**. *E-mail address*: scholar.cm@gmail.com, Ph: +92-3215022236.

IJNS.2011.08.15/509

By using homotopy technique, we make a homotopy $\nu(\tau, q) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$H(\nu, q) = (1-q)\left[L(\nu) - L(u_o)\right] + q\left[P(\nu) - \varphi(\tau)\right] = 0, \quad \tau \in \Omega \tag{2.4}$$

where $q \in [0,1]$ and is known as embedding parameter. The embedding parameter can be considered as an expanding parameter [1, 2], and u_0 is an initial approximation of Eq.(2.1). From Eq.(2.4), for $q = 0$ and $q = 1$ we will have:

$$H(\nu, 0) = L(\nu) - L(u_0) = 0, \tag{2.5}$$

$$H(\nu, 1) = P(\nu) - \varphi(\tau) = 0, \tag{2.6}$$

The variation of q from zero to one is same as change of $\nu(\tau, q)$ from u_o to $u(\tau)$. In topology, this is called deformation, while the terms $L(\nu) - L(u_o)$ and $P(\nu) - \varphi(\tau)$ are called homotopy. By using HPM, we assume that the solution of Eq.(2.4) can be written as a power series in q:

$$\nu = \nu_o + q\nu_1 + q^2\nu_2 + ... \tag{2.7}$$

Now setting $q \to 1$, Eq.(2.7) yields:

$$u = \lim_{q \to 1} \nu = \nu_o + \nu_1 + \nu_2 + ... \tag{2.8}$$

In most cases, the series in Eq.(2.8) is convergent. The unknowns $\nu_0,\ \nu_1,\ \nu_2,\ \nu_3,\ ...$ can be found by comparing the like powers of q in Eq.(2.4).

2.1 Application of HPM

Consider the Cahn Hilliard equation having the form [12]

$$\phi_t = \left(-\phi_{xx} + \phi^3 - \phi\right)_{xx} + \alpha\phi_x \tag{2.9}$$

along with initial condition:

$$\phi(0, x) = \phi_0(x)$$

The initial guess for this problem is:

$$\phi_0(x) = \phi(x, 0) = \tanh\left(\frac{1}{\sqrt{2}}x\right) \tag{2.10}$$

The exact solution of Eq.(2.9) is given as (cf. [12])

$$\phi(x, t) = \tanh\left(\frac{1}{\sqrt{2}}(x + t)\right) \tag{2.11}$$

Let us introduce $\nu(x, t) = \phi(x, t)$, so Eq.(2.9) can be written as:

$$\dot{\nu} + \nu^{(4)} - 6\nu\left(\nu^{(1)}\right)^2 - 3\nu^2\nu^{(2)} + \nu^{(2)} - \alpha\nu^{(1)} = 0 \tag{2.12}$$

$$\dot{\nu} = \frac{\partial\nu}{\partial t},\ \nu^{(1)} = \frac{\partial\nu}{\partial x},\ \nu^{(2)} = \frac{\partial^2\nu}{\partial x^2} \text{ and } \nu^{(4)} = \frac{\partial^4\nu}{\partial x^4}.$$

The homotopy equation for Eq.(2.12) will be:

$$(1-q)\left[\dot{\nu} - \dot{\phi}_0\right] + q\left[\dot{\nu} + \nu^{(4)} - 6\nu\left(\nu^{(1)}\right)^2 - 3\nu^2\nu^{(2)} + \nu^{(2)} - \alpha\nu^{(1)}\right] = 0. \tag{2.13}$$

Now using Eq.(2.7) in Eq.(2.13) and comparing the coefficients of like powers of q, we obtain the following system of equations:

$$\dot{\nu}_0 = \dot{\phi}_0 \tag{2.14}$$

$$\dot{\nu}_1 = -\nu_0^{(4)} + 6\nu_0\left(\nu_0^{(1)}\right)^2 + 3\nu_0^2\nu_0^{(2)} - \nu_0^{(2)} + \alpha\nu_0^{(1)} \tag{2.15}$$

$$\begin{aligned}\dot{\nu}_2 = {} & -\nu_1^{(4)} + 6\nu_1\left(\nu_0^{(1)}\right)^2 + 12\nu_0\nu_0^{(1)}\nu_1^{(1)} + 6\nu_0\nu_1\nu_0^{(2)} \\ & +3\nu_0^2\nu_1^{(2)} - \nu_1^{(2)} + \alpha\nu_1^{(1)}\end{aligned} \tag{2.16}$$

$$\begin{aligned}\dot{\nu}_3 = {} & -\nu_2^{(4)} + 6\nu_2\left(\nu_0^{(1)}\right)^2 + 12\nu_0\nu_0^{(1)}\nu_2^{(1)} + 6\nu_0\left(\nu_1^{(1)}\right)^2 \\ & +\alpha\nu_2^{(1)} - \nu_2^{(2)} + 12\nu_1\nu_0^{(1)}\nu_1^{(1)} + 3\nu_1^2\nu_0^{(2)} \\ & +6\nu_0\nu_1\nu_1^{(2)} + 6\nu_0\nu_2\nu_0^{(2)} + 3\nu_0^2\nu_2^{(2)}\end{aligned} \tag{2.17}$$

$$\begin{aligned}\dot{\nu}_4 = {} & -\nu_3^{(4)} + 6\nu_3\left(\nu_0^{(1)}\right)^2 + 12\nu_0\nu_0^{(1)}\nu_3^{(1)} + 6\nu_1\left(\nu_1^{(1)}\right)^2 \\ & +12\nu_2\nu_0^{(1)}\nu_1^{(1)} + 12\nu_0\nu_1^{(1)}\nu_2^{(1)} + 12\nu_1\nu_0^{(1)}\nu_2^{(1)} \\ & +6\nu_0\nu_1\nu_2^{(2)} + 6\nu_1\nu_2\nu_0^{(2)} + 3\nu_1^2\nu_1^{(2)} + 6\nu_0\nu_2\nu_1^{(2)} \\ & +6\nu_0\nu_3\nu_0^{(2)} + 3\nu_0^2\nu_3^{(2)} - \nu_3^{(2)} + \alpha\nu_3^{(1)}\end{aligned} \tag{2.18}$$

$$\begin{aligned}\dot{\nu}_5 = {} & -\nu_4^{(4)} + 6\nu_4\left(\nu_0^{(1)}\right)^2 + 6\nu_2\left(\nu_1^{(1)}\right)^2 + 6\nu_0\left(\nu_2^{(1)}\right)^2 \\ & +12\nu_1\nu_1^{(1)}\nu_2^{(1)} + 12\nu_0\nu_0^{(1)}\nu_4^{(1)} + 12\nu_3\nu_0^{(1)}\nu_1^{(1)} \\ & +12\nu_1\nu_0^{(1)}\nu_3^{(1)} + 12\nu_2\nu_0^{(1)}\nu_2^{(1)} + 12\nu_0\nu_1^{(1)}\nu_3^{(1)} \\ & +6\nu_1\nu_2\nu_1^{(2)} + 6\nu_0\nu_1\nu_3^{(2)} + 3\nu_0^2\nu_4^{(2)} + 3\nu_2^2\nu_0^{(2)} \\ & +3\nu_1^2\nu_2^{(2)} + 6\nu_0\nu_3\nu_1^{(2)} + 6\nu_0\nu_4\nu_0^{(2)} + 6\nu_1\nu_3\nu_0^{(2)} + 6\nu_0\nu_2\nu_2^{(2)} \\ & -\nu_4^{(2)} + \alpha\nu_4^{(1)}\end{aligned} \tag{2.19}$$

Now on solving Eqs.(2.14-2.19), we get the following solutions:

$$\nu_0 = \tanh\left(\frac{1}{\sqrt{2}}x\right) \tag{2.20}$$

$$\nu_1 = \frac{\alpha t}{\sqrt{2}}\left[1 - \tanh^2\left(\frac{1}{\sqrt{2}}x\right)\right] \tag{2.21}$$

$$\nu_2 = \frac{\alpha^2 t^2}{2}\left[-\tanh\left(\frac{1}{\sqrt{2}}x\right) + \tanh^3\left(\frac{1}{\sqrt{2}}x\right)\right] \tag{2.22}$$

$$\nu_3 = \frac{\alpha^3 t^3}{6\sqrt{2}}\left[-1 + 4\tanh^2\left(\frac{1}{\sqrt{2}}x\right) - 3\tanh^4\left(\frac{1}{\sqrt{2}}x\right)\right] \tag{2.23}$$

$$\nu_4 = \frac{\alpha^4 t^4}{12}\left[\begin{array}{c} 2\tanh\left(\frac{1}{\sqrt{2}}x\right) - 5\tanh^3\left(\frac{1}{\sqrt{2}}x\right) \\ +3\tanh^5\left(\frac{1}{\sqrt{2}}x\right)\end{array}\right] \tag{2.24}$$

$$\nu_5 = \frac{\alpha^5 t^5}{60\sqrt{2}}\left[\begin{array}{c} 2 - 17\tanh^2\left(\frac{1}{\sqrt{2}}x\right) + 30\tanh^4\left(\frac{1}{\sqrt{2}}x\right) \\ -25\tanh^6\left(\frac{1}{\sqrt{2}}x\right)\end{array}\right] \tag{2.25}$$

For higher accuracy we can calculate higher order terms. Thus we have the final solution $\nu(x,t) = \phi(x,t)$, up to fifth order which is given as:

$$\begin{aligned} \phi(x,t) &= \tanh\left(\frac{1}{\sqrt{2}}x\right) + \frac{\alpha t}{\sqrt{2}}\left[1-\tanh^2\left(\frac{1}{\sqrt{2}}x\right)\right] \\ &+\frac{\alpha^2 t^2}{2}\left[-\tanh\left(\frac{1}{\sqrt{2}}x\right)+\tanh^3\left(\frac{1}{\sqrt{2}}x\right)\right] \\ &+\frac{\alpha^3 t^3}{6}\left[-1+4\tanh^2\left(\frac{1}{\sqrt{2}}x\right)-3\tanh^4\left(\frac{1}{\sqrt{2}}x\right)\right] \\ &+\frac{\alpha^4 t^4}{12}\left[\begin{array}{c} 2\tanh\left(\frac{1}{\sqrt{2}}x\right)-5\tanh^3\left(\frac{1}{\sqrt{2}}x\right) \\ +3\tanh^5\left(\frac{1}{\sqrt{2}}x\right) \end{array}\right] \\ &+\frac{\alpha^5 t^5}{60\sqrt{2}}\left[\begin{array}{c} 2-17\tanh^2\left(\frac{1}{\sqrt{2}}x\right) \\ +30\tanh^4\left(\frac{1}{\sqrt{2}}x\right)-25\tanh^6\left(\frac{1}{\sqrt{2}}x\right) \end{array}\right]+\ldots \end{aligned} \tag{2.26}$$

Error analysis

Table 1: Absolute error between exact solution and the solution obtained by HPM for $\alpha = 1$.

t_n/x_n	0.01	0.02	0.03	0.04	0.05
0.1	3×10^{-11}	1×10^{-11}	0.000000	5×10^{-11}	0.000000
0.2	1×10^{-10}	1×10^{-10}	1×10^{-10}	2×10^{-10}	2×10^{-10}
0.3	1×10^{-10}	1×10^{-10}	1×10^{-10}	2×10^{-10}	1×10^{-10}
0.4	1×10^{-10}	2×10^{-10}	2×10^{-10}	2×10^{-10}	4×10^{-10}
0.5	2×10^{-10}	2×10^{-10}	0.000000	3×10^{-10}	2×10^{-10}

3 Variational iteration method (VIM)

In order to describe the basic idea of VIM, we suppose the nonlinear differential equation having the form:

$$L(\psi(x,t)) - N(\psi(x,t)) = \varphi(x,t) \tag{3.1}$$

where L is a linear operator and N is a nonlinear operator, and $\varphi(x,t)$ is nonhomogeneous analytic function. By using VIM, we construct a correction functional given as [9, 10]

$$\psi_{n+1}(x,t) = \psi_n(x,t) + \int_0^t \lambda\left\{L\psi_n(x,\eta) + N\tilde{\psi}_n(x,\eta) - \varphi_n(x,\eta)\right\}d\eta \tag{3.2}$$

where λ is Lagrangian multiplier, which can be found by using optimality condition and variational theory. The term $\psi_n(x,t)$ shows the nth approximation and $\tilde{\psi}_n(x,\eta)$ represents a restricted variation, which gives $\delta\tilde{\psi}_n(x,\eta) = 0$. The successive iterations provide an approximate solution $\psi_{n+1}(x,t)$, $n \geq 0$. Finally the series solution is given as:

$$\phi(x,t) = \lim_{n\to\infty} \phi_n(x,t). \tag{3.3}$$

3.1 Application of VIM

Again consider the equation given in (2.9):

$$\phi_t = \left(-\phi_{xx} + \phi^3 - \phi\right)_{xx} + \alpha\phi_x, \qquad (t,x) \in]0,T] \times [0,L], \tag{3.4}$$

along with initial condition:

$$\phi(0,x) = \phi_0(x), \qquad x \in [0,L] \tag{3.5}$$

The initial guess for this equation is given by Eq.(2.10). By using VIM, we have the correction functional for Eq.(3.4) in the form:

$$\phi_{n+1}(x,t)=\phi_n(x,t)+\int_0^t \lambda\left\{\frac{\partial\phi_n(x,\eta)}{\partial\eta}+\frac{\partial_n^4\tilde{\phi}(x,\eta)}{\partial x^4}-\frac{\partial^2}{\partial x^2}\left(\tilde{\phi}_n^3(x,\eta)-\tilde{\phi}_n(x,\eta)\right)-\alpha\frac{\partial\tilde{\phi}_n(x,\eta)}{\partial x}\right\}d\eta \quad (3.6)$$

The Lagrangian multiplier for Eq.(3.6) is found to be $\lambda=-1$, so the Eq.(3.6) becomes:

$$\phi_{n+1}(x,t)=\phi_n(x,t)-\int_0^t \left\{\frac{\partial\phi_n(x,\eta)}{\partial\eta}+\frac{\partial^4\tilde{\phi}_n(x,\eta)}{\partial x^4}-\frac{\partial^2}{\partial x^2}\left(\tilde{\phi}_n^3(x,\eta)-\tilde{\phi}_n(x,\eta)\right)-\alpha\frac{\partial\tilde{\phi}_n(x,\eta)}{\partial x}\right\}d\eta \quad (3.7)$$

For $n=0$, and $n=1$ in Eq.(3.6), we obtained the following approximations:

$$\phi_1(x,t)=\tanh\left(\frac{1}{\sqrt{2}}x\right)+t\left[\frac{\alpha}{\sqrt{2}}-\frac{\alpha}{\sqrt{2}}\tanh^2\left(\frac{1}{\sqrt{2}}x\right)\right] \quad (3.8)$$

$$\begin{aligned}\phi_2(x,t) &= \tanh\left(\frac{1}{\sqrt{2}}x\right)+t\left[\frac{\alpha}{\sqrt{2}}-\frac{\alpha}{\sqrt{2}}\tanh^2\left(\frac{1}{\sqrt{2}}x\right)\right] \\ &+t^2\left[\frac{\alpha^2}{2}\tanh^3\left(\frac{1}{\sqrt{2}}x\right)-\frac{\alpha^2}{2}\tanh\left(\frac{1}{\sqrt{2}}x\right)\right] \\ &+t^3\left[\begin{array}{c}\frac{15\alpha^2}{2}\tanh^7\left(\frac{1}{\sqrt{2}}x\right)-\frac{37\alpha^2}{2}\tanh^5\left(\frac{1}{\sqrt{2}}x\right)\\ +\frac{29\alpha^2}{2}\tanh^3\left(\frac{1}{\sqrt{2}}x\right)-\frac{7\alpha^2}{2}\tanh\left(\frac{1}{\sqrt{2}}x\right)\end{array}\right] \\ &+t^4\left[\begin{array}{c}-\frac{21\alpha^3}{8\sqrt{2}}\tanh^8\left(\frac{1}{\sqrt{2}}x\right)+\frac{33\alpha^3}{4\sqrt{2}}\tanh^6\left(\frac{1}{\sqrt{2}}x\right)\\ -\frac{9\alpha^3}{\sqrt{2}}\tanh^4\left(\frac{1}{\sqrt{2}}x\right)+\frac{15\alpha^3}{4\sqrt{2}}\tanh^2\left(\frac{1}{\sqrt{2}}x\right)-\frac{3\alpha^3}{8\sqrt{2}}\end{array}\right]\end{aligned} \quad (3.9)$$

Error analysis

Table 2: Absolute error between exact solution and the solution obtained by VIM for $\alpha=1$.

t_n/x_n	0.01	0.02	0.03	0.04	0.05
0.1	1.2913×10^{-7}	1.0542×10^{-6}	3.6296×10^{-6}	8.7732×10^{-6}	1.7467×10^{-5}
0.2	3.4610×10^{-7}	2.7885×10^{-6}	9.4758×10^{-6}	2.2614×10^{-5}	4.4468×10^{-5}
0.3	5.1030×10^{-7}	4.0983×10^{-6}	1.3884×10^{-5}	3.3034×10^{-5}	6.4762×10^{-5}
0.4	6.0680×10^{-7}	4.8651×10^{-6}	1.6456×10^{-5}	3.9095×10^{-5}	7.6529×10^{-5}
0.5	6.3290×10^{-7}	5.0696×10^{-6}	1.7131×10^{-5}	4.0656×10^{-5}	7.9504×10^{-5}

4 Conclusions

We have successfully applied two iterative methods i.e., Homotopy Perturbation Method and Variational Iteration Method to solve nonlinear Cahn-Hilliard equation. The computed solutions agree very well with the exact solution of the problem as shown in table 1 and table 2. The advantage of these methods over other methods like classical perturbation method and homotopy analysis method, is that these are very simple and straightforward in their application.

References

[1] He, J.H.. Comparison of homotopy perturbation method and homtopy analysis method. *Appl. Math. Comput.*, 156(2004): 527-539.

[2] He, J.H.. Homotopy perturbation method for solving boundary value problems. *Phy. Lett. A*, 350(2006): 87-88.

[3] Islam, S., Shah, A., Zhou, Ali, I.. Homotopy perturbation analysis of Slider bearing Lubricated with Powell Eyring fluid.*ZAMP*, 60(6)(2009): 1178-1193.

[4] Chen, W., Lu, Z.. An algorithm for Adomian decomposition method.*Appl. Math. Comput.*, 159(1)(2004): 221-235.

[5] Gangi, D.D., Nourollahi M., Rostamin M.. A Comparison of Variational Iteration Method with Adomian's Decomposition Method in Some Highly Nonlinear Equations. *International Journal of Science and Technology*, 2(2)(2007): 179-188.

[6] Jang, B.. Exact solutions to one dimensional non-homogeneous parabolic problems by the homogeneous Adomian decomposition method. *Appl. Math. Comput.*, 186(2)(2007): 969-979.

[7] Khan, M., Abbas Z., Hayat T.. Analytic solution for flow of Sisko fluid through a porous medium. *Transportin Porous Media*, 71(2008): 23-27.

[8] Khan, M., Munawar S., Abbasbandy S.. Steady flow and heat transfer of a Sisko fluid in annular pipe. *Int. J. Heat and Mass Trans.*, 53(7-8)(2010): 1290-1297.

[9] He, J.H.. Variational iteration method - a kind of nonlinear analytical technique: some examples. *Int. J. Nonlinear Mech.*, 34(1999): 699-708.

[10] He, J.H.. Variational method for autonomous ordinary differential equations. *Appl. Math. Comput.*, 114(2000): 115-123.

[11] Slota, D.. Exact solution of the heat equation with boundary condition of the fourth kind by Hes variational iteration method Original Research Article.*Computers and Mathematics with Applications*,58(11-12)(2009): 2495-2503.

[12] Ugurlu, Y., Kaya D.. Solutions of the Cahn-Hilliard equation. *Computers and Mathematics with Applications*, 56(2008): 3038-3045.