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Travelling Wave Solutions of the OS-BBM Equation by the Simplified G'/G-expansion Method

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Abstract: The Ostrovsky- Benjamin-Bona-Mahony (OS-BBM) equation can bee used to discribe the motion of ocean currents. We construct the traveling wave solutions involving parameters of the OS-BBM equation in terms of the hyperbolic functions and trigonometric functions by using the simplified $G'/G-$expansion method. Hyperbolic function solutions including solitary wave solutions and trigonometric function wave solutions are found.

Keywords: simplified G'/G-expansion method; travelling wave solutions; the OS-BBM equation; exact solutions; solitary wave solutions; hyperbolic function solutions

1 Introduction

It is usually difficult to solve nonlinear partial differential equations (NLPDEs). Seeking exact solutions of NLPDEs has become one of the most exacting and extremely active areas of research investigation. With the development of soliton theory, many powerful methods for obtaining exact solutions of nonlinear partial differential equations (NLPDEs) have been presented, such as inverse scattering method [1], Darboux transformation method [2], bifurcation method of planner dynamical system [3], the G'/G-expansion method [4, 5] and so on.

The Ostrovsky equation [6] is a model of ocean currents motion,

$$(u_t + (u^2)_x - \beta u_{xxx})_x = \gamma u, \quad x \in R \tag{1}$$

where β, γ = const. Parameter β determines the type of dispersion, namely, $\beta = -1$ (negative-dispersion) for surface and internal waves in the ocean and surface waves in a shallow channel with an uneven bottom; $\beta = 1$ (positive dispersion) for capillary waves on the surface of liquid or for oblique magneto-acoustic waves. Parameter $\gamma > 0$ measures the effect of rotation. When $\gamma = 0$, integrating once with respect to x and letting the integral constant be zero, the Ostrovsky equation becomes the well-known KdV equation.

Existence and nonexistence of localized solitary waves of the Ostrovsky equation is classified according to the sign of the dispersion parameter (which can be either positive or negative). Yue Liu and Vladimir Varlamov [7] proved that for the case of positive dispersion the set of solitary waves is stable with respect to perturbations. The issue of passing to the limit as the rotation parameter tends to zero for solutions of the Cauchy problem is investigated on a bounded time interval. V. Varlamov and Yue Liu [8] studied initial-value problems that arises in modelling the unidirectional propagation of long waves in a rotating homogeneous incompressible fluid. Local- and global-in-time solvability is investigated. For the case of positive dispersion a fundamental solution of the Cauchy problem for the linear equation is constructed, and its asymptotics is calculated as $t \to \infty$, $x/t = const$. For the nonlinear problem solutions are constructed in the form of a series and the analogous long-time asymptotics is obtained.

The Benjamin-Bona-Mahony (BBM) equation [9, 10]

$$u_t + u_x - a(u^2)_x - bu_{xxt} = 0, \tag{2}$$

is a well-known equation in physics. The equation has dispersion effect and exists solitary wave behavior.

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In this work, the following the OS-BBM equation

$$\left(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt}\right)_x = \gamma(u + u^2) \tag{3}$$

will be studied. We shall use the simplified G'/G-expansion method [16]. It is based on the explicit linearization of nonlinear differential equations for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. The computations are performed with a computer algebra system to deduce few solutions of the nonlinear equations in an explicit form.

The simplified G'/G-expansion is a modification of the G'/G-expansion method [11] which has been applied to many kinds of PDEs [12–15]. The expression of G'/G in the G'/G expansion method is simplified to $tanh, coth, cot$ and rational forms under some conditions and that why it is called the simplified G'/G-expansion in [16]. The simplified G'/G-method can be thought as a combination of $tanh - coth$ method and cot method. It is pointed out in [17] that the G'/G-expansion method coincides with the truncated expansion method if we use the travelling wave solutions. The G'/G-expansion method is equivalent to application of the simplest equation method with the Riccati equation, to the $tanh$-function method and to the truncated expansion method.

The layout of this paper is organized as follows. In Section 2, we give the description of the simplified G'/G-expansion method. In Section 3, we apply this method to the OS-BBM equation. Conclusions are given in the last section.

2 Description of the simplified (G'/G)-expansion method

We shall first describe the simplified G'/G-expansion method [16] with two independent variables. For a given nonlinear PDE

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \cdots) = 0, \tag{4}$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in u and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. There are mainly four steps in the G'/G-expansion method.

Step 1. The travelling wave variable

$$u(x, t) = \phi(\xi), \quad \xi = x - Vt, \tag{5}$$

permits us to reduce Eq. (4) to an ODE for $u = \phi(\xi)$ in the form

$$P(\phi, -V\phi', \phi', V^2\phi'', -V\phi'', \phi'', \cdots) = 0, \tag{6}$$

where " $'$ " denotes derivative about ξ.

Step 2. Suppose that the solution of ODE (6) can be expressed by a polynomial in G'/G as follows:

$$\phi(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{7}$$

where $G = G(\xi)$ satisfies the following second order linear ODE

$$G'' + \lambda G' + \mu G = 0, \tag{8}$$

where $a_i\ (i = 0, 1, 2, \cdots, m), \lambda$ and μ are constants to be determined later, $a_m \neq 0$. The degree of the polynomial m can be determined by balancing the highest order derivative with nonlinear terms.

Step 3. Substituting (7) into (6) and using the second order linear ODE (8) and then equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations with respect to $a_i\ (i = 0, 1, 2, \cdots, m), V, \lambda$, and μ. Solving the algebraic system, we may find the values of unknowns.

Step 4. Substituting $a_i\ (i = 0, 1, 2, \cdots, m), V, \lambda, \mu$ gotten in Step 3 and the general solutions of Eq. (8) into (7) we can obtain more traveling wave solutions of the nonlinear PDE (4).

Solutions to Eq.(8) takes the simplified form depending on whether $\lambda^2 - 4\mu > 0$, $\lambda^2 - 4\mu > 0$ or $\lambda^2 - 4\mu = 0$,

$$\frac{G'}{G} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2} \tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \tanh\xi_0 = C, |C| < 1, \\ \frac{\sqrt{\lambda^2-4\mu}}{2} \coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \coth\xi_0 = C, |C| > 1, \\ \frac{\sqrt{4\mu-\lambda^2}}{2} \cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \xi_0\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \tan\xi_0 = C \\ \frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0, \end{cases} \tag{9}$$

where C, C_1 and C_2 are arbitrary constants.

We will use (9) to get travelling wave solution of Eq.(3).

3 Traveling wave solutions to the OS-BBM equation

In this section, we will apply the simplified G'/G-expansion method to construct the traveling wave solutions of OS-BBM equation (3).

Combining the independent variables x and t into one variable $\xi = x - Vt$, we suppose that

$$u(x,t) = \phi(\xi), \xi = x - Vt. \tag{10}$$

Then Eq. (3) is converted into an ODE for $u = \phi(\xi)$

$$\beta V\phi^{(4)} + (1 - V - 2\alpha\phi)\phi'' - 2\alpha(\phi')^2 - \gamma\phi - \gamma\phi^2 = 0. \tag{11}$$

Suppose that the solution of the ODE (11) can be expressed by polynomials in terms of G'/G as follows:

$$\phi(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{12}$$

while $G = G(\xi)$ satisfies (8).

The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (11). To determine the degree of the polynomial solutions, we can only substitute the leading term. Let the degree in G'/G of (12) be $D(\phi)$. Then $D(\phi^p \left(\frac{d^s\phi}{d\xi^s}\right)^q) = pD(\phi) + (s + D(\phi)q)$. So we get $m = 2$ and

$$\phi(\xi) = a_2 \left(\frac{G'}{G}\right)^2 + a_1 \left(\frac{G'}{G}\right) + a_0, \tag{13}$$

By substituting (13) and its derivatives into Eq. (11) and collecting all terms with the same power of G'/G together, the left-hand side of Eq. (11) is converted into another polynomial in G'/G. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for $a_2, a_1, a_0, V, \lambda, \mu$ as follows:

$$0: -\gamma a_0 + V\beta\lambda^3\mu a_1 + 8V\beta\lambda\mu^2 a_1 + \lambda\mu a_1 - V\lambda\mu a_1 + 2\mu^2 a_2 + 16V\beta\mu^3 a_2 - 2V\mu^2 a_2 + 14V\beta\lambda^2\mu^2 a_2 - \gamma {a_0}^2 - 2\alpha\lambda\mu a_1 a_0 - 2\alpha\mu^2 {a_1}^2 - 4\alpha\mu^2 a_2 a_0 = 0,$$

$$1: -\gamma a_1 + \lambda^2 a_1 - 2V\mu a_1 + V\beta\lambda^4 a_1 + 16V\beta\mu^2 a_1 - V\lambda^2 a_1 + 2\mu a_1 + 22V\beta\lambda^2\mu a_1 + 6\lambda\mu a_2 + 30V\beta\lambda^3\mu a_2 - 6V\lambda\mu a_2 + 120V\beta\mu^2\lambda a_2 - 2\gamma a_1 a_0 - 2\alpha\lambda^2 a_1 a_0 - 4\alpha\mu a_1 a_0 - 6\alpha\lambda\mu {a_1}^2 - 12\alpha\lambda\mu a_2 a_0 - 12\alpha\mu^2 a_2 a_1 = 0,$$

$$2: 60V\beta\lambda\mu a_1 + 15V\beta\lambda^3 a_1 + 3\lambda a_1 - 3V\lambda a_1 + 8\mu a_2 - 8V\mu a_2 + 16V\beta\lambda^4 a_2 - \gamma a_2 + 232V\beta\lambda^2\mu a_2 + 136V\beta\mu^2 a_2 + 74\lambda^2 a_2 - 4V\lambda^2 a_2 - 6\alpha\lambda a_1 a_0 - \gamma {a_1}^2 - 8\alpha\mu {a_1}^2 - 4\alpha\lambda^2 {a_1}^2 - 2\gamma a_2 a_0 - 16\alpha\mu a_2 a_0 - 8\alpha\lambda^2 a_2 a_0 - 30\alpha\lambda\mu a_2 a_1 - 12\alpha\mu^2 {a_2}^2 = 0,$$

$$3: 2a_1 + 40V\beta\mu a_1 - 2Va_1 + 50V\beta\lambda^2 a_1 + 440V\beta\lambda\mu a_2 - 10V\lambda a_2 + 130V\beta\lambda^3 a_2 + 10\lambda a_2 - 4\alpha a_1 a_0 - 10\alpha\lambda {a_1}^2 - 20\alpha\lambda a_2 a_0 - 36\alpha\mu a_2 a_1 - 18\alpha\lambda^2 a_2 a_1 - 2\gamma a_2 a_1 - 28\alpha\lambda\mu {a_2}^2 = 0,$$

$$4: 60V\beta\lambda a_1 + 240V\beta\mu a_2 + 330V\beta\lambda^2 a_2 - 6Va_2 + 6a_2 - 6\alpha {a_1}^2 - 12\alpha a_2 a_0 - 42\alpha\lambda a_2 a_1 - \gamma {a_2}^2 - 16\alpha\lambda^2 {a_2}^2 - 32\alpha\mu {a_2}^2 = 0,$$

$$5: 24V\beta a_1 + 336V\beta\lambda a_2 - 24\alpha a_2 a_1 - 36\alpha\lambda {a_2}^2 = 0,$$

$$6: -20a_2(\alpha a_2 - 6V\beta) = 0,$$

Solving the algebraic equations yields two sets of solutions:

$$\text{Set A:} \quad \lambda = \lambda, \mu = \frac{\gamma\beta + \alpha + \beta(1+\alpha)\lambda^2}{4\beta(1+\alpha)}, V = \frac{\alpha(1+\alpha)}{\gamma\beta + \alpha}, \\ a_0 = \frac{\gamma\beta + \alpha + 3\beta(1+\alpha)\lambda^2}{2(\gamma\beta + \alpha)}, a_1 = \frac{6\beta\lambda(1+\alpha)}{\gamma\beta + \alpha}, a_2 = \frac{6\beta(1+\alpha)}{\gamma\beta + \alpha} \tag{15}$$

$$\begin{aligned} \text{Set B:}\quad & \lambda=\lambda,\mu=\frac{-(\gamma\,\beta+\alpha)+\beta(1+\alpha)\,\lambda^2}{4\beta\,(1+\alpha)},V=\frac{\alpha\,(1+\alpha)}{\gamma\,\beta+\alpha},\\ & a_0=\frac{-3(\gamma\,\beta+\alpha)+3\beta(1+\,\alpha)\,\lambda^2}{2(\gamma\,\beta+\alpha)},a_1=\frac{6\beta\lambda\,(1+\alpha)}{\gamma\,\beta+\alpha},a_2=\frac{6\beta\,(1+\alpha)}{\gamma\,\beta+\alpha} \end{aligned} \tag{16}$$

Substituting (9) into (13) we can derive the travelling wave solutions of Eq. (3).

3.1 First travelling solution set

To Set A, substituting (15) and (9) into (13), we deduce the following two types of traveling wave solutions of the OS-BBM Eq.(3):

Case 1-1. When $\lambda^2-4\mu=-\frac{\gamma\,\beta+\alpha}{\beta\,(1+\alpha)}>0$, we have the hyperbolic function travelling wave solutions

$$u_1^+(x,t)=\frac{3}{2}\text{sech}^{\,2}\left(\frac{\sqrt{-\frac{\gamma\,\beta+\alpha}{\beta\,(1+\alpha)}}}{2}\xi+\xi_0\right)-1 \tag{17}$$

where $\xi=x-\frac{\alpha\,(1+\alpha)}{\gamma\,\beta+\alpha}t,\xi_0=\tanh^{-1}C$, $|C|<1$, and

$$u_2^+(x,t)=-\frac{3}{2}\text{csch}^{\,2}\left(\frac{\sqrt{-\frac{\gamma\,\beta+\alpha}{\beta\,(1+\alpha)}}}{2}\xi+\xi_0\right)-1 \tag{18}$$

where $\xi=x-\frac{\alpha\,(1+\alpha)}{\gamma\,\beta+\alpha}t,\xi_0=\coth^{-1}C$, $|C|>1$.

Solution (17) is a smooth bell-shape solitary wave solution. A planar profile is shown in Fig.1(a).

(a) smooth bell-shape solitary wave solution (b) smooth valley-shape solitary wave solution

Figure 1: planar profiles of solutions to u. (a) the case for $-\frac{\gamma\,\beta+\alpha}{\beta\,(1+\alpha)}>0$, $|C|>1$. (b) the case for $\frac{\gamma\,\beta+\alpha}{\beta\,(1+\alpha)}>0$, $|C|>1$.

Case 1-2. When $\lambda^2-4\mu=-\frac{\gamma\,\beta+\alpha}{\beta\,(1+\alpha)}<0$, we have the singular trigonometric solution

$$u_1^-(x,t)=\frac{3}{2}\cot^2\left(\frac{\sqrt{\frac{\gamma\,\beta+\alpha}{\beta\,(1+\alpha)}}}{2}\xi+\xi_0\right)+\frac{1}{2}, \tag{19}$$

where $\xi=x-\frac{\alpha\,(1+\alpha)}{\gamma\,\beta+\alpha}t,\xi_0=tan^{-1}C$.

Due to $V=\frac{\alpha\,(1+\alpha)}{\gamma\,\beta+\alpha}$, we can get $\gamma\,\beta+\alpha\neq 0$. There does not exist the situation for $\lambda^2-4\mu=-\frac{\gamma\,\beta+\alpha}{\beta\,(1+\alpha)}=0$.

3.2 Second travelling solution set

To set B, substituting (16) and (9) into (12), two types of traveling wave solutions of the OS-BBM Eq. (3) are obtained:

Case 2-1. When $\lambda^2 - 4\mu = \frac{\gamma\beta+\alpha}{\beta(1+\alpha)} > 0$, we have the hyperbolic function travelling wave solution

$$u_3^+(x,t) = -\frac{3}{2}\text{sech}^2\left(\frac{\sqrt{\frac{\gamma\beta+\alpha}{\beta(1+\alpha)}}}{2}\xi + \xi_0\right) \tag{20}$$

where $\xi = x - \frac{\alpha(1+\alpha)}{\gamma\beta+\alpha}t, \xi_0 = \tanh^{-1} C, |C| < 1$, and

$$u_4^+(x,t) = \frac{3}{2}\text{csch}^2\left(\frac{\sqrt{\frac{\gamma\beta+\alpha}{\beta(1+\alpha)}}}{2}\xi + \xi_0\right). \tag{21}$$

where $\xi = x - \frac{\alpha(1+\alpha)}{\gamma\beta+\alpha}t, \xi_0 = \coth^{-1} C, |C| > 1$.

Solution (20) is a smooth valley-shape solitary wave solution. A planar profile is shown in Fig.1(b).

Case 2-2. When $\lambda^2 - 4\mu = \frac{\gamma\beta+\alpha}{\beta(1+\alpha)} < 0$, we have the singular trigonometric solution

$$u_2^-(x,t) = -\frac{3}{2}\cot^2\left(\frac{\sqrt{-\frac{\gamma\beta+\alpha}{\beta(1+\alpha)}}}{2}\xi + \xi_0\right) - \frac{3}{2}, \tag{22}$$

where $\xi = x - \frac{\alpha(1+\alpha)}{\gamma\beta+\alpha}t, \xi_0 = tan^{-1}C$.

Due to $V = \frac{\alpha(1+\alpha)}{\gamma\beta+\alpha}$, we can get $\gamma\beta + \alpha \neq 0$. There does not exist the situation for $\lambda^2 - 4\mu = \frac{\gamma\beta+\alpha}{\beta(1+\alpha)} = 0$, too.

4 Conclusion

The simplified $G'/G-$expansion method for finding solutions to nonlinear equations delivers solutions in a neat manner and helpful form. In this paper, we have derived many families of exact traveling wave solutions for the OS-BBM equation via the simplified G'/G-expansion method. These travelling wave solutions are expressed by the hyperbolic functions and trigonometric functions.The form of solutions is determined by the equation parameters.

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