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The Modified Sine-Cosine Method and Its Applications to the Generalized K(n,n) and BBM Equations with Variable Coefficients

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Abstract: In this article, a modified sine–cosine method is used to construct many exact solutions to the nonlinear generalized K(n,n) and BBM equations with variable coefficients. Under different parameter conditions, explicit formulas for some new exact solutions are successfully obtained. The proposed solutions are found to be important for the explanation of some practical physical problems.

Keywords: generalized K(n,n) equation with variable coefficients; generalized BBM equation with variable coefficients; exact wave solutions; sine–cosine method

1 Introduction

In the nonlinear science, many important phenomena in various fields can be described by the nonlinear evolution equations (NLEEs). Searching for exact soliton solutions of NLEEs plays an important and a significant role in the study on the dynamics of those phenomena. With the development of soliton theory, many powerful methods for obtaining the exact solutions of NLEEs have been presented, such as the extended tanh-function method [1-5], the tanh-sech method [6-8], the sine-cosine method [9-11], the homogeneous balance method [12,13], the exp-function method [14-17], the Jacobi elliptic function method [18-21], the F-expansion method [22], the homotopy perturbation method [23,24], the variational iteration method [25], the inverse scattering transformation method [26], the Bäcklund transformation method [27], the Hirota bilinear method [28,29] and so on. To our knowledge, most of the aforementioned methods are related to constant coefficients models. Recently, much attention has been paid to the variable-coefficient nonlinear equations which can describe many nonlinear phenomena more realistically than their constant-coefficient ones.

The objective of this article is to apply a modified sine-cosine method using a generalized wave transformation to find the exact solutions of the following two nonlinear dispersive equations with variable coefficients:

1. The generalized K(n,n) equation with variable coefficients [30]

$$u_t + a(t)u_x + b(t)(u^n)_x + k(t)(u^n)_{xxx} = 0, \quad n \neq 0, 1 \tag{1}$$

where $a(t)$, $b(t)$ and $k(t)$ are nonzero functions of t. As a model that characterizes long waves in nonlinear dispersive media, Eq. (1) was formally derived to describe the propagation of surface water waves in a uniform channel. Now it has been established that the equation provides a model for not only the surface waves of long wavelength in liquids, but also hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids. Eq. (1) has been discussed by Wazwaz [31] using sine-cosine method and tanh-method when the functions $a(t)$, $b(t)$ and $k(t)$ are nonzero constants.

2. The generalized Benjamin- Bona-Mahony (BBM) equation with variable coefficients [30]

$$u_t + a(t)u_x + b(t)(u^n)_x + k(t)(u^n)_{xxt} = 0, \quad n \neq 0, 1, \tag{2}$$

where $a(t)$, $b(t)$ and $k(t)$ all are nonzero functions of t. The case n = 1 with constant coefficients when $a(t) = b(t) = k(t) = 1$ corresponds to the BBM equation, which was first proposed by Benjamin et al. [32]. Eq. (2) is an alternative to Eq. (1), and also describes the unidirectional propagation of small-amplitude long waves on the surface of water in a

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channel. This equation is not only convenient for shallow water waves but also for hydromagnetic and acoustic waves, and therefore it has some advantages compared with the KdV equation. When n = 2, Eq. (1) is called the modified BBM equation. X. Lv et al [30] have discussed Eqs. (1) and (2) using the auxiliary differential equation method and obtained various exact traveling wave solutions.

2 Description of the modified sine-cosine method

For a given nonlinear PDEs with independent variables $X = (x, y, z, t)$ and dependent variable u, we consider the PDE

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, ...) = 0. \tag{3}$$

In order to solve Eq. (3), we use the generalized wave transformation

$$u(X) = u(\xi) \ , \qquad \xi = \xi(X). \tag{4}$$

Thus, we suppose that Eq. (3) has the following formal solution:

$$u(X) = \lambda(X) \sin^m(\mu\xi), \qquad |\mu\xi| < \frac{\pi}{2} \tag{5}$$

or,

$$u(X) = \lambda(X) \cos^m(\mu\xi), \qquad |\mu\xi| < \frac{\pi}{2\mu}, \tag{6}$$

where μ and m are nonzero constants, while $\xi(X)$ and $\lambda(X)$ are analytic functions of X to be determined later. To determine $u(X)$ explicitly, we consider the following four steps:

Step1. Substitute (5) or (6) into (3) a trigonometric equations are obtained with either $\sin^j(\mu\xi)\cos^k(\mu\xi)$ or $\cos^j(\mu\xi)\sin^k(\mu\xi)$ terms so that the parameter j can be determined by comparing exponents.

Step2. Equate all coefficients of $\sin^j(\mu\xi)\cos^k(\mu\xi)$ or $\cos^j(\mu\xi)\sin^k(\mu\xi)$ to zero, yield a set of over-determined differential equations for $\lambda(X)$ and $\xi(X)$.

Step3. Solve the system of over-determined differential equations obtained in Step 2 by *Maple* or *Mathematica*.

Step4. Use the results obtained in above steps to derive a series of solutions of Eq. (3).

Remark 1 *Note that throughout this paper we have used the formal solution (5) only and we have not used the formal solution (6) for simplicity which has been left for the readers.*

3 Application

In this section, we will apply the modified sine-cosine method to construct the exact solutions of two nonlinear evolution equations with variable coefficients via the nonlinear generalized K(n,n) equation with variable coefficients (1) and the nonlinear generalized Benjamin- Bona-Mahony (BBM) equation with variable coefficients (2).

3.1 Example 1. The generalized K(n,n) equation with variable coefficients

In order to obtain the exact solutions of Eq. (1), we assume that the solution of this equation can be written in the form

$$u(x,t) = [v(x,t)]^{\frac{1}{n-1}}. \tag{7}$$

Substituting (7) into (1), we have

$$\begin{aligned}&(n-1)^2\left[v(\xi)\,v(\xi)_t + a(t)\,v(\xi)\,v(\xi)_x + nb(t)\,v^2(\xi)\,v(\xi)_x + nk(t)v(\xi)^2\,v(\xi)_{xxx}\right]\\ &+nk(t)\left[(2-n)\,v(\xi)_x^3 + 3(n-1)\,v(\xi)\,v(\xi)_x\,v(\xi)_{xx}\right] = 0,\ n \neq 0, 1,\end{aligned} \tag{8}$$

where

$$v(x,t) = v(\xi) = \lambda(t)\sin^m(\mu\xi), \xi = hx + \int \tau(t)\,dt, \tag{9}$$

while h is a constant and the function $\tau(t)$ is an integrable function of t to be determined later. From (5), we have the derivatives

$$\begin{array}{lcl} \upsilon_t &=& \frac{d\,\lambda(t)}{dt}\sin^m(\mu\xi)+m\mu\lambda(t)\tau(t)\sin^{m-1}(\mu\xi)\cos(\mu\xi),\\ \upsilon_x &=& m\mu h\lambda(t)\sin^{m-1}(\mu\xi)\cos(\mu\xi),\\ \upsilon_{xx} &=& m\mu^2h^2\lambda(t)\left[-m\sin^m(\mu\xi)+(m-1)\sin^{m-2}(\mu\xi)\right],\\ \upsilon_{xxx} &=& m\mu^3h^3\lambda(t)\left[-m^2\sin^{m-1}(\mu\xi)\cos(\mu\xi)+(m-1)(m-2)\sin^{m-3}(\mu\xi)\cos(\mu\xi)\right],\\ \upsilon_{xxt} &=& m\mu^2h^2\frac{d\,\lambda(t)}{dt}\left[-m\sin^m(\mu\xi)+(m-1)\sin^{m-2}(\mu\xi)\right]\\ && +m\mu^3h^2\lambda(t)\tau(t)\left[-m^2\sin^{m-1}(\mu\xi)\cos(\mu\xi)+(m-1)(m-2)\sin^{m-3}(\mu\xi)\cos(\mu\xi)\right]. \end{array} \tag{10}$$

Substituting (10) into (8) gives

$$\begin{array}{l} (n-1)^2\lambda(t)\frac{d\lambda(t)}{dt}\sin^{2m}(\mu\xi)+m\mu(n-1)^2\lambda(t)^2\left[ha(t)+\tau(t)\right]\sin^{2m-1}(\mu\xi)\cos(\mu\xi)\\ +\{m\mu hn(n-1)^2b(t)\lambda(t)^3-m^3\mu^2h^2n\left[\mu h\left(n^2-3n+3\right)\right.\\ \left.+3(n-1)\right]k(t)\lambda(t)^3\}\sin^{3m-1}(\mu\xi)\cos(\mu\xi)+m\mu^2h^2n\left\{3m(m-1)(n-1)\right.\\ \left.+\mu h\left[m^2\left(n^2-3n+3\right)-(3m-2)(n-1)^2\right]\right\}\sin^{3m-3}(\mu\xi)\cos(\mu\xi)=0, \end{array} \tag{11}$$

which are satisfied only if the following conditions hold:

$$\left\{\begin{array}{c} (n-1)^2\lambda(t)\frac{d\,\lambda(t)}{dt}=0,\\ m\mu(n-1)^2\lambda(t)^2\left[ha(t)+\tau(t)\right]+m\mu^2h^2n\left\{3m(m-1)(n-1)\right.\\ \left.+\mu h\left[m^2\left(n^2-3n+3\right)-(3m-2)(n-1)^2\right]\right\}=0,\\ +m\mu hn(n-1)^2b(t)\lambda(t)^3-m^3\mu^2h^2n\left[\mu h\left(n^2-3n+3\right)\right.\\ \left.+3(n-1)\right]k(t)\lambda(t)^3=0,\\ 2m-1=3m-3. \end{array}\right. \tag{12}$$

Solving the system (12) by *Maple,* we have the following case of solution:

$$\begin{array}{l} \mu=\mu,\lambda(t)=c,h=h,\tau(t)=-ha(t)+\frac{chnb(t)[-3(n-1)+2\mu h(n-2)]}{2[\mu h(n^2-3n+3)+3(n-1)]},\\ a(t)=a(t),b(t)=b(t),k(t)=\frac{(n-1)^2b(t)}{4\mu h[\mu h(n^2-3n+3)+3(n-1)]},m=2, \end{array} \tag{13}$$

where c is a constant. In this case, the exact solution of Eq. (1) has the from:

$$u(\xi)=\left[c\sin^2(\mu\xi)\right]^{\frac{1}{n-1}}, \tag{14}$$

where

$$\xi=hx+\int\left\{-ha(t)+\frac{chnb(t)\left[-3(n-1)+2\mu h(n-2)\right]}{2\left[\mu h\left(n^2-3n+3\right)+3(n-1)\right]}\right\}dt. \tag{15}$$

3.2 Example 2. The generalized BBM equation with variable coefficients

In order to obtain the exact solutions of Eq. (2), we assume that the solution of this equation has the same form (7). Substituting (7) into (2), we have

$$\begin{array}{l} (n-1)^2\left[\upsilon(\xi)\upsilon(\xi)_t+a(t)\upsilon(\xi)\upsilon(\xi)_x+nb(t)\upsilon^2(\xi)\upsilon(\xi)_x+nk(t)\upsilon(\xi)^2\upsilon(\xi)_{xxt}\right]\\ +nk(t)\left[(2-n)\upsilon(\xi)_t\upsilon(\xi)_x^2+2(n-1)\upsilon(\xi)\upsilon(\xi)_x\upsilon(\xi)_{xt}\right.\\ \left.+(n-1)\upsilon(\xi)\upsilon(\xi)_t\upsilon(\xi)_{xx}\right]=0,n\neq 0,1. \end{array} \tag{16}$$

Substituting (10) into (16) gives

$$\begin{array}{l} (n-1)^2\lambda(t)\frac{d\,\lambda(t)}{dt}\sin^{2m}(\mu\xi)-m^2\mu^2h^2n\left(n^2-n+2\right)k(t)\lambda(t)^2\frac{d\,\lambda(t)}{dt}\sin^{3m}(\mu\xi)\\ +mn\left[m\left(n^2-n+2\right)-n(n-1)\right]\mu^2h^2k(t)\lambda(t)^2\frac{d\,\lambda(t)}{dt}\sin^{3m-2}(\mu\xi)\\ +m\mu(n-1)^2\lambda(t)^2\left[ha(t)+\tau(t)\right]\sin^{2m-1}(\mu\xi)\cos(\mu\xi)\\ +\left[m\mu hn(n-1)^2b(t)\lambda(t)^3-m^3\mu^3h^2n^3k(t)\lambda(t)^3\right]\sin^{3m-1}(\mu\xi)\cos(\mu\xi)\\ +mn(mn-n+1)(mn-2n+2)\mu^3h^2k(t)\lambda(t)^3\tau(t)\sin^{3m-3}(\mu\xi)\cos(\mu\xi)=0, \end{array} \tag{17}$$

which are satisfied only if the following conditions hold:

$$\left\{\begin{array}{c}(n-1)^2\lambda(t)\frac{d\,\lambda(t)}{dt}+mn\left[m\left(n^2-n+2\right)-n\left(n-1\right)\right]\mu^2h^2k(t)\lambda(t)^2\frac{d\,\lambda(t)}{dt}=0,\\ -m^2\mu^2h^2n\left(n^2-n+2\right)k(t)\lambda(t)^2\frac{d\,\lambda(t)}{dt}=0,\\ m\mu(n-1)^2\lambda(t)^2\left[ha(t)+\tau(t)\right]+mn\left(mn-n+1\right)\left(mn-2n+2\right)\mu^3h^2k(t)\lambda(t)^3\tau(t)=0,\\ \left[m\mu hn(n-1)^2b(t)\lambda(t)^3-m^3\mu^3h^2n^3k(t)\lambda(t)^3\right]=0,\\ 2m=3m-2,\\ 2m-1=3m-3.\end{array}\right. \tag{18}$$

Solving the system (18) by *Maple,* we have the following case of solution:

$$\begin{array}{l}\mu=\mu,\lambda(t)=c,h=h,\tau(t)=\frac{(n-1)^2b(t)}{4h\mu^2n^2k(t)},\\ a(t)=a(t),b(t)=b(t),k(t)=\frac{-(n-1)^2b(t)}{2n\mu^2h^2[2na(t)+c(n+1)b(t)]},m=2,\end{array} \tag{19}$$

where c is a constant. In this case, the exact solution of Eq. (2) has the from:

$$u\left(\xi\right)=\left[c\sin^2\left(\mu\xi\right)\right]^{\frac{1}{n-1}}, \tag{20}$$

where

$$\xi=hx+\int\frac{\left(n-1\right)^2b(t)}{4h\mu^2n^2k(t)}dt. \tag{21}$$

4 Conclusions

In this article, a modified sine–cosine method was applied in order to find the exact solutions of the generalized K(n,n) and the generalized BBM equations with variable coefficients. This method can be considered as a simple method for solving some of nonlinear partial differential equations without resort to any symbolic computation and the obtained results are very concise. The solutions so obtained have also been verified to satisfy the original equation. The proposed method is important because the solutions obtained can be applied to a wide range of problems in science and engineering.

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