On Max Type Difference Equations: Expressions of Solutions

Nouressadat Touafek *, Yacine Halim
Mathematics Department and LPTH Laboratory, Jijel University
Ouled Aissa, BP 98, 18000 (Jijel), Algeria

(Received 28 February 2011, accepted 30 May 2011)

Abstract: In this paper, we give expressions of the solutions of the two max-type difference equations

\[ x_{n+1} = \max \left\{ x_{n-1}^2, \frac{1}{x_{n-1}} \right\}, \quad n = 0, 1, \ldots, \]

where the initial conditions \( x_0, x_{-1} \) are nonzero real numbers.

Keywords: difference equation; periodic solution

1 Introduction

Recently there has been a great interest in the study of the so-called max-type difference equations. See, for example, [4, 5] and the references cited therein.

This type of difference equations arises naturally from, for example, certain models in automatic control theory (See, e.g., [3]).

In [4], Simsek et al. investigate the solutions of the max-type difference equation

\[ x_{n+1} = \max \left\{ x_{n-1}, \frac{1}{x_{n-1}} \right\}. \]

In [1], Elabbasy et al. investigate the solutions of the max-type difference equation

\[ x_{n+1} = \max \left\{ x_{n-2}, \frac{1}{x_{n-2}} \right\}. \]

In this paper we study the following difference equations

\[ x_{n+1} = \max \left\{ x_{n-1}^2, \frac{1}{x_{n-1}} \right\}, \quad x_{n+1} = \max \left\{ x_{n-1}, \frac{1}{x_{n-1}} \right\}, \quad n = 0, 1, \ldots, \]

where the initial conditions \( x_0, x_{-1} \) are nonzero real numbers.

2 The first equation: \( x_{n+1} = \max \left\{ x_{n-1}^2, \frac{1}{x_{n-1}} \right\} \)

In this section we give a specific form of the solutions of the following difference equation

\[ x_{n+1} = \max \left\{ x_{n-1}^2, \frac{1}{x_{n-1}} \right\}, \quad n = 0, 1, \ldots, \]

(1)

Theorem 2.1 Consider the difference equation (1). Let \( x_0, x_{-1} > 0 \).
1) If $x_0, x_{-1} \leq 1$, then
\[ x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^{2^k}, \; k = 0, 1, ... \]
and
\[ x_{2k} = \left( \frac{1}{x_0} \right)^{2^{k-1}}, \; k = 1, 2, ... \]

2) If $x_0, x_{-1} \geq 1$, then
\[ x_{2k+1} = (x_{-1})^{2^{k+1}}, \; k = 0, 1, ... \]
and
\[ x_{2k} = (x_0)^{2^k}, \; k = 1, 2, ... \]

3) If $x_0 \geq 1, x_{-1} \leq 1$, then
\[ x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^{2^k}, \; k = 0, 1, ... \]
and
\[ x_{2k} = (x_0)^{2^k}, \; k = 1, 2, ... \]

4) If $x_0 \leq 1, x_{-1} \geq 1$, then
\[ x_{2k+1} = (x_{-1})^{2^{k+1}}, \; k = 0, 1, ... \]
and
\[ x_{2k} = \left( \frac{1}{x_0} \right)^{2^{k-1}}, \; k = 1, 2, ... \]

**Proof.** 1) Let $x_0 \leq 1, x_{-1} \leq 1$ using the fact that $x_{-1}^2, x_0^2 \leq 1$ and $\frac{1}{x_{-1}}, \frac{1}{x_0} \geq 1$, we get
\[ x_1 = \max \left\{ x_{-1}^2, \frac{1}{x_{-1}}, x_0^2, \frac{1}{x_0} \right\} = \frac{1}{x_{-1}}, x_3 = \max \left\{ x_{-1}^2, \frac{1}{x_{-1}}, x_2^2, \frac{1}{x_2} \right\} = \frac{1}{x_{-1}}, x_5 = \max \left\{ x_{-1}^2, \frac{1}{x_{-1}}, x_4^2, \frac{1}{x_4} \right\} = \frac{1}{x_{-1}}. \]
Also, $x_2 = \max \left\{ x_0^2, \frac{1}{x_0} \right\} = \frac{1}{x_0}, x_4 = \max \left\{ x_2^2, \frac{1}{x_2}, x_3^2, \frac{1}{x_3} \right\} = \frac{1}{x_0}, x_6 = \max \left\{ x_4^2, \frac{1}{x_4}, x_5^2, \frac{1}{x_5} \right\} = \frac{1}{x_0}.$

So, by induction we obtain that
\[ x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^{2^k}, \; k = 0, 1, ... \]
and
\[ x_{2k} = \left( \frac{1}{x_0} \right)^{2^{k-1}}, \; k = 1, 2, ... \]

By the same arguments we can prove cases 2), 3) and 4). ■

Similarly to theorem (2.1), we can easily prove the following theorems.

**Theorem 2.2** Consider the difference equation (1). Let $x_0, x_{-1} < 0$.

1) If $x_0, x_{-1} \leq -1$, then
\[ x_{2k+1} = (x_{-1})^{2^{k+1}}, \; k = 0, 1, ... \]
and
\[ x_{2k} = (x_0)^{2^k}, \; k = 1, 2, ... \]

2) If $x_0, x_{-1} \geq -1$, then
\[ x_1 = x_{-1}^2, x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^{2^k}, \; k = 1, 2, ... \]
and
\[ x_2 = x_0^2, x_{2k} = \left( \frac{1}{x_0} \right)^{2^{k-1}}, \; k = 2, 3, ... \]
3) If $x_0 \geq -1$, $x_{-1} \leq -1$, then

$$x_{2k+1} = (x_{-1})^{2^{k+1}}, \; k = 0, 1, ...$$

and

$$x_2 = x_0^2, \; x_{2k} = \left(\frac{1}{x_0}\right)^{2^{k-1}}, \; k = 2, 3, ...$$

4) If $x_0 \leq -1$, $x_{-1} \geq -1$, then

$$x_1 = x_{-1}^2, \; x_{2k+1} = \left(\frac{1}{x_{-1}}\right)^{2^k}, \; k = 1, 2, ...$$

and

$$x_{2k} = (x_0)^{2^k}, \; k = 1, 2, ...$$

Theorem 2.3 Consider the difference equation (1). Let $x_0 > 0, x_{-1} < 0$.

1) If $x_0 \leq 1$, $x_{-1} \leq -1$, then

$$x_{2k+1} = (x_{-1})^{2^{k+1}}, \; k = 0, 1, ...$$

and

$$x_2 = x_0^2, \; x_{2k} = \left(\frac{1}{x_0}\right)^{2^{k-1}}, \; k = 1, 2, ...$$

2) If $x_0 \geq 1$, $x_{-1} \geq -1$, then

$$x_1 = x_{-1}^2, \; x_{2k+1} = \left(\frac{1}{x_{-1}}\right)^{2^k}, \; k = 1, 2, ...$$

and

$$x_{2k} = (x_0)^{2^k}, \; k = 1, 2, ...$$

3) If $x_0 \geq 1$, $x_{-1} \leq -1$, then

$$x_{2k+1} = (x_{-1})^{2^{k+1}}, \; k = 0, 1, ...$$

and

$$x_{2k} = (x_0)^{2^k}, \; k = 1, 2, ...$$

4) If $x_0 \leq 1$, $x_{-1} \geq -1$, then

$$x_1 = x_{-1}^2, \; x_{2k+1} = \left(\frac{1}{x_{-1}}\right)^{2^k}, \; k = 1, 2, ...$$

and

$$x_{2k} = \left(\frac{1}{x_0}\right)^{2^{k-1}}, \; k = 1, 2, ...$$

Theorem 2.4 Consider the difference equation (1). Let $x_0 < 0, x_{-1} > 0$.

1) If $x_0 \leq -1$, $x_{-1} \leq 1$, then

$$x_{2k+1} = \left(\frac{1}{x_{-1}}\right)^{2^k}, \; k = 0, 1, ...$$

and

$$x_2 = x_0^2, \; x_{2k} = \left(\frac{1}{x_0}\right)^{2^{k-1}}, \; k = 2, 3, ...$$
3) If \( x_0 \geq -1, x_{-1} \leq 1 \), then
\[
x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^{2^k}, \quad k = 0, 1, ...
\]
and
\[
x_2 = x_0^2, \quad x_{2k} = \left( \frac{1}{x_0} \right)^{2^{k-1}}, \quad k = 2, 3, ...
\]

4) If \( x_0 \leq -1, x_{-1} \geq 1 \), then
\[
x_{2k+1} = (x_{-1})^{2^{k+1}}, \quad k = 0, 1, ...
\]
and
\[
x_{2k} = (x_0)^{2^k}, \quad k = 1, 2, ...
\]

3 The second equation: \( x_{n+1} = \max \left\{ x_{n-1}, \frac{1}{x_{n-1}} \right\} \)

In this section we give a specific form of the solutions of the following difference equation
\[
x_{n+1} = \max \left\{ x_{n-1}, \frac{1}{x_{n-1}} \right\}, \quad n = 0, 1, ..., \tag{2}
\]
and in each cases we deduces that every solutions is periodic with period two.

**Theorem 3.1** Consider the difference equation (2). Let \( x_0, x_{-1} > 0 \).

1) If \( x_0, x_{-1} \leq 1 \), then
\[
x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^{2^k}, \quad k = 0, 1, ...
\]
\[
x_{2k} = \left( \frac{1}{x_0} \right)^{2^k}, \quad k = 1, 2, ...
\]
so \( x_{n+2} = x_n \) for \( n \geq 1 \), and the solution is periodic with period two.

2) If \( x_0, x_{-1} \geq 1 \), then
\[
x_{2k+1} = x_{-1}, \quad k = 0, 1, ...
\]
\[
x_{2k} = x_0, \quad k = 1, 2, ...
\]
so \( x_{n+2} = x_n \) for \( n \geq -1 \), and the solution is periodic with period two.

3) If \( x_0 \geq 1, x_{-1} \leq 1 \), then
\[
x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^{2^k}, \quad k = 0, 1, ...
\]
\[
x_{2k} = x_0, \quad k = 1, 2, ...
\]
so \( x_{n+2} = x_n \) for \( n \geq 1 \), and the solution is periodic with period two.

4) If \( x_0 \leq 1, x_{-1} \geq 1 \), then
\[
x_{2k+1} = x_{-1}, \quad k = 0, 1, ...
\]
\[
x_{2k} = \left( \frac{1}{x_0} \right)^{2^k}, \quad k = 1, 2, ...
\]
so \( x_{n+2} = x_n \) for \( n \geq 1 \), and the solution is periodic with period two.
Theorem 3.3

By induction we obtain that

$$x_1 = \max \left\{ x_{-1}, \frac{1}{x_{-1}} \right\} = \frac{1}{x_{-1}}, \quad x_2 = \max \left\{ x_0, \frac{1}{x_0} \right\} = \frac{1}{x_0},$$

and

$$x_3 = \max \left\{ x_1, \frac{1}{x_1} \right\} = \frac{1}{x_{-1}}, \quad x_4 = \max \left\{ x_2, \frac{1}{x_2} \right\} = \frac{1}{x_0}. $$

Proof. 1) Let $x_0 \leq 1, x_{-1} \leq 1$ using the fact that $x_{2k-1} \leq 1$ and $\frac{1}{x_{-1}}, \frac{1}{x_0} \geq 1$, we get

$$x_1 = \max \left\{ x_{-1}, \frac{1}{x_{-1}} \right\} = \frac{1}{x_{-1}}, \quad x_2 = \max \left\{ x_0, \frac{1}{x_0} \right\} = \frac{1}{x_0},$$

and

$$x_3 = \max \left\{ x_1, \frac{1}{x_1} \right\} = \frac{1}{x_{-1}}, \quad x_4 = \max \left\{ x_2, \frac{1}{x_2} \right\} = \frac{1}{x_0}. $$

By induction we obtain that

$$x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^2, \quad k = 0, 1, ...$$

$$x_{2k} = \left( \frac{1}{x_0} \right)^2, \quad k = 1, 2, ...$$

so $x_{n+2} = x_n$ for $n \geq 1$, and the solution is periodic with period two.

By the same arguments we can prove cases 2), 3) and 4). ■

Similarly to theorem (3.1), we can easily prove the following theorems.

Theorem 3.2 Consider the difference equation (2). Let $x_0, x_{-1} < 0$.

1) If $x_0, x_{-1} \leq -1$, then

$$x_1 = \frac{1}{x_{-1}}, \quad x_{2k+1} = (x_{-1})^4, \quad k = 1, 2, ...$$

$$x_2 = \frac{1}{x_0}, \quad x_{2k} = (x_0)^4, \quad k = 2, 3, ...$$

so $x_{n+2} = x_n$ for $n \geq 3$, and the solution is periodic with period two.

2) If $x_0, x_{-1} \geq -1$, then

$$x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^2, \quad k = 0, 1, 2, ...$$

$$x_{2k} = \left( \frac{1}{x_0} \right)^2, \quad k = 1, 2, ...$$

so $x_{n+2} = x_n$ for $n \geq 1$, and the solution is periodic with period two.

3) If $x_0 \geq -1, x_{-1} \leq -1$, then

$$x_1 = \frac{1}{x_{-1}}, \quad x_{2k+1} = (x_{-1})^4, \quad k = 1, 2, ...$$

$$x_{2k} = \left( \frac{1}{x_0} \right)^2, \quad k = 1, 2, ...$$

so $x_{n+2} = x_n$ for $n \geq 2$, and the solution is periodic with period two.

4) If $x_0 \leq -1, x_{-1} \geq -1$, then

$$x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^2, \quad k = 0, 1, ...$$

$$x_2 = \frac{1}{x_0}, \quad x_{2k} = (x_0)^4, \quad k = 2, 3, ...$$

so $x_{n+2} = x_n$ for $n \geq 3$, and the solution is periodic with period two.

Theorem 3.3 Consider the difference equation (1). Let $x_0 > 0, x_{-1} < 0$.

1) If $x_0 \leq 1, x_{-1} \leq -1$, then

$$x_1 = \frac{1}{x_{-1}}, \quad x_{2k+1} = (x_{-1})^4, \quad k = 1, 2, ...$$

$$x_{2k} = \left( \frac{1}{x_0} \right)^2, \quad k = 1, 2, ...$$

so $x_{n+2} = x_n$ for $n \geq 2$, and the solution is periodic with period two.
2) If \( x_0 \geq 1, \ x_{-1} \geq -1 \), then

\[
x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^2, \ k = 0, 1, ...
\]

\[
x_{2k} = x_0, \ k = 1, 2, ...
\]

so \( x_{n+2} = x_n \) for \( n \geq 0 \), and the solution is periodic with period two.

3) If \( x_0 \geq 1, \ x_{-1} \leq -1 \), then

\[
x_1 = \frac{1}{x_{-1}}, \ x_{2k+1} = (x_{-1})^4, \ k = 1, 2, ...
\]

and

\[
x_{2k} = x_0, \ k = 1, 2, ...
\]

so \( x_{n+2} = x_n \) for \( n \geq 2 \), and the solution is periodic with period two.

4) If \( x_0 \leq 1, \ x_{-1} \geq -1 \), then

\[
x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^2, \ k = 1, 2, ...
\]

\[
x_{2k} = \left( \frac{1}{x_0} \right)^2, \ k = 1, 2, ...
\]

so \( x_{n+2} = x_n \) for \( n \geq 1 \), and the solution is periodic with period two.

Theorem 3.4 Consider the difference equation (2). Let \( x_0 < 0, x_{-1} > 0 \).

1) If \( x_0 \leq -1, \ x_{-1} \leq 1 \), then

\[
x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^2, \ k = 0, 1, ...
\]

\[
x_2 = \frac{1}{x_0}, \ x_{2k} = (x_0)^4, \ k = 2, 3, ...
\]

so \( x_{n+2} = x_n \) for \( n \geq 3 \), and the solution is periodic with period two.

2) If \( x_0 \geq -1, \ x_{-1} \geq 1 \), then

\[
x_{2k+1} = x_{-1}, \ k = 0, 1, ...
\]

\[
x_{2k} = \left( \frac{1}{x_0} \right)^2, \ k = 1, 2, ...
\]

so \( x_{n+2} = x_n \) for \( n \geq 1 \), and the solution is periodic with period two.

3) If \( x_0 \geq -1, \ x_{-1} \leq 1 \), then

\[
x_{2k+1} = \left( \frac{1}{x_{-1}} \right)^2, \ k = 0, 1, ...
\]

\[
x_{2k} = \left( \frac{1}{x_0} \right)^2, \ k = 1, 2, ...
\]

so \( x_{n+2} = x_n \) for \( n \geq 1 \), and the solution is periodic with period two.

4) If \( x_0 \leq -1, \ x_{-1} \geq 1 \), then

\[
x_{2k+1} = x_{-1}, \ k = 0, 1, ...
\]

\[
x_2 = \frac{1}{x_0}, \ x_{2k} = (x_0)^4, \ k = 2, 3, ...
\]

so \( x_{n+2} = x_n \) for \( n \geq 3 \), and the solution is periodic with period two.

4 Conclusion

The authors believe that the results in this paper can be extended to the following max-type difference equations

\[
x_{n+1} = \max \left\{ x_{n-k}^{m}, \frac{1}{x_{n-k}} \right\}, \ x_{n+1} = \max \left\{ x_{n-k}, \frac{1}{x_{n-k}}^{m} \right\} \text{ for } k \geq 1 \text{ and } m \geq 1.
\]
References


[2] J. Feuer, K.T. Mcdonnell. On the eventual periodicity of \( x_{n+1} = \max\left(\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right) \) with a perioe-five parameter. Comp. Math. Appl. 56(2008): 1883-890.

