Oscillation Behavior of Certain Fourth Order Linear and Nonlinear Difference Equations

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Abstract: This paper deals with oscillatory behavior of solutions of fourth order difference equation of the form

\[ \Delta^3 \left( \frac{a_n}{q_n} \Delta y_n \right) + q_n y_{n+1} = 0, \quad n = 0, 1, 2, \ldots \]

and

\[ \Delta^3 \left( \frac{a_n}{q_n} \Delta y_n \right) + q_n f(y_{n+1}) = 0, \quad n = 0, 1, 2, \ldots \]

Examples are given to illustrate the results.

Keywords: oscillation; fourth order; linear; nonlinear and difference equations

1 Introduction

We are concerned with the oscillatory behavior of all the solutions of a fourth order difference equations of the form

\[ \Delta^3 \left( \frac{a_n}{q_n} \Delta y_n \right) + q_n y_{n+1} = 0, \quad n \geq n_1. \quad (1.1) \]

and

\[ \Delta^3 \left( \frac{a_n}{q_n} \Delta y_n \right) + q_n f(y_{n+1}) = 0, \quad n \geq n_1. \quad (1.2) \]

where the following conditions are assumed to hold.

(\( H_1 \)) \( \{a_n\} \) and \( \{q_n\} \) are real sequences and \( q_n \neq 0 \) for infinitely many values of \( n \).

(\( H_2 \)) \( r_n = \sum_{s=n_1}^{n-1} \frac{q_s}{a_s} \to \infty \) as \( n \to \infty \)

(\( H_3 \)) \( f : \mathbb{R} \to \mathbb{R} \) is continuous such that \( xf(x) > 0 \) for \( x \neq 0 \) and \( \frac{f(x)}{x} \geq L > 0 \)

By a solution of equation (1.1), we mean a real sequence \( \{y_n\} \) satisfying (1.1) for \( n \geq n_1 \). A solution \( \{y_n\} \) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called non-oscillatory. \( \Delta \) is the forward difference operator defined by \( \Delta y_n = y_{n+1} - y_n \).

In recent years, much research is being done in the study of oscillatory behavior of solutions of third order difference equations.

For more details on oscillatory behavior of difference equations, one may refer [1-17]
2 Main results

In this section, we present some sufficient conditions for the oscillation of all the solutions of (1.1) and (1.2).

Theorem 1 Assume that (H1),(H2) hold and

\[ \sum_{s=n_2}^{\infty} \left( r_n q_n - \frac{a_{s+1}(\Delta r_s)^2}{4r_s q_{s+1}(s-n_1)^2} \right) = \infty \text{ for } n_2 \geq n_1. \]  \hspace{1cm} (2.1)

Then every solution \( \{y_n\} \) of equation (1.1) oscillatory.

Proof. Let \( \{y_n\} \) be a non-oscillatory solution of equation (1.1), without loss of generality, we may assume that \( y_n > 0 \) for \( n \geq n_1 \). From equation (1.1), we have \( \Delta^2 \left( \frac{a_n}{q_n} \Delta y_n \right) \leq 0 \) for \( n \geq n_1 \). Then \( \{y_n\}, \{\Delta y_n\}, \Delta \left( \frac{a_n}{q_n} \Delta y_n \right) \) and \( \Delta^2 \left( \frac{a_n}{q_n} \Delta y_n \right) \) are monotone and eventually of one sign.

We claim \( \Delta^2 \left( \frac{a_n}{q_n} \Delta y_n \right) > 0 \). Suppose to the contrary that \( \Delta^2 \left( \frac{a_n}{q_n} \Delta y_n \right) \leq 0 \) for \( n \geq n_2 \) for \( n_2 \geq n_1 \). Since \( \Delta^2 \left( \frac{a_n}{q_n} \Delta y_n \right) \) is non-increasing there exist a non-negative constant \( k_1 \) and \( n_3 \geq n_2 \) such that \( \Delta^2 \left( \frac{a_n}{q_n} \Delta y_n \right) \leq -k_1 \) for \( n \geq n_3 \) and \( k_1 > 0 \). Summing the last inequality from \( n_3 \) to \( (n-1) \), we obtain

\[ \Delta \left( \frac{a_n}{q_n} \Delta y_n \right) \leq \Delta \left( \frac{a_{n_3}}{q_{n_3}} \Delta y_{n_3} \right) - k_1(n-n_3) \]

Letting \( n \to \infty \), then \( \Delta \left( \frac{a_n}{q_n} \Delta y_n \right) \to -\infty \). Thus, there is an integer \( n_4 \geq n_3 \) such that for \( n \geq n_4 \), \( \Delta \left( \frac{a_n}{q_n} \Delta y_n \right) \leq \Delta (a_{n_4} q_{n_4} \Delta y_{n_4}) < 0 \). Summing the last inequality from \( n_4 \) to \( (n-1) \), we obtain

\[ \frac{a_n}{q_n} \Delta y_n \leq \frac{a_{n_4}}{q_{n_4}} \Delta y_{n_4} - k_2(n-n_4), k_2 > 0. \]

This implies that \( \frac{a_n}{q_n} \Delta y_n \to -\infty \) as \( n \to \infty \). Thus, there is an integer \( n_5 \geq n_4 \) such that for \( n \geq n_5 \), \( \frac{a_n}{q_n} \Delta y_n \leq \frac{a_{n_5}}{q_{n_5}} \Delta y_{n_5} < 0 \). This implies \( \frac{a_n}{q_n} \Delta y_n \leq -k_3 \) for \( n \geq n_5 \) and \( k_3 > 0 \). That is, \( \Delta y_n \leq -k_3 \frac{a_n}{q_n} \). Summing the last inequality from \( n_5 \) to \( (n-1) \), we obtain

\[ y_n \leq y_{n_5} - k_3 \sum_{s=n_5}^{n-1} \frac{q_s}{a_s}. \]

This implies that \( y_n \to -\infty \) as \( n \to \infty \). Which is a contradiction the fact that \( y_n \) is positive. Then \( \Delta^2 \left( \frac{a_n}{q_n} \Delta y_n \right) > 0 \). Define \( w_n \) by

\[ w_n = \frac{r_n}{y_{n+1}} \Delta^2 \left( \frac{a_n}{q_n} \Delta y_n \right) > 0, \]  \hspace{1cm} (2.2)

\[ \Delta w_n = \Delta^2 \left( \frac{a_n+1}{q_n+1} \Delta y_{n+1} \right) \Delta \left( \frac{r_n}{y_{n+1}} \right) + \frac{r_n}{y_{n+1}} \Delta^3 \left( \frac{a_n}{q_n} \Delta y_n \right), \]

\[ \Delta w_n = \frac{y_{n+1} \Delta r_n - r_n \Delta y_{n+1}}{y_{n+1} y_n + 2} \Delta^2 \left( \frac{a_n+1}{q_n+1} \Delta y_{n+1} \right) - r_n q_n, \]

\[ \Delta w_n \leq -r_n q_n + \frac{\Delta r_n}{r_n+1} w_{n+1} - \frac{r_n \Delta y_{n+1}}{y_{n+2}} \Delta^2 \left( \frac{a_n+1}{q_n+1} \Delta y_{n+1} \right). \]

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Consider
\[
\frac{a_n}{q_n} \Delta y_n = \frac{a_{n_1}}{q_{n_1}} \Delta y_{n_1} + \sum_{s=n_1}^{n-1} \Delta \left( \frac{a_s}{q_s} \Delta y_s \right) \geq (n - 1 - n_1) \Delta \left( \frac{a_{n_1}}{q_{n_1}} \Delta y_{n_1} \right) ; \quad n \geq n_1 + 1
\]
\[
\Rightarrow \frac{a_{n+1}}{q_{n+1}} \Delta y_{n+1} = \frac{a_{n+1}}{q_{n+1}} \Delta y_{n+1} + \sum_{s=n_1}^{n-1} \Delta \left( \frac{a_s}{q_s} \Delta y_s \right) \geq (n - 1) \Delta \left( \frac{a_{n+1}}{q_{n+1}} \Delta y_{n+1} \right) ; \quad n \geq n_2 = n_1 + 1.
\]

And
\[
\Delta \left( \frac{a_{n+1}}{q_{n+1}} \Delta y_{n+1} \right) = \Delta \left( \frac{a_{n+1}}{q_{n+1}} \Delta y_{n_1+1} \right) + \sum_{s=n_1}^{n-1} \Delta^2 \left( \frac{a_s}{q_s} \Delta y_{s+1} \right) \geq (n - 1) \Delta^2 \left( \frac{a_{n+1}}{q_{n+1}} \Delta y_{n+1} \right)
\]
\[
\Rightarrow \frac{a_{n+1}}{q_{n+1}} \Delta y_{n+1} \geq (n - 1) \Delta^2 \left( \frac{a_{n+1}}{q_{n+1}} \Delta y_{n+1} \right).
\]

Therefore,
\[
\Delta w_n \leq -r_n q_n + \frac{\Delta r_n}{r_{n+1}} w_{n+1} - \left( \frac{n - 1}{n_1 - 1} \right)^2 r_n q_n + \frac{a_{n+1}}{q_{n+1}} w_{n+1}.
\]
That is
\[
\Delta w_n \leq -r_n q_n + \frac{(\Delta r_n)^2 a_{n+1}}{4r_n q_n (n - n_1)^2} - \left( \frac{n - 1}{n_1 - 1} \right)^2 r_n q_n + \frac{a_{n+1}}{q_{n+1}} w_{n+1}.
\]
\[
\Rightarrow \Delta w_n \leq - \left( \frac{r_n q_n}{4r_n q_n (n - n_1)^2} - \frac{(\Delta r_n)^2 a_{n+1}}{4r_n q_n (n - n_1)^2} \right) w_{n+1}.
\]

Summing the last inequality from \(n_2\) to \((n - 1)\), we have
\[
w_n \leq \sum_{s=n_2}^{n-1} \left( \frac{r_s q_s}{4r_s q_s (s - n_1)^2} - \frac{(\Delta r_s)^2 a_{s+1}}{4r_s q_s (s - n_1)^2} \right).
\]

Leeting \(n \to \infty\), we have, in view of (2.1) that \(w_n \to -\infty\) as \(n \to \infty\), which contradiction \(w_n > 0\) and the proof is complete. \(\blacksquare\)

**Theorem 2** Assume that (H1)-(H3) hold and
\[
\sum_{s=n_2}^{\infty} \left( L r_n q_n - \frac{a_{s+1}}{4r_{s+1} q_{s+1} (s - n_1)^2} \right) = \infty \text{ for } n_2 \geq n_1.
\]

Then every solution \(\{y_n\}\) of equation (1.2) oscillatory.

**Proof.** Proof of this theorem is simillar to the proof of Theorem 1 and hence the details are omitted.

Next, we present some new oscillation results for equation (1.1) and (1.2), we introduce a double sequence \(\{H(m, n)/m \geq n \geq 0\}\) such that

(i) \(H(m, m) = 0\) for \(m \geq 0\)

(ii) \(H(m, n) > 0\) for \(m > n \geq 0\)

(iii) \(\Delta_2 H(m, n) = -h(m, n) \sqrt{H(m, n)^2}, m > n \geq 0\)

Where \(\Delta_2 H(m, n) = H(m, n + 1) - H(m, n) \leq 0\) for \(m \geq n \geq 0\) \(\blacksquare\)
Then which implies that
\[ \lim_{n \to \infty} \sup_{n_2} \frac{1}{H(n, n_2)} \sum_{s=n_2}^{n-1} \left( H(n, s) r_s q_s - \frac{a_{s+1} r_{s+1}^2}{r_s q_{s+1}(s-n_1)^2} \left( h(n, s) - \frac{\Delta r_s}{r_{s+1}} \sqrt{H(n, s)} \right)^2 \right) = \infty \] (2.5)
for \( n_2 \geq n_1 \).

Then ,every solution \( \{y_n\} \) of equation (1.1) oscillatory.

**Proof.** Proceeding in Theorem 1 ,we assume that equation (1.1) has a non-oscillatory solution, say \( y_n > 0 \) for all \( n \geq n_1 \). Defining again \( \{w_n\} \) by equation (2.2), then from Theorem 1, we have \( w_n > 0 \) and equation (2.3) holds. From equation (2.3) we have for \( n \geq n_2 \)

\[ r_n q_n \leq -\Delta w_n + \frac{\Delta r_n}{r_{n+1}} w_{n+1} - \frac{(n-n_1)^2 r_n q_{n+1}}{a_{n+1} r_{n+1}^2} w_{n+1}^2 \]

Therefore, we have

\[ \sum_{s=n_2}^{n-1} H(n, s) r_s q_s \leq -\sum_{s=n_2}^{n-1} H(n, s) \Delta w_s + \sum_{s=n_2}^{n-1} \frac{\Delta r_s}{r_{s+1}} w_{s+1} - \sum_{s=n_2}^{n-1} H(n, s) \frac{(s-n_1)^2 r_s q_{s+1}}{a_{s+1} r_{s+1}^2} w_{s+1}^2, \]

which yields after summing by parts

\[ \sum_{s=n_2}^{n-1} H(n, s) r_s q_s \leq H(n, n_2) w_{n_2} + \sum_{s=n_2}^{n-1} \frac{\Delta_2 H(n, s) w_{s+1}}{r_{s+1}} + \frac{(n-n_1)^2 r_n q_{n+1}}{a_{n+1} r_{n+1}^2} w_{n+1}^2, \]

hence

\[ \sum_{s=n_2}^{n-1} H(n, s) r_s q_s \leq H(n, n_2) w_{n_2} - \sum_{s=n_2}^{n-1} \left( h(n, s) \sqrt{H(n, s)} - \frac{\Delta r_s}{r_{s+1}} H(n, s) \right) w_{s+1} \]

\[ - \sum_{s=n_2}^{n-1} H(n, s) \frac{(s-n_1)^2 r_s q_{s+1}}{a_{s+1} r_{s+1}^2} w_{s+1}^2, \]

\[ \Rightarrow \sum_{s=n_2}^{n-1} H(n, s) r_s q_s \leq H(n, n_2) w_{n_2} + \frac{1}{4} \sum_{s=n_2}^{n-1} \frac{a_{s+1} r_{s+1}^2}{(s-n_1)^2 r_s q_{s+1}} \left( h(n, s) - \frac{\Delta r_s}{r_{s+1}} \sqrt{H(n, s)} \right)^2 \]

\[ - \sum_{s=n_2}^{n-1} \frac{a_{s+1}}{2(s-n_1)} \sqrt{H(n, s) r_s q_{s+1}} \left( h(n, s) \sqrt{H(n, s)} - \frac{\Delta r_s}{r_{s+1}} H(n, s) \right)^2. \]

Then

\[ \sum_{s=n_2}^{n-1} H(n, s) r_s q_s - \frac{a_{s+1} r_{s+1}^2}{4(s-n_1)^2 r_s q_{s+1}} \left( h(n, s) - \frac{\Delta r_s}{r_{s+1}} \sqrt{H(n, s)} \right)^2 < H(n, n_2) w_{n_2}, \]

which implies that

\[ \lim_{n \to \infty} \sup_{n_2} \frac{1}{H(n, n_2)} \sum_{s=n_2}^{n-1} \left( H(n, s) r_s q_s - \frac{a_{s+1} r_{s+1}^2}{r_s q_{s+1}(s-n_1)^2} \left( h(n, s) - \frac{\Delta r_s}{r_{s+1}} \sqrt{H(n, s)} \right)^2 \right) < w_{n_2} < \infty, \]

which contradicts the equation (2.5). Hence the theorem is proved. ■

**B.Selvaraj I. Mohammed Ali Jaffer: Oscillation Behavior of Certain Fourth Order Linear and Nonlinear Difference Equations**

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**Theorem 4**  Assume that (H1)-(H3) hold. Furthermore, assume that there exists a double $H$ such that

$$\lim_{n \to \infty} \sup_{s=n_2} H(n, s) \sum_{s=n_2}^{n-1} \left( L \frac{r_s q_s}{r_{s+1}} \left( h(n, s) - \frac{\Delta r_s}{r_{s+1}} \sqrt{H(n, s)} \right)^2 \right) = \infty \quad (2.6)$$

for $n_2 \geq n_1$.

Then every solution $\{y_n\}$ of equation (1.2) oscillatory.

**Proof.** Proof of this theorem is similar to the proof of Theorem 3 and hence the details are omitted. ■

3 **Examples**

**Example 1:**

The difference equation

$$\Delta^3 (n \Delta y_n) + n^2 y_{n+1} = 0 \quad (3.1)$$

satisfies all condition of Theorem 1 and Theorem 3. Here $H(m, n) = m - n$. Hence all solution of equation (3.1) are oscillatory.

**Example 2:**

The difference equation

$$\Delta^3 ((n+1) \Delta y_n) + (n+5)^2 y_{n+1}^3 = 0 \quad (3.2)$$

satisfies all condition of Theorem 2 and Theorem 4. Here $H(m, n) = (m-n)^\alpha, \alpha \geq 1, m \geq n \geq 0$ and $f(x) = x^3$. Hence all solution of equation (3.2) are oscillatory.

**References**


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