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Numerical Solution of the Falkner-Skan Equation with Stretching Boundary by Collocation Method

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Abstract: In the present paper, a solution for the boundary value problem over a semi-infinite interval has been obtained by transforming the two-dimensional laminar boundary equations into a nonlinear ordinary equation using similarity variables. Moreover, a collocation method is proposed to solve the Falkner-Skan equation. This method is based on rational Legendre functions and convert the Falkner-Skan equation to a system of nonlinear algebraic equations. The results are tabulated and compared with some other methods.

Keywords: Falkner-Skan equation; Collocation method; Rational Legendre functions; Semi-infinite interval; Nonlinear ODE

1 Introduction

Many science and engineering problems arise in unbounded domain. Different spectral methods have been proposed for solving problems in unbounded domains. However, the most common approach is to apply polynomials that are orthogonal over unbounded domains, such as the Hermite spectral method and the Laguerre spectral method [1–8].

Guo [9–11] proposed a method that proceeds by mapping the original problem in an unbounded domain to a problem in a bounded domain, and then used the appropriate Jacobi polynomials to approximate the resulting problems.

Another approach is replacing infinite domain with $[-L, L]$ and semi-infinite interval with $[0, L]$ by choosing L, sufficiently large. This method is named domain truncation [12].

Another effective direct approach for solving such problems is based on rational approximations. Christov [13] and Boyd [14, 15] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [15] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping to the Chebyshev polynomials. Guo et al. [16] introduced a new set of rational Legendre functions which are mutually orthogonal in $L^2(0, +\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half line. Boyd et al. [17] applied pseudospectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre and mapped Fourier sine.

Parand et al. [18–27] applied the spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on rational Tau and collocation methods.

Falkner-Skan equation is categorized as a boundary layer problem. The boundary layer theory was first proposed by Prandtl in 1904. It asserts that the viscous effect would be confined to a thin shear layer adjacent to the solid boundary in the case of a motion with very little viscosity. Hence, the fluid motion is split into two parts: near the boundary the viscosity effect is important and the fluid is said to be viscous, and far away from the boundary the fluid viscous effect is unimportant and can be treated as an inviscid fluid [28].

There is a large body of literature on the solution of Falkner-Skan equation, see Hartree [29], Cebeci and Keller [30], Hildyard [31], Yang and Chien [32], Soundalgekar [33], Forbrich [34], Thomas and Harris [35], Liu and Lyu [36], Brodie and Banks [37], Assassa and Ghazy [38], Summers [39], Massoudi and Ramezani [40], Fazio [41], N. S. Asaithambi [42], Heeg et al. [43], Morgan et al. [44] and Liao [45].

There is a numerical solution to the Falkner-Skan equation in [46], [47] and [28] useding Runge-Kutta integration scheme with shooting method and Lie-group shooting method, respectively to solve this problem. Asiathambi presented

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several numerical methods based on finite-element methods, shooting and finite-differences [48–50]. Yao [51] developed an analytical technique, named Homotopy analysis method, and gave an analytical approximation solution to the Falkner-Skan equation. [52] solved this equation analytically for some value of β. [53] built a solution from the asymptotic method of matched expansions at order one. [54–56] solved the MHD Falkner-Skan using Hankel-padé method, finite-difference scheme and homotopy analysis method, respectively. Kuo adopted the differential transformation method to investigate the velocity and shear stress fields [57] and the temperature field [58] associated with the Flakner-Skan boundary-layer problem. [59] solved the nonlinear ordinary differential equation using Padé Adomian decomposition method (ADM) to obtain series solution.

Here we use rational Legendre collocation method to solve Falkner-Skan equation which is a nonlinear differential equation on a semi-infinite interval. The main point of our analysis lies in the fact that there is no reconstruction of the problem on the finite domain. We show that our results agree well with the results of other methods, which demonstrate the viability of the new technique. In this sense, this method has the potential to provide a wider applicability. On the other hand, the comparison of the results obtained by this method and the others shows that the new method provides accurate solutions.

The organization of the paper is as follows:
In Section 2, we describe the problem formulation. In the next section, we describe the formulation of rational Legendre functions. In section 4, we present our solutions for this problem, summarize the application of this method for Falkner-Skan wedge flow and compare it with the existing methods in the literature. The conclusions are described in the final section.

2 Problem formulation

In this section, we introduce two-dimensional steady-state boundary layer equation with boundary conditions as follows [60, 61]:

$$\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\ &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2},\\ &\begin{cases} y=0 & \text{u=0,v=0,}\\ y=\infty & \text{u=U(x).}\end{cases}\end{aligned} \tag{1}$$

where x , y , u , v are dimensionless variables using reference length x_0 and ν , U are kinematic viscosity and potential velocity respectively. By using newly defined variables and dimensionless variables as follows, where η is a dimensionless variable similarity solution, ψ is the stream function and m is the power of length coordinate x, of the velocity of the potential flow $U(x)=u_1x^m$ [60, 61]:

$$\eta=\sqrt{\frac{m+1}{2}}\sqrt{\frac{U}{\nu x}}y;\ \ \psi=\sqrt{\frac{2}{m+1}}\sqrt{\nu xU}f(\eta) \tag{2}$$

$$u=Uf'(\eta),\ \ v=-\sqrt{\frac{m+1}{2}}\sqrt{\frac{\nu U}{x}}\{f+\frac{m-1}{m+1}\eta f'\} \tag{3}$$

$$m=\frac{\beta}{2-\beta}\Rightarrow\frac{2m}{m+1}=\beta \tag{4}$$

thus Eq. (1) is converted to the third-order nonlinear Falkner-Skan equation as follows:

$$f'''+ff''+\beta(1-{f'}^2)=0, \tag{5}$$

with the boundary conditions

$$f(0)=0,\ f'(0)=0,\ f'(\infty)=1, \tag{6}$$

The wall mass transfer $f(0)$ which is defined here as zero, can sometimes can get nonzero value, namely $f(0)=\gamma$, showing the strength of the mass transfer at the wall, i.e. $\gamma<0$ corresponds to injection and $\gamma>0$ corresponds to suction and β is a parameter of the streamwise pressure gradient.

3 Rational Legendre functions properties

The well-known Legendre polynomials are orthogonal in the interval $[-1\ ,\ 1]$ with respect to the weight function $\rho(y) = 1$ and can be determined with the aid of the following recurrence formula:

$$\begin{cases} P_0(y) = 1, \\ P_1(y) = y, \\ (n+1)P_{n+1}(y) = (2n+1)yP_n(y) - nP_{n-1}(y) \qquad n\ \geq\ 1. \end{cases} \tag{7}$$

The rational Legendre (RL) functions are defined as follows

$R_n(x) = P_n(\frac{x-L}{x+L})$, where the constant parameter L sets the length scale of the mapping. Boyd [62] offered guidelines for optimizing the map parameter L for rational Chebyshev functions, which is useful for rational Legendre functions, too.

$R_n(x)$ is the nth eigenfunction of the singular Sturm-Liouville problem

$$\frac{(x+L)^2}{L}(xR_n'(x))' + n(n+1)R_n(x) = 0, \quad x \in [0, \infty), \quad n = 0,\ 1,\ 2,\ ... \tag{8}$$

Thus RL functions satisfy:

$$\begin{cases} R_0(x) = 1, \\ R_1(x) = \frac{x-L}{x+L}, \\ (n+1)R_{n+1}(x) = (2n+1)\frac{x-L}{x+L}R_n(x) - nR_{n-1}(x) \qquad n\ \geq\ 1. \end{cases} \tag{9}$$

3.1 Function approximation

Let $w(x) = \frac{2L}{(x+L)^2}$ denotes a non-negative, integrable, real-valued function over the interval $I = [0, \infty)$. We define:

$$L^2_w(I) = \{\ v:\ I \to\ \Re\ |\ v\ is\ measurable\ and\ ||v||_w < \infty\ \}, \tag{10}$$

where

$$||v||_w = (\int_0^\infty |v(x)|^2 w(x)dx)^{\frac{1}{2}}, \tag{11}$$

is the norm induced by the scalar product

$$< u, v >_w = \int_0^\infty u(x)v(x)w(x)dx. \tag{12}$$

Thus $\{R_n(x)\}_{n\geq 0}$ denote a system which is mutually orthogonal under (12), i.e.,

$$< R_n, R_m >_w = \frac{2}{2n+1}\delta nm, \tag{13}$$

where δ_{nm} is the Kronecker delta function. This system is complete in $L^2_w(I)$. For any function $u \in L^2_w(I)$ the following expansion holds

$$u(x) = \sum_{k=0}^{+\infty} a_k R_k(x), \tag{14}$$

with

$$a_k = \frac{< u, R_k >_w}{||R_k||^2_w}. \tag{15}$$

The a_ks are the expansion coefficients associated with the family $\{R_k(x)\}$.

3.2 Rational Legendre interpolation approximation

Canuto et al. [63] and Gottlieb et al. [64] intoduced Gauss-integration. Further, Guo et al. [16] introduced rational Legendre-Gauss points. Let

$$\Re_N = span\{R_0, R_1, ..., R_N\}, \tag{16}$$

and $\{y_j\}_{j=0}^{N}$ be $N+1$ roots of the polynomial $P_{N+1}(x)$. these points are known as Legendre-Gauss points. We define

$$x_j = L\frac{1+y_j}{1-y_j}\ , \quad j = 0, 1, ..., N, \tag{17}$$

which are named as rational Legendre-Gauss nodes. In fact, these points are zeros of the function $R_{N+1}(x)$. The relations between rational Legendre orthogonal systems and rational Gauss integrations are as follows:

$$\int_0^\infty u(x)w(x)dx = \int_{-1}^{1} u(L\frac{1+y}{1-y})\rho(y)dy = \sum_{j=0}^{N} u(x_j)w_j, \qquad \forall u \in \Re_{2N+1}, \tag{18}$$

where

$$w_j = \frac{2L}{(x_j+L)^2 x_j [R'_{N+1}(x_j)]^2}, \quad j = 0, ..., N, \tag{19}$$

are the corresponding weights with the $N+1$ rational Legendre-Gauss nodes. The interpolating function of a smooth function u on a semi-infinite interval is denoted by $I_N u$. It is an element of $\Re_N$ and is defined as

$$I_N u(x) = \sum_{j=0}^{N} a_k R_k(x). \tag{20}$$

$I_N u$ is the orthogonal projection of u upon $\Re_N$ with respect to the inner product (12) and the norm (11). Thus by the orthogonality of rational Legendre function we have

$$< I_N u - u, R_i >_w = 0, \forall R_i \in \Re_N. \tag{21}$$

To obtain the order of convergence of rational Legendre approximation, we first define the space

$$H^r_{w,A}(I) = \{v : v\ is\ measurable\ and\ \|v\|_{r,w,A} < \infty\}, \tag{22}$$

where the norm is induced by

$$\|v\|_{r,w,A} = \left(\sum_{k=0}^{r} \left\| (x+1)^{\frac{r}{2}+k} \frac{\mathrm{d}^k}{\mathrm{d}x^k} v \right\|_w^2 \right)^{\frac{1}{2}}, \tag{23}$$

and A is the Sturm-Liouville operator as follows:

$$Av(x) = -w^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}v(x)\right). \tag{24}$$

We have the following theorem for the convergence:

Theorem 1 *For any $v \in H^r_{w,A}(I)$ and $r \geq 0$,*

$$\|I_N v - v\|_w \leq cN^{-r}\|v\|_{r,w,A}. \tag{25}$$

A complete proof of the theorem and discussion on convergence is given in [16]. To apply a pseudospectral approach, we consider the residual $Res(x)$ when the expansion is substituted into the governing equation. It requires that a_k's be selected so that the boundary conditions are satisfied, but make the residual zero at as many (suitable chosen) spatial points as possible.

4 Solving Falkner-Skan wedge flow equation with Rational Legendre functions

In this section, we apply the collocation method to find solutions of Falkner-Skan equation. To solve this equation, we expect to approximate this equation with

$$I_N f(x) = \sum_{i=0}^{N} a_i R_i(x), \tag{26}$$

and form residual function, but should pay attention that

$$\lim_{x\to\infty} R_i'(x) = 0, \tag{27}$$

so

$$\lim_{x\to\infty} \frac{dI_N f(x)}{dx} = 0. \tag{28}$$

it means that the third boundary condition doesn't satisfy. Hence, we should use secondary function $f_0(x) = \frac{x^3}{(x+L)^2}$ and approximate $f(x)$ as

$$f(x) \approx f_0(x) + I_N f(x), \tag{29}$$

so that the third boundary condition will satisfy.
now we form the residual function:

$$Res(x) = \tfrac{d^3}{dx^3} f_0(x) + \tfrac{d^3}{dx^3} I_N f(x) + (f_0(x) + I_N f(x))(\tfrac{d^2}{dx^2} f_0(x) + \tfrac{d^2}{dx^2} I_N f(x)) + \beta[1 - (\tfrac{d}{dx} f_0(x) + \tfrac{d}{dx} I_N f(x))^2] = 0, \tag{30}$$

$I_N y$ is a good approximation of function y if it is zero on the whole domain. In other words, we should select coefficients a_k's so that the residual function approaches zero on the most of the domain. The collocation method for Falkner-Skan equation is to find $I_N y \in \Re_N$ such that

$$Res(x_j) = 0, \qquad j = 0, \ldots, N-2, \tag{31}$$

$$I_N y(x_0) = \gamma, \tag{32}$$

$$\left.\frac{dI_N y(x)}{dx}\right|_{x=0} = 0. \tag{33}$$

where the x_js are rational Legendre-Gauss nodes. This generates a set of $N+1$ nonlinear equations that can be solved by Newton method for unknown coefficients a_ks.
As shown in Tables 1-3 where the comparison of collocation method solution of $f''(0)$ with the analytical method, namely HAM [51] and numerical results by 4th-order Runge-Kutta method, combined with Newton-Raphson technique with the integral distance $\eta_\infty = 10$, the integral step size $\delta\eta = 0.001$ and the error of $f''(0)$ less than 1×10^{-6} is arised when γ varies from -1 to 7 and some fixed βs are evaluated to be 0, 1 and 2, respectively. It is found that the results of the present work agree well with those by other methods [42, 51], This verifies the validity of the present work.

Besides, the influences of β and γ on the solution curves of $f'(\eta)$ are further analyzed, where some fixed βs are evaluated to be 0, 1 and 2, respectively, when the parameter $\gamma = 5$ is chosen, and some fixed γa are evaluated to be -1, 0, 1 and 5, respectively, when the parameter $\beta = 1$ is unchanged as well, as shown in Figs. 1 and 2. This also shows the validity of the present work.

5 Conclusion

In this study, the collocation method with Rational Legendre functions is employed to solve the Falkner-Skan equation, which is a nonlinear boundary value problem. A comparison is also made between the results of the present method and other numerical/analytical methods. It is found that the results of the present work agree well with those obtained by other methods. The validity of this method is based on this assumption that it converges by increasing the number of collocation points.
Our aim was to apply an accurate and well-conditioned method which gives more accurate answers without reformulating the equation to bounded domains. Numerical results indicate the convergence and effectiveness of the present approach.

Table 1: Results for collocation method using rational Legendre for $\beta = 0$ in comparison with numerical result by [42] and analytical result by [51].

γ	N	L	$Rational\ Legendre$	Yao[51]	Numerical result[42]
-1	32	1.1	0.0034	-0.0642666	0
-0.5	32	1.116	0.14849	0.144418	0.14848
0	32	1.07	0.4692	0.469164	0.4692
0.5	32	0.9	0.85792	0.857873	0.85792
1	24	0.55	1.2836	1.28363	1.2836
1.5	24	0.95	1.7319	1.73191	1.7319
2	24	0.8	2.1945	2.19451	2.1945
2.5	24	0.7	2.6666	2.66657	2.6666
3	24	1	3.1451	3.14510	3.1451
3.5	24	1	3.6282	3.62821	3.6282
4	24	1	4.1146	4.11464	4.1146
4.5	24	1	4.6035	4.60353	4.6035
5	24	0.75	5.0943	5.09430	5.0943
7	24	1	7.0692	7.06919	7.0692

Table 2: Results for collocation method using rational Legendre for $\beta = 1$ and comparison with numerical result by [42] and analytical result by [51].

γ	N	L	$Rational\ Legendre$	Yao[51]	Numerical result[42]
-1	16	1	0.75657	0.754547	0.75657
-0.5	16	1.7	0.96923	0.968547	0.96923
0	16	1.5	1.2326	1.23245	1.2326
0.5	16	1.5	1.5418	1.54173	1.5418
1	16	1	1.8893	1.88931	1.8893
1.5	16	1	2.2676	2.26765	2.2676
2	16	1	2.6700	2.67006	2.6701
2.5	16	1	3.0911	3.09112	3.0911
3	16	1	3.5266	3.52664	3.5266
3.5	16	1.3	3.9734	3.97339	3.9734
4	16	1	4.4289	4.42895	4.4289
4.5	16	1.2	4.8915	4.89146	4.8915
5	16	1	5.3595	5.35954	5.3595
7	16	1.45	7.2693	7.26728	7.2693

Table 3: Results for collocation method using rational Legendre for $\beta = 2$ and comparison with numerical result by [42] and analytical result by [51].

γ	N	L	$Rational\ Legendre$	Yao[51]	Numerical result[42]
-1	16	1	1.19999	1.19817	1.2000
-0.5	16	1	1.4254	1.42592	1.4255
0	16	1	1.6872	1.68736	1.6872
0.5	16	1	1.9831	1.98314	1.9831
1	16	1	2.3098	2.30982	2.3098
1.5	16	1	2.6633	2.66332	2.6633
2	16	1	3.0396	3.03961	3.0396
2.5	16	1	3.43496	3.43497	3.4350
3	16	1	3.8461	3.84614	3.8461
3.5	16	1	4.27037	4.27037	4.2704
4	16	1	4.7053	4.70537	4.7054
4.5	16	1	5.1492	5.14927	5.1493
5	16	1	5.6006	5.60055	5.6006
7	16	1	7.4581	7.31203	7.4581

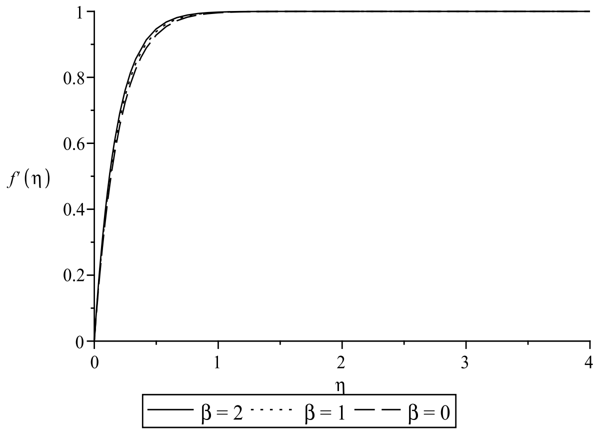

Figure 1: $f'(\eta)$ by collocation method using rational Legendre for different value of β and $\gamma = 5$

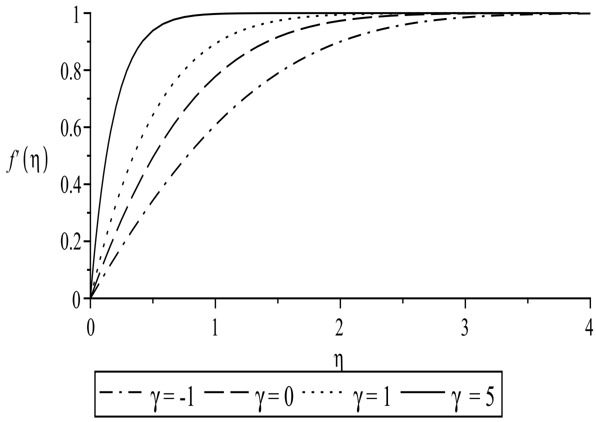

Figure 2: $f'(\eta)$ by collocation method using rational Legendre for different value of γ and $\beta = 1$

References

[1] O Coulaud, D Funaro, O Kavian. Laguerre spectral approximation of elliptic problems in exterior domains. *Comput. Method. Appl. Mech. Eng.*, 80(1990):451–458.

[2] D Funaro, O Kavian. Approximation of some diffusion evolution equations in unbounded domains by Hermite functions. *Math. Comp.*, 57(1990):597–619.

[3] D Funaro. Computational aspects of pseudospectral Laguerre approximations. *Appl. Numer. Math.*, 6(1990):447–457.

[4] B Y Guo. Error estimation of Hermite spectral method for nonlinear partial differential equations. *Math. Comput.*, 68(1999):1067–1078.

[5] B Y Guo, J Shen. Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval. *Numer. Math.*, 86(2000):635–654.

[6] Y Maday, B Pernaud-Thomas, H Vandeven. Reappraisal of Laguerre type spectral methods. *La. Rech. Aerospatiale.*, 6(1985):13–35.

[7] J Shen. Stable and efficient spectral methods in unbounded domains using Laguerre functions. *SIAM J. Numer. Anal.*, 38(2001):1113–1133.

[8] H I Siyyam. Laguerre tau methods for solving higher order ordinary differential equations. *J. Comput. Anal. Appl.*, 3(2001):173–182.

[9] B Y Guo. Gegenbauer approximation and its applications to differential equations on the whole line. *J. Math. Anal. Appl.*, 226(1998):180–206.

[10] B Y Guo. Jacobi spectral approximations to differential equations on the half line. *J. Comput. Math.*, 18(2000):95–112.

[11] B Y Guo. Jacobi approximations in certain Hilbert spaces and their applications to singular differential equations. *J. Math. Anal. Appl.*, 243(2000):373–408.

[12] J P Boyd. Chebyshev and Fourier Spectral Methods, Second Edition. Dover., New York(2000).

[13] CI Christov. A complete orthogonal system of functions in $L^2(-\infty, \infty)$ space. *SIAM J. Appl. Math.*, 42(1982):1337–1344.

[14] J P Boyd. Spectral methods using rational basis functions on an infinite interval. *J. Comput. Phys.*, 69(1987):112–142.

[15] J P Boyd. Orthogonal rational functions on a semi-infinite interval. *J. Comput. Phys.*, 70(1987):63–88.

[16] B Y Guo, J Shen, Z Q Wang. A rational approximation and its applications to differential equations on the half line. *J. Sci. Comput.*, 15(2000):117–147.

[17] J P Boyd, C Rangan, P H Bucksbaum. Pseudospectral methods on a semi-infinite interval with application to the Hydrogen atom: a comparison of the mapped Fourier-sine method with Laguerre series and rational Chebyshev expansions. *J. Comput. Phys.*, 188(2003):56–74.

[18] K Parand, M Razzaghi. Rational Chebyshev tau method for solving Volterra's population model. *Appl. Math. Comput.*, 149(2004):893-900.

[19] K Parand, M Razzaghi. Rational Chebyshev tau method for solving higher-order ordinary differential equations. *Int. J. Comput. Math.*, 81(2004):73–80.

[20] K Parand, M Razzaghi. Rational Legendre approximation for solving some physical problems on semi-infinite intervals. *Phys. Scripta.*, 69(2004):353–357.

[21] K Parand, M Shahini. Rational Chebyshev pseudospectral approach for solving Thomas-Fermi equation. *Phys. Lett. A*, 373(2009):210–213.

[22] K Parand, A Taghavi. Rational scaled generalized Laguerre function collocation method for solving the Blasius equation. *J. Comput. Appl. Math.*, 233(2009):980–989.

[23] K Parand, M Shahini, M Dehghan. Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane-Emden type. *J. Comput. Phys.*, 228(2009):8830–8840.

[24] K Parand, A Taghavi, M Shahini. Comparison between rational Chebyshev and modified generalized Laguerre functions pseudospectral methods for solving Lane–Emden and unsteady gas equations. *Acta. Phys. Pol. B*, 40(2009):1749–1763.

[25] K Parand, A Taghavi, A Shams, H Ghezel sofloo. Spectral method for solving differential equation of gas flow through a micro-nano porous media. *Computation and Theoretical Nanoscience*, in press.

[26] K Parand, Z Delafkar, N Pakniat, A Pirkhedri, M Haji. Collocation method using sinc and Rational Legendre functions for solving Volterra's population model *Commun. Nonlinear. Sci.*, 16(2011):18111819.

[27] K Parand, S Abbasbandy, S Kazem, J A Rad. A novel application of radial basis functions for solving a model of first-order integro-ordinary differential equation *Commun. Nonlinear. Sci.*, in press.

[28] C S Liu, J R Chang. The Lie-group shooting method for multiple-solutions of Falkner-Skan equation under suction-injection conditions. *Int. J. Nonlin. Mech.*, 43(2008):844–851.

[29] D R Hartree. On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer. *Proc. Cambr. Philos. Soc.*, 33(1937):223–239.

[30] T Cebeci, H B Keller. Shooting and parallel shooting methods for solving the Falkner-Skan boundary-layer equation. *J. Comput. Phys.*, 7(1971):289–300.

[31] L T Hildyard. Falkner-Skan problem in magnetohydrodynamics. *Phys. Fluids.*, 15(1972)1023–1027.

[32] H T Yang, L C Chien. Analytic solutions of the Falkner-Skan equation when beta equals minus 1 AND gamma equals 0. *SIAM. J. Appl. Math.*, 29(1975):558–569.

[33] V M Soundalgekar, H S Takhar, M Singh. Velocity and temperature field in MHD Falkner-Skan flow. *J Phys. Soc. Jpn.*, 50(1981):3139–3143.

[34] C A Forbrich. Improved solutions to the Falkner-Skan boundary-layer equation. *AIAA. J.*, 20(1982):1306–1307.

[35] H T Sharp, W L Harris. A pseudo-spectral method and parametric differentiation applied to the Falkner-Skan equation. *J. Comput. Phys.*, 55(1984):377–386.

[36] Y Liu, Y Lyu. New series expansion method for the solution of the Falkner-Skan equation. *AIAA. J.*, 27(1989):1453–1455.

[37] P Brodie, W H H Banks. Further properties of the Falkner-Skan equation. *Acta. Mech.*, 65(1987):205–211.

[38] G M R Assassa. Numerical solution of Falkner and Skan equation by optimization techniques. *Advances in modelling and simulation*, 5(1986):33–49.

[39] D M Summers. A random vortex simulation of Falkner-Skan boundary layer flow. *J. Comput. Phys.*, 85(1989):86–103.

[40] M Massoudi, M Ramezan. Effect of injection or suction on the Falkner-Skan flows of second grade fluids. *Int. J. nonlin. Mech.*, 24(1989):221–227.

[41] R Fazio. The falkneer-skan equation: Numerical solutions within group invariance theory. *Calcolo*, 31(1994):115–124.

[42] N S Asaithambi. A numerical method for the solution of the Falkner-Skan equation. *Appl. Math. Comput.*, 81(1997):259–264.

[43] R S Heeg, D Dijkstra, P J Zandbergen. The stability of Falkner-Skan flows with several inflection points. *Z. Angew. Math. Phys.*, 50(1999):82–93.

[44] P E Morgan, S G Rubin, P K Khosla. Application of the reduced Navier-Stokes methodology to flow stability of Falkner-Skan class flows. *Computers and Fluids*, 28(1999):307-321.

[45] S J Liao. A uniformly valid analytic solution of two-dimensional viscous flow over a semi-infinite flat plate. *J. Fluid. Mech.*, 385(1999):101-128.

[46] I Sher, A Yakhot. New approach to solution of the Falkner-Skan equation. *AIAA. J.*, 39(2001):965-967.

[47] Y J Kim. The Falkner- Skan wedge flows of power-law fluids embedded in a porous medium. *Transport. Porous. Med.*, 44(2001):267-279.

[48] A Asaithambi. A second-order finite-difference method for the Falkner-Skan equation. *Appl. Math. Comput.*, 156(2004):779-786.

[49] A Asaithambi. Numerical solution of the Falkner-Skan equation using piecewise linear functions. *Appl. Math. Comput.*, 159(2004):267–273.

[50] A Asaithambi. Solution of the Falkner-Skan equation by recursive evaluation of Taylor coefficients. *J. Comput. Appl. Math.*, 176(2005):203–214.

[51] B Yao. Approximate analytical solution to the Falkner-Skan wedge flow with the permeable wall of uniform suction. *Commun. Nonlinear. Sci.*, 14(2009):3320–3326.

[52] M B Zaturska, W H H Banks. A new solution branch of the Falkner-Skan equation. *Acta. Mech.*, 152(2001):197–201.

[53] S Saintlos, J Bretteville. A uniformly valid approximation for a Falkner-Skan flow. *Cr. Mecanique.*, 330(2002):673–682.

[54] S Abbasbandy, T Hayat. Solution of the MHD Falkner-Skan flow by Hankel-Padé method. *Physics Letters, Section A: General Atomic and Solid State Physics*, 373(2009):731–734.

[55] S Abbasbandy, T Hayat. Solution of the MHD Falkner-Skan flow by homotopy analysis method. *Commun. Nonlinear. Sci.*, 14(2009):3591–3598.

[56] A Ishak, R Nazar, I Pop. MHD boundary-layer flow of a micropolar fluid past a wedge with constant wall heat flux. *Commun. Nonlinear. Sci.*, 14(2009):109–118.

[57] B L Kuo. Application of the differential transformation method to the solutions of Falkner-Skan wedge flow. *Acta. Mech.*, 164(2003):161–174.

[58] B L Kuo. Heat transfer analysis for the Falkner-Skan wedge flow by the differential transformation method. *Int. J. Heat. Mass. Tran.*, 48(2005):5036–5046.

[59] N S Elgazery. Numerical solution for the Falkner-Skan equation. *Chaos, Solitons and Fractals*, 35(2008):738–746.

[60] E Alizadeh, M Farhadi, K Sedighi, H R Ebrahimi-Kebria, A Ghafourian. Solution of the FalknerSkan equation for wedge by Adomian Decomposition Method. *Commun. Nonlinear. Sci.*, 14(2009):724–733.

[61] H Schilichting. Boundary-layer theory (8th ed.). McGraw-Hill. Inc(2004).

[62] J P Boyd.The optimization of convergence for Chebyshev polynomial methods in an unbounded domain. *J. Comput. Phys.*, 45(1982):43–79.

[63] C Canuto, M Y Hussaini, A Quarteroni, T A Zang. Spectral Methods in Fluid Dynamic. Springer-Verlag, New York(1987).

[64] D Gottlieb, M Y Hussaini, S Orszag. Theory and Applications of Spectral Methods in Spectral Methods for Partial Differential Equations, in: R. Voigt, D. Gottlieb, M.Y. Hussaini (Eds.), Spectral Methods for Partial Differential Equations. SIAM, Philadelphia(1984).