On the Unsteady Linearly Accelerating Flow of a Fractional Second Grade Fluid Through a Circular Cylinder

M. Kamran1 *, M. Athar2, M. Imran2
1COMSATS Institute of Information Technology, Wah Cantt
2Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan
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Abstract: In this paper we obtain the velocity field and the adequate shear stress corresponding to the unsteady flow of a second grade fluid with fractional calculus due to a linearly accelerating circular cylinder, by means of the finite Hankel and Laplace transforms. The solutions that have been obtained satisfy both the governing equations and all imposed initial and boundary conditions. For \( \beta \to 1 \) and for \( \beta \to 1 \) and \( \alpha_1 \to 0 \), the corresponding solutions for a second grade fluid, respectively, for the Newtonian fluid, performing the same motion, are obtained from our general solutions. Finally, the influences of the pertinent parameters on the fluid motion, as well as a comparison between models, is underlined by graphical illustrations.

Keywords: fractional second grade fluid; velocity field; shear stress; exact solutions

1 Introduction

Navier-Stokes equations are nonlinear partial differential equations. For this reason, there exists only a limited number of exact solutions in which the nonlinear inertial terms do not disappear automatically. Exact solutions are very important not only because they are solutions of some fundamental flows but also because they serve as accuracy checks for experimental, numerical, and asymptotic methods. The inadequacy of the classical Navier-Stokes theory to describe rheologically complex fluids such as polymer solutions, blood, paints, certain oils and greases, has led to the development of several theories of non-Newtonian fluids. Amongst the many models which have been used to describe the non-Newtonian behavior exhibited by certain fluids, the fluids of differential type [1] have received special attention. The fluid of second grade, which form a subclass of the fluids of the differential type, have been studied successfully in various types of flow situations. Here, we mention some of the studies such as [2-6]. Since the equations governing the flow of second-grade fluids are one order higher than the Navier-Stokes equations, one would require boundary conditions in addition to the “nonslip” condition to have a well-posed problem.

In recent years, fractional calculus has encountered much success in the description of complex dynamics. Fractional derivative models are used quite after to describe viscoelastic behavior of polymers in the glass transition and the glassy state. The starting point is usually classical differential equation which is modified by replacing the time derivatives of an integer order by the so-called left-hand Liouville, or the Riemann-Liouville operators. This generalization allows one to define precisely noninteger order or derivatives [7, 8]. Fractional derivative constitutive equations have been found to be quite flexible in describing linear viscoelastic behavior of polymers from glass transition to the main or a relaxation in the glassy state. Recently, fractional calculus has encountered much success in the description of viscoelasticity [9, 10]. More recently, Tan and Xu [11] discussed the generalized second-grade flow due to the impulsive motion of a flat plate. The relevant studies involving generalized second grade fluid in the bounded domains are [12-20].

In this work, we consider the viscoelastic fluid to be modeled as a second grade fluid with fractional derivatives (SGFFD) and study the flow starting from rest due to the sliding of the cylinder along its axis with a linear acceleration. The velocity and adequate shear stress, obtained by means of the finite Hankel and Laplace transforms, are presented under series form in terms of the generalized G-function. The similar solutions for the ordinary second grade and Newtonian fluids, performing the same motion, are obtained as special cases when \( \beta \to 1 \), respectively \( \beta \to 1 \) and \( \alpha_1 \to 0 \).

*Corresponding author. E-mail address: getkamran@gmail.com

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2 Basic governing equations

The flows to be here considered have the velocity \( \mathbf{v} \) of the form \[21\]
\[ \mathbf{v}(r,t) = \mathbf{v}(r,t)e_z, \]
where \( e_z \) is the unit vector in the \( z \)-direction of the cylindrical coordinates system \( r, \theta \) and \( z \). For such flows, the constraint of incompressibility is automatically satisfied. Furthermore, if the fluid is at rest up to the moment \( t = 0 \), then
\[ \mathbf{v}(r,0) = 0. \]

The governing equations, corresponding to such motions for second grade fluid, are \[21, 22\]
\[ \tau(r,t) = (\mu + \alpha_1 \frac{\partial}{\partial t}) \frac{\partial v(r,t)}{\partial r}, \quad \frac{\partial v(r,t)}{\partial t} = (\nu + \alpha \frac{\partial}{\partial t}) \left( \frac{\partial^2}{r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) v(r,t), \]
where \( \mu \) is the dynamic viscosity of the fluid, \( \alpha_1 \) is a material constant (one of the two material moduli which define a second grade fluid), \( \nu = \mu/\rho \) is the kinematic viscosity of the fluid (\( \rho \) being its constant density), and \( \tau(r,t) = S_{rz}(r,t) \) is the shear stress which is different of zero.

The governing equations corresponding to an incompressible SGFFD, performing the same motion, are obtained by replacing the inner time derivatives with respect to \( t \) from Eqs. (3), by the fractional differential operator \[23\]
\[ D^\beta_t f(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} d\tau, & 0 \leq \beta < 1; \\ \frac{d}{dt} f(t), & \beta = 1, \end{cases} \]
where \( \Gamma(\cdot) \) is the Gamma function. Consequently, the governing equations to be used here are
\[ \frac{\partial v(r,t)}{\partial t} = (\nu + \alpha D^\beta_t) \left( \frac{\partial^2}{r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) v(r,t), \]
\[ \tau(r,t) = (\mu + \alpha_1 D^\beta_t) \frac{\partial v(r,t)}{\partial r}, \]
where the new material constants \( \alpha \) and \( \alpha_1 \) (although we keep the same notation) reduce to the previous ones for \( \beta \to 1 \). In the following the fractional differential equations with appropriate initial and boundary conditions will be solved by means of Hankel and Laplace transforms.

3 Flow through a circular cylinder

Suppose that an incompressible SGFFD is situated at rest in a circular cylinder of radius \( R( > 0) \). At time \( t = 0^+ \) the cylinder suddenly begins to move with a velocity \( Vt^2 \) along its axis. Due to the shear the inner fluid is gradually moved, its velocity being of the form (1)\(_1\). The governing equation is given by Eq. (5) and the appropriate initial and boundary conditions are
\[ v(r,0) = 0; \quad r \in [0,R], \]
\[ v(R,t) = Vt^2; \quad t > 0, \]
where \( V \) is a constant.

The partial differential equation (5), also containing fractional derivatives, can be solved in principle by several methods, the integral transforms technique representing a systematic, efficient and powerful tool. In the following we shall use the Laplace transform to eliminate the time variable and the finite Hankel transform for the spatial variable. However, in order to avoid the burdensome calculations of residues and contour integrals, we shall apply the discrete inverse Laplace transform method.
Taking the inverse Hankel transform of Eq. (14), it leads to

\[ \mathcal{H}^{-1} \left[ \frac{q^2}{q^2 + \frac{1}{r} \frac{\partial \mathcal{L}[v]}{\partial r}} \right] = \frac{2V}{q^3}, \]

where \( \mathcal{H}^{-1} \) denotes the inverse Hankel transform, \( \mathcal{H}[v] \) denotes the Hankel transform of the function \( v(r,t) \). We shall denote by \[24\]

\[ \mathcal{H}[v] = \int_0^R r \mathcal{H}[v](rr_n)dr, \]

the finite Hankel transform of the function \( \mathcal{H}[v] \), and use the identity

\[ \int_0^R r \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) J_0(rr_n)dr = Rr_n J_1(Rr_n) v(R,t) - r_n^2 v_n(r_n,t), \]

where \( r_n \) is the positive root of the transcendental equation \( J_0(Rr_n) = 0 \), and \( J_0(.) \) is the Bessel function of the first kind of order zero.

Multiplying Eq. (9) by \( r J_0(r) \), then integrating the result w.r.t. \( r \) from 0 to \( R \) and using Eqs. (10) and (12), we find that

\[ \mathcal{H}[v](r_n,q) = (\nu + \alpha q^3)2V R r_n J_1(Rr_n) \frac{1}{q^3(q + \nu r_n^2 + \alpha q^3 r_n^2)} \]

It can be also written in the suitable form

\[ \mathcal{H}[v](r_n,q) = \mathcal{H}[v_1](r_n,q) + \mathcal{H}[v_2](r_n,q), \]

where

\[ \mathcal{H}[v_1](r_n,q) = 2V R r_n J_1(Rr_n) \frac{1}{q^3}, \quad \mathcal{H}[v_2](r_n,q) = -2V R r_n J_1(Rr_n) \frac{1}{q^3(q + \nu r_n^2 + \alpha q^3 r_n^2)}. \]

Applying the inverse Hankel transform to Eqs. (15) and using the known formulae

\[ \int_0^R r J_0(rr_n)dr = \frac{R}{r_n} J_1(Rr_n), \quad u(r,t) = \frac{2}{R^2} \sum_{n=1}^\infty u_n(r,t) J_0(rr_n) \]

we get

\[ \mathcal{H}[v_1](r,q) = 2V \frac{1}{q^3}; \quad \mathcal{H}[v_2](r,q) = \frac{4V}{R} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(Rr_n)} \frac{1}{q^3(q + \nu r_n^2 + \alpha q^3 r_n^2)}. \]

Using the identity

\[ \frac{1}{q^3(q + \nu r_n^2 + \alpha q^3 r_n^2)} = \sum_{k=0}^{\infty} \frac{(-\nu r_n^2)^k q^{-\beta k - \beta - 2}}{(q^{1-\beta} + \alpha r_n^2)^{k+1}}, \]

Eqs. (16) can be further simplified to give

\[ \mathcal{H}[v_1](r,q) = 2V \frac{1}{q^3}; \quad \mathcal{H}[v_2](r,q) = -\frac{4V}{R} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(Rr_n)} \sum_{k=0}^{\infty} \frac{(-\nu r_n^2)^k q^{-\beta k - \beta - 2}}{(q^{1-\beta} + \alpha r_n^2)^{k+1}}. \]

Taking the inverse Hankel transform of Eq. (14), it leads to

\[ \mathcal{H}[v](r,q) = 2V \frac{1}{q^3} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n J_1(Rr_n)} \sum_{k=0}^{\infty} \frac{(-\nu r_n^2)^k q^{-\beta k - \beta - 2}}{(q^{1-\beta} + \alpha r_n^2)^{k+1}}. \]
by now taking the inverse Laplace transform of Eq. (19), the velocity field \( v(r, t) \) is given by
\[
v(r, t) = \frac{Vt^2}{R} - 4V \sum_{n=1}^{\infty} \frac{J_0(\alpha r n)}{r_n J_1(\alpha R n)} \sum_{k=0}^{\infty} (-\nu \alpha^2_n)^k G_{1-\beta,-\beta k-\beta-2,k+1}(-\alpha^2_n, t),
\]
(20)
where the generalized function \( G_{a, b, c}(\cdot, \cdot) \) is defined by [15, Eqs. (97) and (101)]
\[
G_{a, b, c}(d, t) = L^{-1}\left\{ \frac{q^{b}}{(q^a - dt)^c} \right\} = \frac{1}{\Gamma(c + k)} \frac{d^k}{\Gamma(k + 1)} \frac{t^{(c + k)a - b - 1}}{\Gamma((c + k)a - b) \Gamma(ac - b) > 0, \mid \frac{d}{q^a} \mid < 1.
\]

3.2 Calculation of the shear stress
Applying the Laplace transform to Eq. (6), we find that
\[
\tau(r, q) = (\mu + \alpha_1 q^\beta) \frac{\partial v(r, q)}{\partial r}.
\]
In view of Eq. (19), it results that
\[
\tau(r, q) = (\mu + \alpha_1 q^\beta) \frac{4V}{R} \sum_{n=1}^{\infty} \frac{J_1(\alpha r n)}{J_1(\alpha R n)} \sum_{k=0}^{\infty} (-\nu \alpha^2_n)^k q^{-\beta k - \beta - 2} \sum_{k=0}^{\infty} (q^{-\beta} + \alpha^2_n)^k + 1.
\]
(23)
Now taking the inverse Laplace transform of both sides of Eq. (23) and using Eq. (21), we find the shear stress \( \tau(r, t) \) under the suitable form
\[
\tau(r, t) = \frac{4V}{R} \sum_{n=1}^{\infty} \frac{J_1(\alpha r n)}{J_1(\alpha R n)} \sum_{k=0}^{\infty} (-\nu \alpha^2_n)^k \left[ \mu G_{1-\beta,-\beta k-\beta-2,k+1}(-\alpha^2_n, t) + \alpha_1 G_{1-\beta,-\beta k-\beta-2,k+1}(-\alpha^2_n, t) \right].
\]
(24)

4 Special cases
4.1 The special case \( \beta \rightarrow 1 \)
Making \( \beta \rightarrow 1 \) into Eqs. (20) and (24), we obtain the similar solutions
\[
v(r, t) = \frac{Vt^2}{R} - 4V \sum_{n=1}^{\infty} \frac{J_0(\alpha r n)}{r_n J_1(\alpha R n)} \sum_{k=0}^{\infty} (-\nu \alpha^2_n)^k G_{0,-k-3,k+1}(-\alpha^2_n, t),
\]
(25)
\[
\tau(r, t) = \frac{4V}{R} \sum_{n=1}^{\infty} \frac{J_1(\alpha r n)}{J_1(\alpha R n)} \sum_{k=0}^{\infty} (-\nu \alpha^2_n)^k \left[ \mu G_{0,-k-3,k+1}(-\alpha^2_n, t) + \alpha_1 G_{0,-k-2,k+1}(-\alpha^2_n, t) \right],
\]
(26)
for a second grade fluid performing the same motion. These solutions can also be simplified to give (see also Eqs. \( (A_1) - (A_2) \) from Appendix)
\[
v(r, t) = \frac{Vt^2}{\nu R} \sum_{n=1}^{\infty} \frac{J_0(\alpha r n)}{r_n^2 J_1(\alpha R n)} \left[ t - \frac{1 + \alpha^2_n}{\nu \alpha^2_n} \left\{ 1 - \exp\left( -\frac{\nu \alpha^2_n t}{1 + \alpha^2_n} \right) \right\} \right],
\]
(27)
\[
\tau(r, t) = \frac{4\nu V}{R} \sum_{n=1}^{\infty} \frac{J_1(\alpha r n)}{r_n^2 J_1(\alpha R n)} \left[ t - \frac{1}{\nu \alpha^2_n} \left\{ 1 - \exp\left( -\frac{\nu \alpha^2_n t}{1 + \alpha^2_n} \right) \right\} \right].
\]
(28)
As from, these last expressions of our solutions are in accordance with those obtained in [14, Eqs. (4.3) and (4.4)] by a different technique.
4.2 Newtonian case

Making \( \alpha_1 \to 0 \) and then \( \alpha \to 0 \) into Eqs. (27) and (28), we obtain the velocity field

\[
v(r,t) = Vt^2 - \frac{4V}{\nu R} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r^2_n J_1(Rr_n)} \left[ t - \frac{1}{\nu r_n^2} \{ 1 - e^{-\nu r_n^2 t} \} \right],
\]

(29)

and the associated shear stress

\[
\tau(r,t) = \frac{4\rho V}{R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r^2_n J_1(Rr_n)} \left[ t - \frac{1}{\nu r_n^2} \{ 1 - e^{-\nu r_n^2 t} \} \right],
\]

(30)

corresponding to a Newtonian fluid performing the same motion.

5 Conclusion

In this paper we determined the velocity field and the adequate shear stress corresponding to the flow of a second grade fluid with fractional derivatives due to an infinite circular cylinder subject to a translation of linear acceleration, along its axis. The solutions have been obtained by means of Laplace and finite Hankel transforms, and are presented under series form in terms of generalized \( G \) functions. In the special cases, when \( \beta \to 1 \) or \( \beta \to 1 \) and \( \alpha_1 \to 0 \), the corresponding solutions for a second grade fluid, respectively, for the Newtonian fluid, performing the same motion, are obtained from general solutions. Direct computations show that they satisfy both the governing equations and all imposed initial and boundary conditions.

Now, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity \( v(r,t) \) as well as those of the shear stress \( \tau(r,t) \), are depicted against \( t \) for different values of the time \( t \) and of the pertinent parameters. Figs. 1a. and 1b. clearly show that both the velocity and the shear stress are increasing functions of \( t \). They are also increasing functions of \( r \), excepting \( \tau(r,t) \) on a small interval. The influences of the kinematic viscosity \( \nu \) and of the material parameter \( \alpha \) on the velocity \( v(r,t) \) are shown in Figs. 2a. and 3a. Qualitatively, their influence is the same but it differs quantitatively. Anyway, the velocity is an increasing function with respect to \( \nu \) and \( \alpha \), but \( \tau(r,t) \) is a decreasing function of \( \nu \) and \( \alpha \) near the boundary as shown in Figs. 2b. and 3b.

![Figure 1: Profiles of the velocity \( v(r,t) \) and shear stress \( \tau(r,t) \) given by Eqs. (20) and (24) for \( R = 0.5 \), \( V = 1 \), \( \nu = 0.003 \), \( \mu = 2 \), \( \alpha = 0.0003 \), \( \beta = 0.2 \) and different values of \( t \).](image)

The influence of the fractional parameter \( \beta \) on the fluid motion is presented in Figs. 4a. and 4b. Its effect is opposite to that of time \( t \). More exactly the velocity \( v(r,t) \) and the shear stress \( \tau(r,t) \) are decreasing functions with regards to \( \beta \).

Finally, for comparison, the profiles of \( v(r,t) \) and \( \tau(r,t) \) corresponding to the motion of the three models (Newtonian, ordinary second grade and fractional second grade) are together depicted in Figs. 5 for the same values of \( t \) and of the common material parameters. The Newtonian fluid, as it results from these figures, is the slowest and second grade fluid with fractional derivatives is the swiftest on the whole flow domain. The units of the material constants are SI units within all figures, and the roots \( r_n \) have been approximated by \( (4n - 1)\pi / (4R) \).

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Figure 2: Profiles of the velocity $v(r, t)$ and shear stress $\tau(r, t)$ given by Eqs. Eqs. (20) and (24) for $t = 5$, $R = 0.5$, $V = 1$, $\nu = 0.006$, $\mu = 1$, $\beta = 0.2$ and different values of $\alpha$.

Figure 3: Profiles of the velocity $v(r, t)$ and shear stress $\tau(r, t)$ given by Eqs. Eqs. (20) and (24) for $t = 10$, $R = 0.5$, $V = 1$, $\alpha = 0.003$, $\mu = 1$, $\beta = 0.2$ and different values of $\nu$.

Figure 4: Profiles of the velocity $v(r, t)$ and shear stress $\tau(r, t)$ given by Eqs. Eqs. (20) and (24) for $t = 5$, $R = 0.5$, $V = 1$, $\alpha = 0.003$, $\mu = 1$, $\nu = 0.004$ and different values of $\beta$. 

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Figure 5: Profiles of the velocity $v(r,t)$ and shear stress $\tau(r,t)$ corresponding to the Newtonian, second grade and fractional second grade fluid, for $t = 5$, $R = 0.5$, $V = 1$, $\alpha = 0.003$, $\nu = 0.003$, $\mu = 2.5$ and $\beta = 0.2$.

Appendix

\[ \sum_{k=0}^{\infty} (-\nu r^2)_n^k G_{0,-k-2,k+1}(-\alpha r^2_n,t) = \frac{1}{\nu r^2_n} \left\{ 1 - \exp \left( - \frac{\nu r^2_n t}{1 + \alpha r^2_n} \right) \right\} , \]

\[ \sum_{k=0}^{\infty} (-\nu r^2)_n^k G_{0,-k-3,k+1}(-\alpha r^2_n,t) = \frac{1}{\nu r^2_n} \left[ t - \frac{1 + \alpha r^2_n}{\nu r^2_n} \left\{ 1 - \exp \left( - \frac{\nu r^2_n t}{1 + \alpha r^2_n} \right) \right\} \right] . \]

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