An Iterative Formula with a Parameter Function for Solving Nonlinear Equations

Fan Sha\textsuperscript{1} *, Xueyuan Tan\textsuperscript{2}

\textsuperscript{1} School of Mathematical Sciences Nanjing Normal University, Nanjing 210097, People’s Republic of China  
\textsuperscript{2} Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences Nanjing Normal University,  
Nanjing 210097, People’s Republic of China

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Abstract: We introduce a class of iterative methods with a positive parameter function. Convergence analysis show that the proposed methods are cubic convergent. Numerical results are given to illustrate the effectiveness of our method.

Keywords: Nonlinear equations, Newton method, convergence.

1 Introduction

Consider the following system of a nonlinear equation
\[ f(x) = 0 \]  
where \( x \in R \).

The best known and the most widely used method for solving (1) is the Newton’s method given by
\[ x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}, \quad k = 0, 1, 2, \ldots \]  
(2)

It converges quadratically to simple zeros and linearly to multiple zeros.

Newton-type iterative methods have been studied extensively in [3–15]. In this work, we present a class of Newton-type iterative methods with a parameter function in the form
\[ x^{(k+1)} = x^{(k)} - \frac{\frac{3}{2}f'(x^{(k)}) - \frac{1}{2}f'(\phi(x^{(k)})))f(x^{(k)})}{\tau(x^{(k)})(f(x^{(k)}))^2 + (f'(x^{(k)}))^2}, \quad k = 0, 1, 2, \ldots, \]  
(3)

where \( k = 1, 2, \cdots \) and \( \tau(x) > 0 \).

In the following section, we will prove that the scheme is cubic convergent for simple zero and linear convergent for multiple zeros. Then, in the final section, numerical examples are given to support our theory analysis.

2 The main results

The following Lemma [2] is required in the subsequent derivation.

\textbf{Lemma 1} Assume that \( f(x) = 0 \) has a single root \( x^* \). \( g(x) \) is \( m \) order continuous differentiable at a neighborhood of \( x^* \), and
\[ g^{(j)}(x^*) = 0, j = 1, \cdots, m - 1, g^{(m)}(x^*) \neq 0, \quad (m \geq 2). \]

Then, there exists \( r > 0 \), for any \( x_0 \in [x^* - r, x^* + r] \), the iterative scheme \( x^{(k+1)} = g(x^{(k)}), k = 1, 2, \cdots \), converges to \( x^* \) with order \( m \).

* Corresponding author. E-mail address: shafan1989826@126.com
Then the convergence theorems can be obtained as follows.

**Theorem 2** Assume that \( f(x) = 0 \) has a single root \( x^* \), \( f(x) \) is three order continuous differentiable at a neighborhood of \( x^* \), \( \phi(x) \) and \( \tau(x) \) are two order differentiable, \( x^* = \phi(x^*) \), and \( \phi(x^*) = 0 \) which means \( x^{(k)} = \phi(x^{k-1}) \) is quadratic convergent. Then, the proposed Newton-like method given in (3) is locally cubic convergent.

**Proof.** By the given conditions and Lemma 1, it is sufficient to prove that when \( x = x^* \),

\[
(x - \frac{\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f(x)}{\tau(x)(f(x))^2 + (f'(x))^2}, \quad \text{and} \quad (x - \frac{\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f(x)}{\tau(x)(f(x))^2 + (f'(x))^2})''.
\]

In fact

\[
(x - \frac{\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f(x)}{\tau(x)(f(x))^2 + (f'(x))^2}) = 1 - \frac{2f(x)f'(x)\tau(x) + f^2(x)\tau'(x) + 2f'(x)f''(x)S}{(\tau(x)f^2(x) + (f'(x))^2)^2} - \frac{(\frac{3}{2}f''(x) - \frac{1}{2}f''(\phi(x))\phi'(x))f(x) + (\frac{3}{2}f''(x) - \frac{1}{2}f''(\phi(x))\phi'(x))f'(x) + (\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f''(x)}{\tau(x)(f^2(x) + f'(x))^2}
\]

\[
+ \frac{T((\frac{3}{2}f''(x) - \frac{1}{2}f''(\phi(x)))f'(x) + (\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f''(x))}{(\tau(x)(f^2(x) + (f'(x))^2)^2}
\]

\[
+ \frac{T'((\frac{3}{2}f''(x) - \frac{1}{2}f''(\phi(x)))f'(x) + (\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f''(x))}{(\tau(x)(f^2(x) + f'(x))^2)^2}
\]

\[
- 2T^2((\frac{3}{2}f''(x) - \frac{1}{2}f''(\phi(x)))f'(x)\tau(x) + f^2(x)\tau'(x) + 2f(x)f'(x)).
\]

where

\[
S = (\frac{3}{2}f''(x) - \frac{1}{2}f''(\phi(x))\phi'(x))f(x) + (\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f'(x)
\]

\[
T = 2f(x)f'(x)\tau(x) + f^2(x)\tau'(x) + 2f(x)f'(x).
\]

Also note that

\[
\phi(x^*) = x^*, \quad \text{and} \quad \phi'(x^*) = 0,
\]

we have

\[
(x - \frac{\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f(x)}{\tau(x)(f(x))^2 + (f'(x))^2})|_{x=x^*} = 1 - \frac{2f(x^*) - 4f'(x^*)f'(x^*)}{f'(x^*)^2} = 0,
\]

and

\[
(x - \frac{\frac{3}{2}f'(x) - \frac{1}{2}f'(\phi(x)))f(x)}{\tau(x)(f(x))^2 + (f'(x))^2})''|_{x=x^*} = 2f''(x^*) - 4f''(x^*)f'(x^*) + 2f'(x^*)f'(x^*) = 0.
\]

The proof is completed. \( \blacksquare \)
Theorem 3 Assume that \( f(x) = 0 \) has a multiple root \( x^* \) with multiplicity \( m \geq 2 \). \( f(x) \), \( \phi(x) \) and \( \tau(x) \) are sufficiently differentiable at a neighborhood of \( x^* \). \( x^* = \phi(x^*) \), and \( \phi'(x^*) \neq 0 \) which means \( x^{(k)} = \phi(x^{(k-1)}) \) is linear convergent. Then, the proposed Newton-like method given in (3) is at least locally linear convergent.

Proof. Using Taylor expansion of \( f(x) \) about \( x^* \), and denote \( e_n = x^{(n)} - x^* \) and \( \tilde{e}_n = \phi(x^{(n)}) - \phi(x^*) \), we have

\[
\begin{align*}
f(x^{(n)}) &= f^{(m)}(x^*) \frac{e_n^m}{m!} (1 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + O(e_n^5)), \\
f'(x^{(n)}) &= f^{(m-1)}(x^*) \frac{e_n^{m-1}}{(m-1)!} (1 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + B_4 e_n^4 + O(e_n^5)), \\
f'(\phi(x^{(n)})) &= f^{(m)}(\phi(x^*)) \frac{e_n^{m-1}}{(m-1)!} (1 + B_1 \tilde{e}_n + B_2 \tilde{e}_n^2 + B_3 \tilde{e}_n^3 + B_4 \tilde{e}_n^4 + O(e_n^5)), \\
\phi(x^{(n)}) - \phi(x^*) &= D_1 e_n + D_2 e_n^2 + O(e_n^3),
\end{align*}
\]

where \( D_1 \neq 0 \).

Then

\[
\begin{align*}
f^2(x^{(n)}) &= f^{(m)}(x^*)^2 \frac{e_n^m}{m!} (1 + 2A_1 e_n + (A_1^2 + 2A_2)e_n^2 + O(e_n^3)), \\
f'(x^{(n)})^2 &= f^{(m-1)}(x^*)^2 \frac{e_n^{m-1}}{(m-1)!} (1 + 2B_1 e_n + (B_1^2 + 2B_2)e_n^2 + O(e_n^3)), \\
f'(\phi(x^{(n)}))^2 &= f^{(m)}(\phi(x^*))^2 \frac{e_n^{m-1}}{(m-1)!} (1 + B_1 \tilde{e}_n + B_2 \tilde{e}_n^2 + B_3 \tilde{e}_n^3 + B_4 \tilde{e}_n^4 + O(e_n^5)), \\
(\frac{3}{2} f'(\phi(x^{(n)})) f(x^{(n)}) &= f^{(m)}(\phi(x^*)) \frac{e_n^{m-1}}{(m-1)!} (\frac{3}{2} - \frac{1}{2}D_1^{m-1}) e_n + O(e_n^3).
\end{align*}
\]

As a result, we obtain

\[
e_{n+1} = K_1 e_n + O(e_n^3),
\]

where \( K_1 = 1 - \frac{3}{2m} + \frac{1}{2m} D_1^{m-1} \). This completes the proof. \( \blacksquare \)

Noting that, based the above proof, we have

\[
\frac{f(x^{(n)})}{f'(x^{(n)})} = \frac{e_n}{m} (1 - \frac{A_1}{m} e_n + (-\frac{2}{m} A_2 + \frac{n}{m^2} A_1^2) e_n^2 + O(e_n^3)),
\]

then it is easy to come to the following conclusion.

Corollary 4 Under the condition given in the Theorem 3. If we choose \( \phi(x) = x - \frac{f(x)}{f'(x)} \), then we have

\[
e_{n+1} = (1 - \frac{3}{2m} + \frac{1}{2m} \frac{m-1}{m}) e_n + O(e_n^3).
\]

3 Numerical results

In this section, we present the results of some numerical tests to illustrate the effectiveness of our method. We test the following examples.

1. \( \sin^2(x) - x^2 + 1 \), \( x \in [0, 3] \),
2. \( (x-1)^3 - 1 \), \( x \in [0, 3] \),
3. \( (x-1)^6 - 1 \), \( x \in [0, 3] \),
4. \( xe^x - \sin^2x + 3\cos x + 5 \), \( x \in [-2, 0] \),
5. \( e^{x^2 + 7x - 30} - 1 \), \( x \in [2, 4] \),
6. \( (x - 2)^3(x + 2)^4 \), \( [1, 3], [-3, -1] \).
Numerical computations reported here have been carried out in MATLAB environment on a PC. We choose the parameter function as \( \tau(x) = \frac{1}{2}(e^x + e^{-x}) \). The stopping criterion has been taken as \( |f(x^{(k)})| \leq 10^{-6} \).

In the following table, the number of iterations (IT) and \( \text{Error} = |f(x^{(k)})| \) are listed. Here, we choose Newton iterative formula for \( x^{(k)} = \phi(x^{(k-1)}) \).

As far as the numerical results are considered, the initial approximations are very important. Although the roots of the example 6 are not simple, and our method and Newton method converge linearly, the numerical results have show the effectiveness of our method in case of multiple roots. From the table, we can find that our proposed method is superior to the Newton method. Furthermore, the other advantage of our method is that the zero denominator can be avoided.

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## References


