Solution of The Smoluchowski’s Equation by Homotopy Analysis Method

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Abstract: This paper presents the application of the homotopy analysis method (HAM) as a numerical so-
lution to homogeneous Smoluchowski’s equation. The Smoluchowski’s coagulation equation is a mean-field
model for the growth of clusters (particles, droplets, etc.) by binary coalescence; that is, the driving growth
mechanism is the merger of two particles into a single one. The HAM contains the auxiliary parameter h, taht
provides a powerful tool to analyze strongly linear and nonlinear problems. The obtained results have been
compared with the HPM and ADM. It is shown that the results of the HAM for a special case are the same as
those obtained by HPM and ADM.

Keywords: Homotopy analysis method; homogeneous Smoluchowski’s equation; coagulation

1 Introduction

Many problems in physics, fluid mechanics, biological models, applied science and engineering are usually difficult to
solve analytically. Perturbation method is one of the well known methods to solve nonlinear problems, it is based on
the existence of small/large parameters, the so-called perturbation quantity. Many nonlinear problems do not such kind
of perturbation quantity, and we can use non-perturbation methods, such as the artificial small parameter method, the
δ-expansion method, and the homotopy perturbation method (HPM) [2-7,9]. However, both of the perturbation and non-
perturbation methods cannot provide us with a simple way to adjust and control the convergence region and rate of given
approximate series. To overcome such problems, the homotopy analysis method (HAM) is developed and proposed by

HAM is attractive in the sense that it is independent of whether a small parameter is present or not. Thus, the HAM is
valid for nonlinear problems, either weak or strong.

The Smoluchowski’s equation is a system of partial differential equations modelling the diffusion and binary cogula-
tion of a large collection of tiny particles [8]. In a colloid, a population of comparatively massive particles is agitated by
the bombardment of much smaller particles in the ambient environment: the motion of the colloidal particles may then be
modelled by Brownian motion. The Smoluchowski’s coagulation equation provides a mean-field description of a system
of an infinite number of particles growing by successive mergers, each particle being fully identified by its mass ranging in
the set of positive real numbers. In fact, the only mechanism taken into account in this model is the coalescence of two
particles to form a larger one. Denoting by $c(t, x) \geq 0$ the concentration of particles of mass $x \in (0, \infty)$ at time $t \geq 0$,
the dynamics of $c$ is given by

$$\frac{\partial c(t, x)}{\partial t} = L_c(c(t, ))(x), \quad (t, x) \in (0, \infty) \times (0, \infty),$$

where the coagulation reaction term $L_c$ is defined by

$$L_c(c)(x) = \frac{1}{2} \int_0^\infty K(y, x - y) c(y) c(x - y) \, dy - \int_0^\infty c(x) c(y) K(x, y) \, dy$$

for $x \in (0, \infty)$. In (1), $K(x, y)$ models the likelihood that two particles with respective masses $x$ and $y$ merge into a single
one (with mass $x + y$) and the coagulation kernel $K$ is a non-negative and symmetric function from $(0, \infty) \times (0, \infty)$ into

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0, \infty). The first term of the right-hand side of (2) describes the formation of particles of mass \( x \) resulting from the coalescence of two particles with respective masses \( y \) and \( x - y, y \in (0, x) \). The second term accounts for the disappearance of particles of mass \( x \) by coalescence with other particles.

Smoluchowski’s equation is widely applied to describe the time evolution of the cluster-size distribution during aggregation processes. Analytical solutions for this equation, however, are known only for a very limited number of kernels. Therefore, numerical methods have to be used to describe the time evolution of the cluster-size distribution. A numerical technique is presented for the solution of the homogeneous Smoluchowski’s coagulation equation with constant kernel. The aim of this paper is to solve the Smoluchowski’s coagulation equation using the homotopy analysis method. Results shows that the Adomian decomposition method and homotopy perturbation method are a special case of the homotopy analysis method [1].

2 Basic idea of HAM

To describe the basic ideas of the HAM, we consider the following differential equation

\[ N[u(x, t)] = 0, \]  
(3)

where \( N \) is a nonlinear operator, \( x, t \) denotes independent variables, \( u(x, t) \) is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [10] constructs the so-called zero-order deformation equation

\[ (1 - q)L[\phi(x, t; q) - u_0(x, t)] = q h H(x, t)N[\phi(x, t; q)], \]  
(4)

where \( q \in [0, 1] \) is the embedding parameter, \( h \neq 0 \) is a non-zero auxiliary parameter, \( H(x, t) \neq 0 \) is an auxiliary function, \( L \) is an auxiliary linear operator, \( u_0(x, t) \) is an initial guess of \( u(x, t) \), \( \phi(x, t; q) \) is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when \( q = 0 \) and \( q = 1 \), it holds

\[ \phi(x, t; 0) = u_0(x, t), \phi(x, t; 1) = u(x, t), \]  
(5)

respectively. Thus, as \( q \) increases from 0 to 1, the solution \( u(x, t; q) \) varies from the initial guess \( u_0(x, t) \) to the solution \( u(x, t) \). Expanding \( u(x, t; q) \) in Taylor series with respect to \( q \), we have

\[ \phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t)q^m, \]  
(6)

where

\[ u_m(x, t) = \left. \frac{\partial^m \phi(x, t; q)}{\partial q^m} \right|_{q=0}. \]  
(7)

If the auxiliary linear operator, the initial guess, the auxiliary parameter \( h \), and the auxiliary function are so properly chosen, the series (4) converges at \( q = 1 \), then we have

\[ u(x, t) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t), \]  
(8)

which must be one of solutions of original nonlinear equation, as proved by Liao [9]. As \( h = -1 \) and \( H(x, t) = 1 \), Eq. (2) becomes

\[ (1 - q)L[\phi(x, t; q) - u_0(x, t)] + q N[\phi(x, t; q)] = 0, \]  
(9)

which is used mostly in the homotopy perturbation method [4], where as the solution obtained directly, without using Taylor series. According to the definition (5), the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

\[ \vec{u}_n = \{u_0(x, t), u_1(x, t), \ldots, u_n(x, t)\}. \]

Differentiating equation (2) \( m \) times with respect to the embedding parameter \( q \) and then setting \( q = 0 \) and finally dividing them by \( m! \), we have the so-called \( m \)th-order deformation equation

\[ L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = h H(x, t)R_m(\vec{u}_{m-1}), \]  
(10)
where
\[ R_m(\vec{u}_{m-1}) = \frac{1}{(m - 1)!} \frac{\partial^{m-1} N[\phi(x, t; q)]}{\partial q^{m-1}} |_{q=0}. \] (11)

and
\[ \chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases} \] (12)

It should be emphasized that \( u_m(x, t) \) for \( m \geq 1 \) is governed by the linear equation (8) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab.

For the convergence of the above method we refer the reader to Liao’s work. If Eq. (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3 Application

In this part, we will apply the HAM to the equations of the homogeneous Smoluchowski’s. Smoluchowski’s equation comes in two flavours: discrete and continuous. In the discrete version, the cluster mass may take values in the set of positive integers, whereas, in the continuous version, the cluster mass take values in \( \mathbb{R}^+ \).

We consider the continuous homogenous Smoluchowski’s equation without diffusion part, which is defined as follows:
\[ \frac{\partial c(x, t)}{\partial t} = \frac{1}{2} \int_0^x K(x - y, y) c(y, t) c(x - y, t) \, dy - \int_0^\infty K(x, y) c(x, t) c(y, t) \, dy, \] (13)

**Example 3.1.**

We first consider Eq. (13) with constant kernel \( K(x, y) = 1 \). To solve Eq. (13) by means of HAM, we consider the following process after separating the linear and nonlinear parts of the equation.

From Eq. (13), we define the nonlinear operator
\[ N[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t} - \frac{1}{2} \int_0^x \phi(y, t; q) \phi(x - y, t; q) \, dy + \int_0^\infty \phi(x, t; q) \phi(y, t; q) \, dy, \] (14)

We choose the \( c_0(x, t) \) as the initial approximation of \( c(x, t) \), as follows
\[ c_0(x, t) = e^{-x}, \] (15)

We choose the linear operator
\[ L[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t}, \] (16)

with the property \( L[e] = 0 \), where \( e \) is coefficient.

To ensure this, let \( h \neq 0 \) denote an auxiliary parameter, \( q \in [0, 1] \) an embedding parameter. We have the zeroth-order deformation equation
\[ (1 - q) L[\phi(x, t; q) - c_0(x, t)] = q h H(x, t) N[\phi(x, t; q)], \] (17)

obviously, when \( q=0 \) and \( q=1 \),
\[ \phi(x, t; 0) = c_0(x, t), \quad \phi(x, t; 1) = c(x, t), \] (18)

Thus, \( \phi(x, t; q) \) can be expanded in the Maclaurin series with respect to \( q \) in the form
\[ \phi(x, t; q) = c_0(x, t) + \sum_{m=1}^\infty c_m(x, t) q^m, \] (19)

where
\[ c_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t; q)}{\partial q^m} |_{q=0}, \] (20)
Note that the zeroth-order deformation Eq.(17) contains the auxiliary parameter \( h \), so that \( \phi(x, t; q) \) is dependent on \( h \). Assuming that \( h \) is so properly chosen that the series Eq.(19) is convergent at \( q = 1 \), we obtain from Eq.(19) that

\[
c(x, t) = c_0(x, t) + \sum_{m=1}^{\infty} c_m(x, t),
\]

Fore the sake of simplicity, introduce

\[
c_m^* = \{c_1, c_2, c_3, \cdots c_m\}, \tag{21}
\]

We differentiate the zeroth-order deformation Eq.(17) \( m \) times with respect to \( q \), then set \( q = 0 \). Dividing the obtained equation by \( m! \), we get the so-called \( m \)-th order deformation equation:

\[
L[c_m(x, t) - \chi_m c_{m-1}(x, t)] = h H(x, t) R_m(c_{m-1}^-), \tag{22}
\]

subject to the initial condition

\[
c_m(x, t) = 0, \quad (m \geq 2) \tag{23}
\]

where

\[
R_m(c_{m-1}^-) = \frac{\partial c_{m-1}^-}{\partial t} - \frac{1}{2} \int_0^x \sum_{j=0}^{m-1} c_j(x - y, t) c_{m-1-j}(y, t) dy
\]

\[
+ \int_0^x \sum_{j=0}^{m-1} c_j(x, t) c_{m-1-j}(y, t) dy, \tag{24}
\]

We now successively obtain the solution to each high order deformation equation:

\[
c_m(x, t) = \chi_m c_{m-1}(x, t) + L^{-1} [h H(x, t) R_m(c_{m-1}^-)], \quad m \geq 1, \tag{25}
\]

We start with an initial approximation \( c_0(x, t) = e^{-x} \). Therefore, We now successively obtain directly the other components as:

\[
c_1 = h e^{-x} t (\frac{1}{2} x + 1),
\]

\[
c_2 = \frac{1}{8} h t e^{-x} (-4x + 8 - 6ht x + hx^2 t + 6th - 4hx + 8h),
\]

\[
c_3 = \frac{-1}{48} h t e^{-x} (24x - 48 + 72ht x - 12hx^2 t - 72h + 48hx - 96h + 36h^2 h^2 x
\]

\[-12ht^2 h^2 x^2 + t^2 h^2 x^3 - 24t^2 h^2 + 72h^2 x - 12th^2 x^2 - 72h^2 t - 48h^2 + 2t^2 h^2 x),
\]

\[
c_4 = \frac{1}{348} h t e^{-x} (384 + 1728h^2 t + 576h^2 h^2 + 288t^2 h^2 x^2 - 192r - 576h^2 x
\]

\[+ 288t^2 h^2 x^2 + 120h^3 h^3 x^2 - 20h^3 h^3 x^3 - 240h^3 h^3 x - 864h^3 x - 864t^3 h^3 x
\]

\[+ 864t^4 h - 576hx + 1152h + 864h^3 + 576t^2 h^3 + 120t^3 h^3 - 192h^3 x
\]

\[- 864ht x + 144hx^2 t + t^3 h^3 x^4 - 24t^3 h^3 x^3 + 144h^3 x^2 + 1152h^2
\]

\[- 864t^3 h^2 x - 1728h^2 t x - 24t^3 h^2 x^3 + 288th^2 x^2 + 384h^3 x)
\]

\[;
\]

**Example 3.2.**

We now consider Eq. (13) with the multiplicative coagulation kernel \( k(x, t) = xy \).

\[
\frac{\partial c(x, t)}{\partial t} = \frac{1}{2} \int_0^x (x - y) g c(y, t) c(x - y, t) dy - \int_0^\infty xy c(x, t) c(y, t) dy, \tag{26}
\]

Now to solve the Eqs. (26) by means of homotopy analysis method, we choose the linear operator

\[
L[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t}, \tag{27}
\]

\[I\]
and the initial approximation
\[ c_0(x, t) = \frac{e^{-x}}{x}, \]  
(28)

We now define a nonlinear operators as:
\[ N[\phi(x, t; q)] = \frac{\partial\phi(x, t; q)}{\partial t} - \frac{1}{2} \int_0^x \phi(y, t; q) \phi(x - y, t; q) dy + \int_0^{\infty} \phi(x, t; q) \phi(y, t; q) dy, \]  
(29)

To ensure this, let \( h \neq 0 \) denote an auxiliary parameter, \( q \in [0, 1] \) an embedding parameter. We have the zeroth-order deformation equation
\[ (1 - q)L[\phi(x, t; q) - c_0(x, t)] = q h H(x, t) N[\phi(x, t; q)], \]  
(30)

Thus, \( \phi(x, t; q) \) can be expanded in the Maclaurin series with respect to \( q \) in the form
\[ \phi(x, t; q) = c_0(x, t) + \sum_{m=1}^{\infty} c_m(x, t)q^m, \]  
(31)

where
\[ c_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t; q)}{\partial q^m}|_{q=0}. \]  
(32)

Note that the zeroth-order deformation Eq.(30) contains the auxiliary parameter \( h \), so that \( \phi(x, t; q) \) is dependent on \( h \). Assuming that \( h \) is so properly chosen that the series Eq.(31) is convergent at \( q = 1 \), we obtain from Eq.(31) that
\[ c(x, t) = c_0(x, t) + \sum_{m=1}^{\infty} c_m(x, t), \]

For the sake of simplicity, introduce
\[ \overline{c_m} = \{c_1, c_2, c_3, \cdots c_m\}. \]  
(33)

We differentiate the zeroth-order deformation Eq.(30) \( m \) times with respect to \( q \), then set \( q = 0 \). Dividing the obtained equation by \( m! \), we get the so-called \( m \)th-order deformation equation:
\[ L[c_m(x, t) - \chi_m c_{m-1}(x, t)] = h H(x, t) R_m(\overline{c_{m-1}}), \]  
(34)

subject to the initial condition
\[ c_m(x, t) = 0, \quad (m \geq 2) \]  
(35)

where
\[ R_m(\overline{c_{m-1}}) = \frac{\partial c_{m-1}}{\partial t} - \frac{1}{2} \int_0^x (x - y)y \sum_{j=0}^{m-1} c_j(x - y, t) c_{m-1-j}(y, t) dy 
+ \int_0^{\infty} xy \sum_{j=0}^{m-1} c_j(x, t) c_{m-1-j}(y, t) dy, \]  
(36)

We now successively obtain the solution to each high order deformation equation:
\[ c_m(x, t) = \chi_m c_{m-1}(x, t) + L^{-1}[h H(x, t) R_m(\overline{c_{m-1}})], m \geq 1, \]  
(37)
We start with an initial approximation \( \sum_{n=0}^{\infty} c_n x^n \). Therefore, we now successively obtain directly the other components as:

\[
c_1 = h e^{-x} t \left( \frac{1}{2} x + 1 \right),
\]
\[
c_2 = \frac{1}{12} h t e^{-x} (-6x + 12 - 6hx^2 + h^2t + 6thx - 6hx + 12h),
\]
\[
c_3 = \frac{-1}{144} h t e^{-x} (72x - 144 + 144hx^2t - 24hx^3t - 144htx + 144hx - 288h
\]
\[
-12t^2h^2x^4 + 36t^2h^2x^3 + x^5t^2h^2 - 24t^2h^2x^2 + 144th^2x^2 - 24th^2x^3,
\]
\[
c_4 = \frac{1}{2880} h t e^{-x} (2880 + 1440t^2h^2x^2 - 1440x - 4320h^2x + 440t^2h^3x^2
\]
\[
+120t^4h^3x^3 + 4320th^3x^2 + 720hx^3t - 4320hx + 840h - 1440h^3x
\]
\[
+4320hx - 4320hx^2 - 24t^4h^3x_4 - 2160t^2h^3x^3 - 4320th^3x^2
\]
\[
+720th^3x^3 + t^4h^3x^7 + 720th^3x^4 + 120t^4h^3x^5 + 840h^2 + 840h^2tx
\]
\[
-2160th^2x^3 - 840th^2x^2 + 2880h^3 - 20x^5t^3h^3 + 60x^5t^2h^3 + 720t^2h^2x^4
\]
\[
+144th^2x^3 - 60x^5t^2h^2)
\]

\[ \vdots \]

4 Numerical results

In this section we have presented the numerical results for solving homogeneous Smoluchowski’s equation by the homotopy analysis method.

Since the HPM, ADM solutions are indeed a special case of the HAM solution when \( h = -1 \). This fact has been pointed out by Liao [11]. Furthermore, by HAM, it is easy to discover the valid region of \( h \), which corresponds to the line segments nearly parallel to the horizontal axis. The convergent regions of \( h \) are presented by the \( h \)-curves as shown in Fig. 2, Fig. 4. From \( h \)-curve of example 3.1, \( h \)-curve of example 3.2, find that the valid region of \( h \) is about \(-0.8 \leq h \leq -0.4, -0.2 \leq h \leq -1.3 \), respectively. For this figures of \( h \)-curves, we fixed the values of \( (x, t) \) to be \((0, 1), (1, 1)\), respectively. It is clear that \( h = -1 \) is not a valid value to ensure the convergence of solution series. This is the reason why Liao [11] introduced the convergence-control parameter \( h \) to improve the early version of the HAM. Therefore, unlike the homotopy perturbation method, Adomian decomposition method and the variational iteration method, the HAM provides a convenient way to ensure the convergence of solution series. Thus, HPM solution in [1] is not a suitable option for homogeneous Smoluchowski’s equation.

In Figs. 1 and 3, one can also see the comparison between obtained results HAM whit HPM, ADM are given in [1].

5 Conclusion

In this Letter, we have successfully developed HAM for solving homogeneous Smoluchowski’s equation. It is apparently seen that HAM is a very powerful and efficient technique in finding analytical solutions for wide classes of linear and nonlinear problems. The results got from the performance of HAM over homogeneous Smoluchowski’s equation, was specified that the solution of HAM in special case is similar to numerical results of HPM, ADM.

Matlab has been used for computations in this paper.

References


LINS homepage: http://www.nonlinearscience.org.uk/
Fig1: The comparison of the 10th-order HAM results for $h = -0.6, -1$ (example 3.1).

Fig2: The $h$-curve of $c(0, 1)$ based on the 8th-order HAM (example 3.1).

Fig3: The comparison of the 10th-order HAM results for $h = -0.79, -1$ (example 3.2).

Fig4: The $h$-curve of $c(1, 1)$ based on the 8th-order HAM (example 3.2).


