ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.10(2010) No.4,pp.430-437

Subordination Results for a Class of Analytic Functions Involving the Hurwitz-Lerch Zeta Function

G. Murugusundaramoorthy *, K. Vijaya, K. Uma
School of Advanced Sciences , VIT University,
Vellore - 632014, India.
(Received 5 July 2010, accepted 4 September 2010)

Abstract: In this paper, we introduce a generalized class of starlike functions and obtain the subordination results for various subclasses of starlike functions. Further, we obtain the integral means inequalities for various subclasses of starlike functions.

Keywords: univalent; starlike; convex; subordinating factor sequence; integral means; Hadamard product; Hurwitz-Lerch Zeta function

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open disc $U = \{z : |z| < 1\}$. Also denote by T a subclass of $\mathcal{A}$ consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0, \; z \in U \tag{2}$$

introduced and studied by Silverman[16]. For functions $\phi \in \mathcal{A}$ given by $\phi(z) = z + \sum\limits_{n=2}^{\infty} \phi_n z^n$ and $\psi \in \mathcal{A}$ given by $\psi(z) = z + \sum\limits_{n=2}^{\infty} \psi_n z^n$, we define the Hadamard product (or convolution) of ϕ and ψ by

$$(\phi * \psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n, \quad z \in U. \tag{3}$$

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [21],p. 121).

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{4}$$

$$(a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| < 1)$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}); \mathbb{N} := \{1, 2, 3, ...\})$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [3], Ferreira and Lopez [5], Garg et al. [7], Lin and Srivastava [11], Lin et al. [12], and others. Srivastava and Attiya [20] (see also Raducanu and Srivastava [15], and Prajapat and Goyal [14]) introduced and investigated the linear operator:

***Corresponding author**. *E-mail address*: gmsmoorthy@yahoo.com

IJNS.2010.12.30/425

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{b,\mu} * f(z) \tag{5}$$

$(z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$G_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \quad (z \in U). \tag{6}$$

We recall here the following relationships (given earlier by [14], [15]) which follow easily by using (1), (5) and (6)

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} C_n(b,\mu) a_n z^n \tag{7}$$

where

$$C_n(b,\mu) = \left| \left(\frac{1+b}{n+b} \right)^{\mu} \right| \tag{8}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}$. For various choices of μ and b we get different operators and are listed below.

1. For $\mu = 0$

$$\mathcal{J}_b^0 f(z) := f(z). \tag{9}$$

2. For $\mu = 1$ and $b = 0$

$$\mathcal{J}_0^1 f(z) = \mathcal{L} f(z) := \int_0^z \frac{f(t)}{t} dt := z + \sum_{n=2}^{\infty} \left(\frac{1}{n} \right) a_n z^n. \tag{10}$$

3. For $\mu = 1$ and $b = \nu (\nu > -1)$

$$\mathcal{J}_{\nu}^1 f(z) := \mathcal{J}_{\nu}(f)(z) = \frac{1+\nu}{z^{\nu}} \int_0^z t^{\nu-1} f(t) dt := z + \sum_{n=2}^{\infty} \left(\frac{1+\nu}{n+\nu} \right) a_n z^n. \tag{11}$$

4. For $\mu = \sigma (\sigma > 0)$ and $b = 1$

$$\mathcal{J}_1^{\sigma} f(z) := z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1} \right)^{\sigma} a_n z^n = \mathcal{I}^{\sigma}(f)(z), \tag{12}$$

where $\mathcal{L}f$ and $\mathcal{J}_{\nu} f$ are the integral operators introduced by Alexander [1] and Bernardi [2], respectively, and $\mathcal{I}^{\sigma}(f)$ is the Jung-Kim-Srivastava integral operator [8] closely related to some multiplier transformation studied by Flett [6]. Making use of the operator $\mathcal{J}_b^{\mu}$, we introduce a new subclass of analytic functions with negative coefficients and discuss some some usual properties of the geometric function theory of this generalized function class.

For $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$ and $k \geq 0$, we let $\mathcal{J}_b^{\mu}(\lambda, \gamma, k)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\text{Re} \left\{ \frac{z(\mathcal{J}_b^{\mu} f(z))'}{(1-\lambda)z + \lambda \mathcal{J}_b^{\mu} f(z)} - \gamma \right\} > k \left| \frac{z(\mathcal{J}_b^{\mu} f(z))'}{(1-\lambda)z + \lambda \mathcal{J}_b^{\mu} f(z)} - 1 \right|, \quad z \in U, \tag{13}$$

where $\mathcal{J}_b^{\mu} f(z)$ is given by (5). We further let $T\mathcal{J}_b^{\mu}(\lambda, \gamma, k) = \mathcal{J}_b^{\mu}(\lambda, \gamma, k) \cap T$. It is of interest to note that when $\mu = 0, k = 0, \lambda = 1$ and $b = 0$ yield subclasses of starlike functions of order γ denoted by $T^*(\gamma)$ and convex functions of order γ denoted by $C(\gamma)$ of T stduied extensively by Silverman[16].

Motivated by earlier works of [4, 13, 16, 17] and [19] in this paper, we investigate certain characteristic properties and obtain the subordination results for the class of functions $f \in \mathcal{J}_b^{\mu}(\lambda, \gamma, k)$ and integral means results for the class of functions $f \in T\mathcal{J}_b^{\mu}(\lambda, \gamma, k)$.

2 Basic Properties

In this section we obtain the characterization properties for the classes $\mathcal{J}_b^\mu(\lambda, \gamma, k)$ and $T\mathcal{J}_b^\mu(\lambda, \gamma, k)$.

Theorem 1 *A function $f(z)$ of the form (1) is in $\mathcal{J}_b^\mu(\lambda, \gamma, k)$ if*

$$\sum_{n=2}^{\infty}[n(1+k) - \lambda(\gamma+k)]C_n(b,\mu)\,|a_n| \leq 1-\gamma, \tag{14}$$

where $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $k \geq 0$, and $C_n(b,\mu)$ is given by (8).

Proof. It suffices to show that

$$\begin{aligned} &k\left|\frac{z(\mathcal{J}_b^\mu f(z))'}{(1-\lambda)z+\lambda\mathcal{J}_b^\mu f(z)} - 1\right| \\ &\quad - \operatorname{Re}\left\{\frac{z(\mathcal{J}_b^\mu f(z))'}{(1-\lambda)z+\lambda\mathcal{J}_b^\mu f(z)} - 1\right\} \\ &\leq 1-\gamma. \end{aligned}$$

We have

$$\begin{aligned} &k\left|\frac{z(\mathcal{J}_b^\mu f(z))'}{(1-\lambda)z+\lambda\mathcal{J}_b^\mu f(z)} - 1\right| \\ &\quad - \operatorname{Re}\left\{\frac{z(\mathcal{J}_b^\mu f(z))'}{(1-\lambda)z+\lambda\mathcal{J}_b^\mu f(z)} - 1\right\} \\ &\leq (1+k)\left|\frac{z(\mathcal{J}_b^\mu f(z))'}{(1-\lambda)z+\lambda\mathcal{J}_b^\mu f(z)} - 1\right| \\ &\leq \frac{(1+k)\sum\limits_{n=2}^{\infty}(n-\lambda)C_n(b,\mu)|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda C_n(b,\mu)|a_n||z|^{n-1}} \\ &< \frac{(1+k)\sum\limits_{n=2}^{\infty}(n-\lambda)C_n(b,\mu)|a_n|}{1-\sum\limits_{n=2}^{\infty}\lambda C_n(b,\mu)|a_n|}. \end{aligned}$$

The last expression is bounded above by $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[n(1+k) - \lambda(\gamma+k)]C_n(b,\mu)|a_n| \leq 1-\gamma$$

and the proof is complete. ■

In the following theorem, we obtain necessary and sufficient conditions for functions in $T\mathcal{J}_b^\mu(\lambda, \gamma, k)$.

Theorem 2 *Let $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $k \geq 0$, then a function f of the form (2) to be in the class $T\mathcal{J}_b^\mu(\lambda, \gamma, k)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+k) - \lambda(\gamma+k)]C_n(b,\mu)\,|a_n| \leq 1-\gamma, \tag{15}$$

where $C_n(b,\mu)$ are given by (8).

Proof.

In view of Theorem 1, we need only to prove the necessity. If $f \in T\mathcal{J}_b^\mu(\lambda, \gamma, k)$ and z is real then

$$\operatorname{Re}\left\{\frac{1-\sum\limits_{n=2}^{\infty}nC_n(b,\mu)\,a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda C_n(b,\mu)\,a_n z^{n-1}} - \gamma\right\} > k\left|\frac{\sum\limits_{n=2}^{\infty}(n-\lambda)C_n(b,\mu)\,a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda C_n(b,\mu)\,a_n z^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)\,|a_n| \leq 1-\gamma,$$

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$, and $C_n(b,\mu)$ are given by (8). ■

Corollary 3 *If $f \in T\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$, then*

$$|a_n| \leq \frac{1-\gamma}{[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)}, \;\; 0 \leq \lambda < 1, \;\; 0 \leq \gamma < 1, k \geq 0, \tag{16}$$

where $C_n(b,\mu)$ are given by (8). Equality holds for the function

$$f(z) = z - \frac{1-\gamma}{[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)}z^n.$$

Theorem 4 (Extreme Points) *Let*

$$\begin{aligned} f_1(z) &= z \quad \text{and} \\ f_n(z) &= z - \frac{1-\gamma}{[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)}z^n, \;\; n \geq 2, \end{aligned} \tag{17}$$

for $0 \leq \gamma < 1$, $0 \leq \lambda < 1$, $k \geq 0$, $C_n(b,\mu)$ are given by (8). Then $f(z)$ is in the class $T\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty}\omega_n f_n(z), \tag{18}$$

where $\omega_n \geq 0$ and $\sum\limits_{n=1}^{\infty}\omega_n = 1$.

Proof. Suppose $f(z)$ can be written as in (18). Then

$$f(z) \;\; = \;\; z - \sum_{n=2}^{\infty}\omega_n\frac{1-\gamma}{[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)}z^n.$$

Now,

$$\sum_{n=2}^{\infty}\frac{[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)}{1-\gamma}\omega_n\frac{1-\gamma}{[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)}$$

$= \sum_{n=2}^{\infty}\omega_n = 1-\omega_1 \leq 1$.

Thus $f \in T\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$. Conversely, let us have $f \in T\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$. Then by using (16), we set

$$\omega_n = \frac{[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)}{1-\gamma}a_n, \quad n \geq 2$$

and $\omega_1 = 1 - \sum\limits_{n=2}^{\infty}\omega_n$. Then we have $f(z) = \sum\limits_{n=1}^{\infty}\omega_n f_n(z)$ and hence this completes the proof of Theorem 4. ■

For Convenience we let

$$\Phi_n(\lambda,\gamma,k) = [n(1+k)-\lambda(\gamma+k)]C_n(b,\mu). \tag{19}$$

where $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $k \geq 0$, and $C_n(b,\mu)$ is given by (8).

Remark: Usual manner adopting the techniques of Silverman[16] and Murugusundaramoorthy[13],one can prove the results on distortion bounds , radii of convexity and starlikeness for functions $f(z) \in T\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$ we skip the details.

3 Integral Means Inequalities

In this section, we obtain integral means inequalities for the functions in the family $T\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$.

Definition 1 (Subordination Principle) *For analytic functions g and h with $g(0)=h(0)$, g is said to be subordinate to h, denoted by $g\prec h$, if there exists an analytic function w such that $w(0)=0$, $|w(z)|<1$ and $g(z)=h(w(z))$, for all $z\in U$.*

Lemma 5 *[10] If the functions f and g are analytic in U with $g\prec f$, then for $\eta>0$, and $0<r<1$,*

$$\int_0^{2\pi}\left|g(re^{i\theta})\right|^{\eta}d\theta\le\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\eta}d\theta. \tag{20}$$

In [16], Silverman found that the function $f_2(z)=z-\frac{z^2}{2}$ is often extremal over the family T and applied this function to resolve his integral means inequality, conjectured in [17] and settled in [18], that

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\eta}d\theta\le\int_0^{2\pi}\left|f_2(re^{i\theta})\right|^{\eta}d\theta,$$

for all $f\in T$, $\eta>0$ and $0<r<1$. In [18], Silverman also proved his conjecture for the subclasses of starlike functions of order γ denoted by $T^*(\gamma)$ and convex functions of order γ denoted by $C(\gamma)$ of T.

Applying Lemma 5, Theorem 2 and Theorem 4, we prove the following result.

Theorem 6 *Suppose $f\in T\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$, $\eta>0$, $0\le\lambda\le1$, $0\le\gamma<1$, $k\ge0$ and $f_2(z)$ is defined by*

$$f_2(z)=z-\frac{1-\gamma}{\Phi_2(\lambda,\gamma,k)}z^2,$$

where

$$\Phi_2(\lambda,\gamma,k)=[2(1+k)-\lambda(k+\gamma)]C_2(b,\mu) \tag{21}$$

and $C_2(b,\mu)$ is given by (27). Then for $z=re^{i\theta}$, $0<r<1$, we have

$$\int_0^{2\pi}|f(z)|^{\eta}d\theta\le\int_0^{2\pi}|f_2(z)|^{\eta}d\theta. \tag{22}$$

Proof. For $f(z)=z-\sum\limits_{n=2}^{\infty}|a_n|z^n$, (22) is equivalent to proving that

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}|a_n|z^{n-1}\right|^{\eta}d\theta\le\int_0^{2\pi}\left|1-\frac{(1-\gamma)}{\Phi_2(\lambda,\gamma,k)}z\right|^{\eta}d\theta.$$

By Lemma 5, it suffices to show that

$$1-\sum_{n=2}^{\infty}|a_n|z^{n-1}\prec1-\frac{1-\gamma}{\Phi_2(\lambda,\gamma,k)}z.$$

Setting

$$1-\sum_{n=2}^{\infty}|a_n|z^{n-1}=1-\frac{1-\gamma}{\Phi_2(\lambda,\gamma,k)}w(z), \tag{23}$$

and using (15), we obtain $w(z)$ is analytic in $U, w(0)=0$, and

$$\begin{aligned}|w(z)|&=\left|\sum_{n=2}^{\infty}\frac{\Phi_2(\lambda,\gamma,k)}{1-\gamma}|a_n|z^{n-1}\right|\\&\le|z|\sum_{n=2}^{\infty}\frac{\Phi_n(\lambda,\gamma,k)}{1-\gamma}|a_n|\\&\le|z|,\end{aligned}$$

where $\Phi_n(\lambda,\gamma,k)=[n(1+k)-\lambda(\gamma+k)]C_n(b,\mu)$. This completes the proof by Theorem 6. ■

4 Subordination Results

Following Frasin [4] and Singh [19] , we obtain subordination results for the new class $\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$.

Definition 2 (Subordinating Factor Sequence) *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z)=\sum\limits_{n=1}^{\infty}a_nz^n$, $a_1=1$ is regular, univalent and convex in U, we have*

$$\sum_{n=1}^{\infty}b_na_nz^n \prec f(z), \quad z\in U. \tag{24}$$

Lemma 7 *[22] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\ \left\{1+2\sum_{n=1}^{\infty}b_nz^n\right\}>0, \quad z\in U. \tag{25}$$

Theorem 8 *Let $f\in\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$ and $g(z)$ be any function in the usual class of convex functions C, then*

$$\frac{\Phi_2(\lambda,\gamma,k)}{2[1-\gamma+\Phi_2(\lambda,\gamma,k)]}(f*g)(z)\prec g(z) \tag{26}$$

where $0\le\gamma<1$; $k\ge 0$ and $0\le\lambda\le 1$, with

$$C_2(b,\mu)=\left(\frac{1+b}{2+b}\right)^{\mu} \tag{27}$$

and

$$Re\ \{f(z)\}>-\frac{1-\gamma+\Phi_2(\lambda,\gamma,k)}{\Phi_2(\lambda,\gamma,k)}, \quad z\in U. \tag{28}$$

The constant $\frac{\Phi_2(\lambda,\gamma,k)}{2[1-\gamma+\Phi_2(\lambda,\gamma,k)}$ is the best estimate.

Proof. Let $f\in\mathcal{J}_b^{\mu}(\lambda,\gamma,k)$ and suppose that $g(z)=z+\sum\limits_{n=2}^{\infty}c_nz^n\in C$. Then

$$\begin{aligned}&\frac{\Phi_2(\lambda,\gamma,k)}{2[1-\gamma+\Phi_2(\lambda,\gamma,k)]}(f*g)(z)\\ &\qquad=\frac{\Phi_2(\lambda,\gamma,k)}{2[1-\gamma+\Phi_2(\lambda,\gamma,k)]}\left(z+\sum_{n=2}^{\infty}c_na_nz^n\right).\end{aligned} \tag{29}$$

Thus, by Definition 2, the subordination result holds true if

$$\left\{\frac{\Phi_2(\lambda,\gamma,k)}{2[1-\gamma+\Phi_2(\lambda,\gamma,k)]}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 7, this is equivalent to the following inequality

$$\text{Re}\ \left\{1+\sum_{n=1}^{\infty}\frac{\Phi_2(\lambda,\gamma,k)}{[1-\gamma+\Phi_2(\lambda,\gamma,k)]}a_nz^n\right\}>0, \quad z\in U. \tag{30}$$

Now, for $|z| = r < 1$, we have

$$\begin{aligned}
&\text{Re}\left\{1+\frac{\Phi_2(\lambda,\gamma,k)}{[1-\gamma+\Phi_2(\lambda,\gamma,k)]}\sum_{n=1}^{\infty}a_nz^n\right\}\\
&=\text{Re}\left\{1+\frac{\Phi_2(\lambda,\gamma,k)}{[1-\gamma+\Phi_2(\lambda,\gamma,k)]}z+\frac{\sum\limits_{n=1}^{\infty}\Phi_2(\lambda,\gamma,k)a_nz^n}{[1-\gamma+\Phi_2(\lambda,\gamma,k)]}\right\}\\
&\geq 1-\frac{\Phi_2(\lambda,\gamma,k)}{[1-\gamma+\Phi_2(\lambda,\gamma,k)]}r-\frac{1}{[1-\gamma+\Phi_2(\lambda,\gamma,k)]}\sum_{n=1}^{\infty}\Phi_n(\lambda,\gamma,k)a_nr^n\\
&\geq 1-\frac{\Phi_2(\lambda,\gamma,k)}{[1-\gamma+\Phi_2(\lambda,\gamma,k)]}r-\frac{1-\gamma}{[1-\gamma+\Phi_2(\lambda,\gamma,k)]}r\\
&> 0,
\end{aligned}$$

and by noting the fact that $\Phi_n(\lambda,\gamma,k)$ is increasing function for $n \geq 2$. Thus we have also made use of the assertion (14) of Theorem 1. This evidently proves the inequality (30) and hence also the subordination result (26) asserted by Theorem 8. The inequality (28) follows from (26) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Next we consider the function

$$F(z) := z - \frac{1-\gamma}{\Phi_2(\lambda,\gamma,k)}z^2$$

where $0 \leq \gamma < 1,\ k \geq 0, 0 \leq \lambda < 1$ and $\Phi_2(\lambda,\gamma,k)$ is given by (21). Clearly $F \in \mathcal{J}_b^{\mu}(\lambda,\gamma,k)$. For this function (26)becomes

$$\frac{\Phi_2(\lambda,\gamma,k)}{2[1-\gamma+\Phi_2(\lambda,\gamma,k)]}F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\text{Re}\left(\frac{\Phi_2(\lambda,\gamma,k)}{2[1-\gamma+\Phi_2(\lambda,\gamma,k)]}F(z)\right)\right\} = -\frac{1}{2}, \quad z \in U.$$

This shows that the constant $\frac{\Phi_2(\lambda,\gamma,k)}{2[1-\gamma+\Phi_2(\lambda,\gamma,k)]}$ is best possible. ■

Acknowledgments

The authors express their sincerest thanks to the referee for his useful comments.

References

[1] J. W. Alexander. Functions which map the interior of the unit circle upon simple regions. *Ann. of Math.* 17(1915):12-22.

[2] S. D. Bernardi. Convex and starlike univalent functions. *Trans. Amer. Math. Soc.* 135(1969):429-446.

[3] J. Choi and H. M. Srivastava. Certain families of series associated with the Hurwitz-Lerch Zeta function. *Appl. Math. Comput.* 170(2005):399-409.

[4] B. A. Frasin. Subordination results for a class of analytic functions defined by a linear operator. *J. Ineq. Pure and Appl. Math.* 7 (2006)(4):1-7.

[5] C. Ferreira and J. L. Lopez. Asymptotic expansions of the Hurwitz-Lerch Zeta function. *J. Math. Anal. Appl.* 298(2004):210-224.

[6] T. M. Flett. The dual of an inequality of Hardy and Littlewood and some related inequalities. *J. Math. Anal. Appl.* 38(1972):746-765.

[7] M. Garg, K. Jain and H. M. Srivastava. Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions. *Integral Transform. Spec. Funct.* 17(2006): 803-815.
[8] I. B. Jung, Y. C. Kim AND H. M. Srivastava. The Hardy space of analytic functions associated with certain one-parameter families of integral operators. *J. Math. Anal. Appl.* 176(1993):138-147.
[9] R. J. Libera. Some classes of regular univalent functions. *Proc. Amer. Math. Soc.* 16(1965): 755-758.
[10] J. E. Littlewood. On inequalities in theory of functions. *Proc. London Math. Soc.* 23(1925): 481-519.
[11] S.-D. Lin and H. M. Srivastava. Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations. *Appl. Math. Comput.* 154(2004):725-733.
[12] S.-D. Lin, H. M. Srivastava and P.-Y. Wang. Some espansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, *Integral Transform. Spec. Funct.* 17(2006): 817-827.
[13] G.Murugusundaramoorthy. Unified class of analytic functions with negative coefficients involving the Hurwitz-Lerch Zeta function. *Bulletin of Mathematical Analysis and Applications.* 1(2009)(3):71-84.
[14] J. K. Prajapat and S. P. Goyal. Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions. *J. Math. Inequal.* 3(2009): 129-137.
[15] D. Raducanu and H. M. Srivastava. A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function. *Integral Transform. Spec. Funct.* 18(2007):933-943.
[16] H. Silverman. Univalent functions with negative coefficients. *Proc. Amer. Math. Soc.* 51 (1975):109-116.
[17] H. Silverman, A survey with open problems on univalent functions whose coefficients are negative. *Rocky Mt. J. Math.* 21(1991):1099-1125.
[18] H. Silverman. Integral means for univalent functions with negative coefficients. H*ouston J. Math.* 23(1997): 169-174.
[19] S. Singh. A subordination theorem for spirallike functions. *Internat. J. Math. and Math. Sci.* 24(2000)(7) :433-435.
[20] H. M. Srivastava and A. A. Attiya. An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination. *Integral Transform. Spec. Funct.* 18(2007):207-216.
[21] H. M. Srivastava and J. Choi. Series associated with the Zeta and related functions. Dordrecht, Boston, London: Kluwer Academic Publishers. 2001.
[22] H. S. Wilf. Subordinating factor sequence for convex maps of the unit circle. *Proc. Amer. Math. Soc.* 12(1961):689-693.