Approximate Controllability of Impulsive Differential Equations with Nonlocal Conditions *

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Abstract: This paper is concerned with the approximate controllability for the impulsive differential equation with nonlocal conditions in Hilbert spaces. Based on the semigroup theory and fixed point approach, a sufficient condition for the approximate controllability of the impulsive differential equation with nonlocal conditions is established.

Keywords: nonlocal condition; impulsive differential equations; approximate controllability; mMeasure of noncompactness

1 Introduction

It is well known that the issue of controllability plays an important role in control theory and engineering (see [1]-[5]) because they have close connections to pole assignment, structural decomposition, quadratic optimal control, observer design etc. In recent years, the problem of controllability for various kinds of differential and impulsive differential systems has been extensively studied by many authors (see [6]-[13], [4]) using different approaches. Recently, Chang et al. [14] studied the controllability of impulsive neutral functional differential systems with infinite delay in Banach spaces by using Dhage’s fixed point theorem. More recently, Chang et al. [15] proved the existence of solutions for non-densely defined neutral impulsive differential inclusions with nonlocal conditions by using the Leray-Schauder theorem of the alternate for kakutani maps. In all these works ([6], [7], [9], [4], [13], [14]) contain the assumption of compactness of the semigroup, as well as the supposition of the controllability of corresponding linear system, i.e., the invertibility of the linear controllability operator W. But it is known (see [16], [17]) that in infinite-dimensional case these hypotheses are in contradiction to each other. Actually, in this situation the controllability may be provided only on the subspace RangeW.

Motivated by the above approach, the goal of the present paper is to study the approximate controllability for the following impulsive differential equations with nonlocal conditions

\[ \begin{aligned}
   x'(t) &= Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J := [0, b], \quad t \neq t_i, \\
   x(0) &= x_0 - g(x), \\
   \Delta x(t_i) &= J_i(x(t_i)), \quad i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < b,
\end{aligned} \]

where \( A \) is the infinitesimal generator of a compact semigroup \( T(t) \) in a Hilbert space \( X \); the operator \( B \) is a bounded linear operator from a Hilbert space \( U \) into \( X \); the control function \( u(\cdot) \) is given in \( L^2(J, U) \); \( f, g \) are appropriate continuous functions to be specified later; \( I_i : X \to X \) is a nonlinear map, \( \Delta x|_{t=t_i} = x(t_i^+) - x(t_i^-) \), for all \( i = 1, 2, \ldots, p, \quad 0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = b \), here \( x(t_i^-) \) and \( x(t_i^+) \) represent the left and the right limits of \( x(t) \) at \( t = t_i \), respectively.

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The remainder of this paper is organized as follows. In section 2, we give some definitions and facts. In section 3, for $0 < \alpha \leq 1$, we give the existence of solutions for problem (1) based on the semigroup theory and fixed point approach. In section 4, sufficient conditions for approximately controllability of impulsive differential equations are established.

## 2 Preliminaries

Throughout this paper, $U, X$ are Hilbert spaces which are identified with their duals. $T(t)$ is a strongly continuous semigroup on $X$, with generator $A : D(A) \to X$. We denote by $C(0, b; X)$ the space of $X$-valued continuous functions on $[0, b]$ with the norm $\|x\| = \sup\{\|x(t)\|, t \in [0, b]\}$ and by $L^1(0, b; X)$ the space of $X$-valued Bochner integral functions on $[0, b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\| dt$. Let

$$PC(0, b; X) = \{x : [0, b] \to X : x(t) \text{ be continuous at } t \neq t_i \text{ and }\}.$$  

It is easy to check that $PC(0, b; X)$ is a Banach space with the norm $\|x\|_{PC} = \sup_{t \in [0, b]} \|x(t)\|$.

Consider the infinite-dimensional linear control system

$$\begin{aligned}
x'(t) &= Ax(t) + Bu(t), t \in J := [0, b], \\
x(0) &= x_0,
\end{aligned}$$  

where $u(t) \in L^2(J, U), A : X \to X, B : U \to X$.

Let $B \in L(U, X)$ and $b \geq 0$, we define

$$\Phi_b u = \int_0^b T(b-s)Bu(s)ds.$$  

**Definition 2.1**[18] Let $b > 0$.

- The system (2) (or the pair $(A, B)$) is exactly controllable in time $b$ if $\text{Ran} \Phi_b = X$.
- $(A, B)$ is approximately controllable in time $b$ if $\text{Ran} \Phi_b$ is dense in $X$.
- The pair $(A, B)$ is null-controllable in time $b$ if $\text{Ran} \Phi_b \supseteq \text{Ran} T_b$.

It is easy to see that the exact controllability in time $b$ is equivalent to the following property: for any $x_0, x_1 \in X$ there exists $u \in L^2(0, b; U)$ such that the solution $x$ of

$$\begin{aligned}
x'(t) &= Ax(t) + Bu(t), \\
x(0) &= x_0,
\end{aligned}$$  

satisfies $x(b) = x_1$. Approximate controllability in time $b$ is equivalent to the following: for any $x_0, x_1 \in X$ and $\epsilon > 0$, there exists $u \in L^2(0, b; U)$ such that the solution $x$ of (3) satisfies $\|x(b) - x_1\| < \epsilon$. Null-controllability in time $b$ is equivalent to the following: for any $x_0 \in X$, there exists a $u \in L^2(0, b; U)$ such that the solution $x$ of (3) satisfies $x(b) = 0$.

It is convenient at this point to define operators

$$\Gamma_b^\alpha = \int_0^b T(b-s)BB^*T^*(b-s)ds,$$$$
R(\alpha, \Gamma_b^\alpha) = (\alpha I + \Gamma_b^\alpha)^{-1}.$$  

$(S_1)$ $\alpha R(\alpha, \Gamma_b^\alpha) \to 0$ as $\alpha \to 0$ in the strong operator topology.

The assumption $(S_1)$ holds if and only if the linear system (2) is approximately controllable on $J$, see [19].

Next, we introduce the Hausdorff measure of noncompactness $\beta(\cdot)$ defined on each bounded subset $\Omega$ of Banach space $Y$ by

$$\beta(\Omega) = \inf \{\epsilon > 0 : \Omega \text{ has a finite } \epsilon - \text{net in } Y\}.$$  

Some basic properties of the Hausdorff measure of noncompactness $\beta(\cdot)$ are given in the following lemma.

**Lemma 2.2**[20] Let $Y$ be a real Banach space and $B, C \subseteq Y$ be bounded, then the following properties are satisfied:
(1) $B$ is precompact if and only if $\beta(B) = 0$;
(2) $\beta(B) = \beta(\overline{B}) = \beta(\text{conv} B)$, where $\overline{B}$ and $\text{conv} B$ mean the closure and convex hull of $B$ respectively;
(3) $\beta(B) \leq \beta(C)$ when $B \subseteq C$;
(4) $\beta(B + C) \leq \beta(B) + \beta(C)$, where $B + C = \{x + y : x \in B, y \in C\}$;
(5) $\beta(\lambda B) = |\lambda|\beta(B)$ for any $\lambda \in \mathbb{R}$;
(6) If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant $k$, then $\beta(QB) \leq k\beta(B)$ for any bounded subset $B \subseteq D(Q)$, where $Z$ is a Banach space.

The map $Q : W \subseteq Y \to Y$ is said to be a $\beta$-contraction if there exists a positive constant $k < 1$ such that $\beta(QC) \leq k\beta(C)$ for any bounded closed subset $C \subseteq W$, where $Y$ is a Banach space.

We will also use the sequential MNC $\beta_0$ generated by $\beta$, that is, for any bounded subset $\Omega \subset X$, we define

$$
\beta_0(\Omega) = \sup\{\beta([x_n : n \geq 1]) : \{x_n\}_{n=1}^{+\infty} \text{ is a sequence in } \Omega\}.
$$

It follows that

$$
\beta_0(\Omega) \leq \beta(\Omega) \leq 2\beta_0(\Omega). \quad (4)
$$

In addition, when $X$ is separable, we have $\beta_0(\Omega) = \beta(\Omega)$.

Throughout the paper, we denote $\beta$ by the Hausdorff measure of noncompactness of $X$ and denote $\beta_c$ by the Hausdorff measure of noncompactness of $C(0, b; X)$.

To the proof of the main result, we need the following statement (see [20]).

**Proposition 2.3** If $D \subseteq C(0, b; X)$ is bounded, then for all $t \in [0, b]$,

$$
\beta(D(t)) \leq \beta_c(D),
$$

where $D(t) = \{x(t) : x \in D\} \subseteq X$. Furthermore, if $D$ is equicontinuous on $[0, b]$, then $\beta(D(t))$ is continuous on $[0, b]$ and

$$
\beta_c(D) = \sup\{\beta(D(t)) : t \in [0, b]\}.
$$

The following fixed point theorem plays a key role in the proof of our main results.

**Lemma 2.4**([20], Darbo-Sadovskii) If $W \subseteq Y$ is bounded closed and convex, the continuous map $Q : W \to W$ is a $\beta$-contraction, then the map $Q$ has at least one fixed point in $W$.

In the following sections, it will be shown that the system (1) is approximately controllable if for all $\alpha > 0$ there exists a function $x(t) \in PC(0, b; X)$ such that

$$
\begin{align*}
\dot{u}(t) & = B^* T^* (b - t) R(\alpha, \Gamma_0^b) p(x(\cdot)), \\
x(t) & = T(t)[x_0 - g(x)] + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)Bu(s)ds \\
& \quad + \sum_{0 < t_i < t} T(t - t_i) I_i(x(t_i)),
\end{align*}
$$

where

$$
p(x(\cdot)) = x_1 - T(b)[x_0 - g(x)] - \int_0^b T(b-s)f(s, x(s))ds \\
\quad - \sum_{i=1}^p T(b - t_i) I_i(x(t_i)).
$$

### 3 Existence results

The main purpose in this section is to find conditions for solvability of system (5) and (6) for $\alpha > 0$.

Let $r$ be a finite positive constant. We consider the sets $B_r = \{x \in X : ||x|| \leq r\}$, $W_r = \{x \in PC(0, b; X) : x(t) \in B_r, t \in [0, b]\}$.

For convenience, let us introduce some notations.

$$
M = \sup_{t \in [0, b]} ||T(t)||, \quad M_1 = ||B||, \quad k = \max\{1, MM_1, M^2 M_1, \sqrt{5} MM_1\},
$$

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The nonlocal impulsive problem (1) has a solution on $\mathcal{J}$ if the continuous function $\Omega$ is such that

$$f \in \text{PC} \left( [0, b], \mathbb{R}^n \right),$$

where

$$I_i(x(t)) := \begin{cases} 0 & \text{for } t \in [0, a), \\ M(t) & \text{for } t = a, \\ 0 & \text{for } t \in [a, b], \end{cases}$$

with $M(t) = \max \{b(t), c(t)\}$ for $t \in [a, b]$. The linear operator $A : D(A) \subset X \to X$ generates a compact semigroup $\{T(t) : t \geq 0\}$, i.e., $T(t)$ is compact for each $t > 0$.

(HA) The linear operator $A : D(A) \subset X \to X$ generates a compact semigroup $\{T(t) : t \geq 0\}$, i.e., $T(t)$ is compact for each $t > 0$.

(Hf1) $f : [0, b] \times X \to X$; for a.e. $t \in [0, b]$, the function $f(t, \cdot) : X \to X$ is continuous and for all $x \in X$, the function $f(\cdot, x) : [0, b] \to X$ is measurable. Moreover, there exists a function $m \in L^1(0, b; \mathbb{R}^+)$ and a nondecreasing continuous function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, x)\| \leq m(t)\Omega(\|x\|)$$

for all $x \in X$ and $t \in [0, b]$.

(Hf2) For each $\alpha > 0$,

$$\lim_{r \to \infty} \sup \left( r - \frac{c}{\alpha}r - \frac{l}{\alpha}\Omega(r) \right) = \infty,$$

and

$$M \left( k + \sum_{i=1}^{p} k_i \right) \left( 1 + \frac{2\sqrt{b}}{\alpha}M_1^2M_2 \right) < 1.$$

(HI1) $I_i : X \to X$ is Lipschitz continuous with Lipschitz constant $k_i$ for $i = 1, 2, \ldots, p$.

(HI2) $I_i : X \to X$ is compact for $i = 1, 2, \ldots, p$.

(Hg1) $g : \text{PC}(0, b; X) \to X$ is Lipschitz continuous with Lipschitz constant $k$.

(Hg2) $g : \text{PC}(0, b; X) \to X$ is a compact mapping.

Theorem 3.1 Assume that the conditions (HA), (Hf1), (Hf2), (HI1) and (Hg1) are satisfied. Then for all $0 < \alpha \leq 1$, the nonlocal impulsive problem (1) has a solution on $\mathcal{J}$.

Proof. For $\alpha > 0$, we define the operator $\Gamma_\alpha : \text{PC}(0, b; X) \to \text{PC}(0, b; X)$ by

$$\Gamma_\alpha(x) = (\Gamma_\alpha^1 x) + (\Gamma_\alpha^2 x) + (\Gamma_\alpha^3 x) + (\Gamma_\alpha^4 x),$$

with

$$\begin{align*}
(\Gamma_\alpha^1 x)(t) &= T(t)[x_0 - g(x)], \\
(\Gamma_\alpha^2 x)(t) &= \int_0^t T(t-s)f(s, x(s))ds, \\
(\Gamma_\alpha^3 x)(t) &= \int_0^t T(t-s)Bu(s)ds, \\
(\Gamma_\alpha^4 x)(t) &= \sum_{0 < t_i < t} T(t-t_i)I_i(x(t_i)),
\end{align*}$$

where

$$u(t) = B^*T^*(b-t)R(\alpha, \Gamma_0^b)p(x(\cdot)),$$

and

$$p(x(\cdot)) = x_1 - T(b)[x_0 - g(x)] - \int_0^b T(b-s)f(s, x(s))ds - \sum_{i=1}^{p} T(b-t_i)I_i(x(t_i)).$$

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It will be shown that for all \( \alpha > 0 \) the operator \( \Gamma_\alpha \) from \( PC(0, b; X) \) into itself has a fixed point by using Lemma 2.4.

Firstly, for all \( \alpha > 0 \), we prove that the mapping \( \Gamma_\alpha \) is continuous on \( PC(0, b; X) \). For this purpose, let \( \{x_n\}_{n=1}^\infty \) be a sequence in \( PC(0, b; X) \) with \( \lim_{n \to \infty} x_n = x \) in \( PC(0, b; X) \). By the continuity of \( f \) with respect to the second argument, we deduce that for each \( s \in [0,b] \), \( f(s, x_n(s)) \) converges to \( f(s, x(s)) \) in \( X \), and we have

\[
||\Gamma_\alpha x_n - \Gamma_\alpha x||_{PC} \\
\leq M||g(x_n) - g(x)|| + M \int_0^b ||f(s, x_n(s)) - f(s, x(s))|| ds \\
+ M \sum_{i=1}^p ||I_i(x_n(t_i)) - I_i(x(t_i))|| + \frac{b}{\alpha} M^3 M^3 \|g(x_n) - g(x)\| \\
+ \int_0^b ||f(s, x_n(s)) - f(s, x(s))|| ds + \sum_{i=1}^p ||I_i(x_n(t_i)) - I_i(x(t_i))||,
\]

then by the continuity of \( g, I_i \), and using the dominated convergence theorem, we get \( \lim_{n \to \infty} \Gamma_\alpha x_n = \Gamma_\alpha x \) in \( PC(0, b; X) \), which implies that for all \( \alpha > 0 \), the mapping \( \Gamma_\alpha \) is continuous on \( PC(0, b; X) \).

Secondly, for an arbitrary \( \alpha > 0 \), there exists a positive constant \( r_0 = r(\alpha) \) such that \( \Gamma_\alpha : W_{r_0} \to W_{r_0} \). By assumption \((Hf2)\), there exists \( r_0 > 0 \) such that

\[
\frac{d}{\alpha} + \frac{c}{\alpha} r_0 + \frac{l}{\alpha} \Omega(r_0) \leq r_0.
\]

If \( x(t) \in W_{r_0} \subseteq PC(0, b; X) \), then we obtain

\[
||u(t)|| \leq \frac{1}{\alpha} M M_1 \left( ||x|| + M ||x_0|| + M ||g(0)|| \right) \\
+ M (k + \sum_{i=1}^p k_i) r_0 + M ||m||_{L^1} \Omega(r_0) + M \sum_{i=1}^p ||I_i(0)|| \\
\leq \frac{1}{\alpha} M M_1 \left( ||x|| + M ||x_0|| + M ||g(0)|| + M \sum_{i=1}^p ||I_i(0)|| \right) \\
+ \frac{1}{\alpha} M^2 M_1 \left( k + \sum_{i=1}^p k_i \right) r_0 + \frac{1}{\alpha} M^2 M_1 ||m||_{L^1} \Omega(r_0) \\
\leq \frac{d}{3k\alpha} + \frac{c}{3k\alpha} r_0 + \frac{l}{3\alpha} \Omega(r_0) \leq \frac{r_0}{3k}, \tag{7}
\]

and

\[
||\Gamma_\alpha (x)(t)|| \leq ||T(t)[x_0 - g(x)]|| + \int_0^t ||T(t-s) f(s, x(s))|| ds \\
+ \int_0^t ||T(t-s) Bu(s)|| ds + \sum_{0,t_i < t} ||T(t-t_i) I_i(x(t_i))|| \\
\leq M \left[ ||x_0|| + ||g(0)|| + \sum_{i=1}^p ||I_i(0)|| \right] \\
+ M \left( k + \sum_{i=1}^p k_i \right) r_0 + M ||m||_{L^1} \Omega(r_0) + \sqrt{b} M M_1 ||u||_{L^2} \\
\leq \frac{d}{3} + \frac{c}{3} r_0 + \frac{l}{3} \Omega(r_0) + k ||u||_{L^2} \\
\leq \frac{1}{3} \left[ d + cr_0 + l \Omega(r_0) \right] + \frac{1}{3} r_0 \leq \frac{2r_0}{3}.
\]

Therefore, we get

\[
||\Gamma_\alpha x(t)|| \leq r_0.
\]
Thus $\Gamma_\alpha$ maps $W_{r_0}$ into itself.

Finally, according to Lemma 2.4, it remains to prove that $\Gamma_\alpha$ is a $\beta$-contraction in $W_{r_0}$.

Step 1. By using the conditions (H1) and (H1), for each $0 < \alpha \leq 1$, we get that $\Gamma_\alpha^1 + \Gamma_\alpha^1 : W_{r_0} \rightarrow PC(0, b; X)$ is Lipschitz continuous with constant $M(k + \sum_{i=1}^{p} k_i)$. In fact, for $x, y \in W_{r_0}$, we have

$$\| (\Gamma_\alpha^1 + \Gamma_\alpha^1)x - (\Gamma_\alpha^1 + \Gamma_\alpha^1)y \|_{PC} \leq M \left[ \sup_{t \in [0, b]} \| T(t) (g(x) - g(y)) \| + \sum_{i=1}^{p} \| T(t - t_i) (I_i(x(t_i)) - I_i(y(t_i))) \| \right] \leq M \left( k + \sum_{i=1}^{p} k_i \right) \| x - y \|,$$

according to Lemma 2.2, we obtain

$$\beta((\Gamma_\alpha^1 + \Gamma_\alpha^1)W_{r_0}) \leq M(k + \sum_{i=1}^{p} k_i)\beta(W_{r_0}).$$

Step 2. It is easy to prove that the operator $\Gamma_\alpha^2 : W_{r_0} \rightarrow PC(0, b; X)$, is compact by using Ascoli-Arzela’s theorem. (Also see Theorem 3.1 in [22]). Thus

$$\beta(\Gamma_\alpha^2 W_{r_0}) = 0. \quad (8)$$

Step 3. We give an estimate for $\beta(\Gamma_\alpha^3 W_{r_0})$. By the formula (4), for any $\epsilon > 0$, there are sequences $\{x_k\}_{k=1}^{+\infty} \subset W_{r_0}$ such that

$$\beta_\epsilon(\int_{0}^{t} T(t - s) B u(s) ds) \leq 2\beta_\epsilon(\{ \int_{0}^{t} T(t - s) B u_k(s) ds : k \geq 1 \}) + \epsilon,$$

where

$$u_k(t) = B^* T^* (b - t) R(\alpha, \Gamma_\alpha^0)[x_1 - T(b)[x_0 - g(x_k)] - \int_{0}^{b} T(b - s) f(s, x_k(s)) ds - \sum_{i=1}^{p} T(b - t_i) I_i(x_k(t_i))].$$

Let

$$\beta(\{u_k\}_{k=1}^{+\infty}) = \lambda,$$

then for all $\lambda' > \lambda$, there exists a finite family $\{v_1, v_2, \ldots, v_j\} \subset L^2(J; U)$, such that for any $u_k \in L^2(J; U)$, there exists $i \in \{1, 2, \ldots, j\}$ with

$$\| u_k - v_i \|_{L^2(J; U)} \leq \lambda'.$$

From the above inequality we can deduce that

$$\| \int_{0}^{t} T(t - s) B u_k(s) ds - \int_{0}^{t} T(t - s) B v_i(s) ds \| \leq \int_{0}^{t} M_1 | u_k(s) - v_i(s) | ds \leq M_1 \| v_i - u_k \|_{L^2(J; U)} \leq \sqrt{b} M_1 \| v_i - u_k \|_{L^2(J; U)} \leq \sqrt{b} M_1 \lambda',$$

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and thus
\[
\beta_c(\{ \int_0^T (t-s)Bu_k(s)ds: k \geq 1 \}) \leq \sqrt{b} M M_1(\{u_k\}_{k=1}^{\infty}).
\]
Since
\[
\beta(\{u_k\}_{k=1}^{\infty}) \leq \frac{1}{\alpha} M^2 M_1(\{k\}_{i=1}^{p} k_i)\beta(W_{r_0}),
\]
which implies that
\[
\beta_c(\Gamma^3_{\alpha} W_{r_0}) \leq \frac{2\sqrt{b}}{\alpha} M^2 M^3(\{k\}_{i=1}^{p} k_i)\beta(W_{r_0}) + \epsilon.
\]
Since \(\epsilon\) is arbitrary, we obtain
\[
\beta_c(\Gamma^3_{\alpha} W_{r_0}) \leq \frac{2\sqrt{b}}{\alpha} M^2 M^3(\{k\}_{i=1}^{p} k_i)\beta(W_{r_0}).
\]
Consequently,
\[
\beta(\Gamma^3_{\alpha} W_{r_0}) \leq \beta((\Gamma^1_{\alpha} + \Gamma^4_{\alpha}) W_{r_0}) + \beta(\Gamma^2_{\alpha} W_{r_0}) + \beta(\Gamma^3_{\alpha} W_{r_0}) \leq M\left(k + \sum_{i=1}^{p} k_i\right)\beta(W_{r_0}) + \frac{2\sqrt{b}}{\alpha} M^2 M^3\left(k + \sum_{i=1}^{p} k_i\right)\beta(W_{r_0}) = M\left(k + \sum_{i=1}^{p} k_i\right)\left(1 + \frac{2\sqrt{b}}{\alpha} M^2 M^2\right)\beta(W_{r_0}).
\]
According to the assumption \((Hf2)\),
\[
M\left(k + \sum_{i=1}^{p} k_i\right)\left(1 + \frac{2\sqrt{b}}{\alpha} M^2 M^2\right) < 1,
\]
the mapping \(\Gamma_{\alpha}\) is a \(\beta\)-contraction in \(W_{r_0}\). By Darbo-Sadovskii’s fixed point theorem, the operator \(\Gamma_{\alpha}\) has a fixed point in \(W_{r_0}\). Thus the problem \((1)\) has a solution on \(J\).

4 Approximate Controllability of impulsive system

In this section, we will consider the approximately controllable for the nonlocal impulsive problem \((1)\). For this purpose, we need the following Lemma.

**Lemma 4.1** [23] Suppose that \(A: D(A) \rightarrow X\) is the infinitesimal generator of a compact semigroup of uniformly bounded linear operators \(\{T(t), t \geq 0\}\) in Banach space \(X\). Then the operator \(Q: L^p(J; X) \rightarrow C(J; X)\) with \(p > 1\), given by
\[
(Qf)(\cdot) = \int_0^T T(\cdot - s)f(s)ds
\]
is strongly continuous.

**Theorem 4.2** Assume that the linear system \((2)\) is approximately controllable on \(J\). If the conditions \((HA), (Hf1), (Hf2), (H11), (H12), (Hg1)\) and \((Hg2)\) are satisfied then the nonlocal impulsive problem \((1)\) is approximately controllable.

**proof:** Let \(\hat{x}_{\alpha}(\cdot)\) be a fixed point of \(\Gamma_{\alpha}\) in \(W_{r_0}\). Any fixed point of \(\Gamma_{\alpha}\) is a mild solution of \((1)\) under the control
\[
\hat{u}_{\alpha}(t) = B^* T^*(b - t) R(\alpha, \Gamma^0_{\alpha}) p(\hat{x}_{\alpha}),
\]

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Thus the system (1) is approximately controllable on $J$.

Define

$$\hat{x}_\alpha(b) = x_1 + \alpha R(\alpha, \Gamma^b_0)p(\hat{x}_\alpha),$$

where

$$p(\hat{x}_\alpha) = x_1 - T(b)[x_0 - g(\hat{x}_\alpha)] - \int_0^b T(b-s)f(s, \hat{x}_\alpha(s))ds - \sum_{i=1}^m T(b-t_i)I_i(\hat{x}_\alpha(t_i)).$$

Let $f_\alpha(t) = f(t, \hat{x}_\alpha(t))$, by $(Hf1)$, we obtain that $f$ is bounded continuous operator from $J$ into $X$, hence $f_\alpha \in L^2(J,X)$. Furthermore, $\{f_\alpha(\cdot)\} \subseteq X$, $\{f_\alpha(\cdot)\}$ is bounded in $L^2(J,X)$, there is a subsequence, relabeled as $\{f_\alpha(\cdot)\}$, and $f(s) \in L^2(J,X)$ such that

$$f_\alpha(\cdot) \longrightarrow (w)f(\cdot) \text{ in } L^2(J,X).$$

By Lemma 4.1, we have

$$Qf_\alpha \longrightarrow (s)Qf \text{ in } C(J,X).$$

In addition, since $\{\hat{x}_\alpha\}$ is bounded in Hilbert space $X$, then there exists a subsequence, relabeled as $\{\hat{x}_\alpha\}$, and $x \in X$, such that

$\hat{x}_\alpha \longrightarrow (w)x, \text{ in } X.$

Define

$$w = x_1 - T(b)[x_0 - g(x)] - \int_0^b T(b-s)f(s)ds - \sum_{i=1}^m T(b-t_i)I_i(x(t_i)).$$

Then

$$\|p(\hat{x}_\alpha) - w\| \leq M\|g(\hat{x}_\alpha) - g(x)\| + \int_0^b T(b-s)\|f(s, \hat{x}_\alpha(s)) - f(s)\|ds$$

$$+ \sum_{i=1}^m T(b-t_i)\|I_i(\hat{x}_\alpha(t_i)) - I_i(x(t_i))\|.$$ 

Using $(Hg2)$, $(Hf2)$ and Lemma 4.1 again, we have

$$p(\hat{x}_\alpha) \rightarrow w, \text{ as } \alpha \rightarrow 0^+.$$

It comes from

$$\|\hat{x}_\alpha(b) - x_1\| \leq \|\alpha R(\alpha, \Gamma^b_0)(w)\| + \|\alpha R(\alpha, \Gamma^b_0)\|\|p(\hat{x}_\alpha) - w\|$$

$$\leq \|\alpha R(\alpha, \Gamma^b_0)(w)\| + \|p(\hat{x}_\alpha) - w\|,$$

that

$$\hat{x}_\alpha(b) \rightarrow x_1, \text{ as } \alpha \rightarrow 0^+.$$

Thus the system (1) is approximately controllable on $J$. 

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References


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